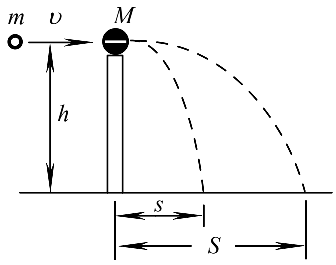
历届国际物理奥林匹克竞赛试题与解答

第 1 届

（1967 年于波兰的华沙）

【题 1】质量 M=0.2kg 的小球静置于垂直柱上，柱高 h=5m。一粒质量 m=0.01kg、以速度 $_0$=500m/s 飞行的子弹水平地穿过球心。球落在距离柱 s=20m 的地面上。问子弹落在地面何处？子弹动能中有多少转换为热能？



解：在所有碰撞情况下，系统的总动量均保持不变：

$$mv_0 = mv + MV$$

其中 v 和 V 分别是碰撞后子弹的速度和小球的速度. 两者的飞行时间都是 $t=\sqrt{\dfrac{2h}{g}}=1.01$ s

球在这段时间沿水平方向走过 20m 的距离，故它在水平方向的速度为：

$$V=\frac{20}{1.01}=19.8\ （\text{m/s}）$$

由方程 0.01×500=0.01v+0.2×19.8

可求出子弹在碰撞后的速度为：v=104m/s

子弹也在 1.01s 后落地，故它落在与柱的水平距离为 S=vt=104×1.01=105m 的地面上。

碰撞前子弹的初始动能为 $\dfrac{1}{2}mv_0^2$ =1250 J

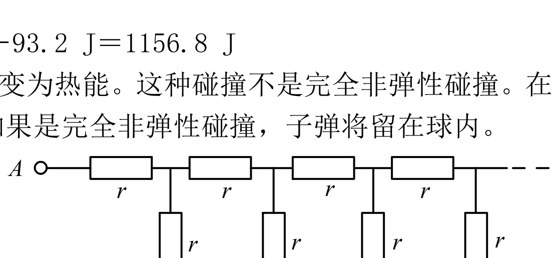
球在刚碰撞后的动能为 $\dfrac{1}{2}MV^2$ =39.2 J

子弹在刚碰撞后的动能为 $\dfrac{1}{2}mv^2$ =54 J

与初始动能相比，两者之差为 1250 J－93.2 J=1156.8 J

这表明原来动能的 92.5%被系统吸收而变为热能。这种碰撞不是完全非弹性碰撞。在完全弹性碰撞的情形下，动能是守恒的。而如果是完全非弹性碰撞，子弹将留在球内。

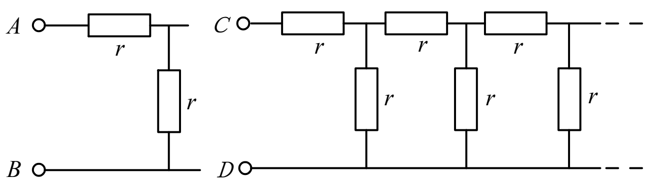
【题 2】右图（甲）为无限的电阻网络，其中每个电阻均为 r，求 A、B 两点间的总电阻。



图（甲）

解：如图（乙）所示

A、B 两点间的总电阻应等于 C、D 两点间的总电阻与电阻 r 的并联，再与 r 串联后的等效电阻。



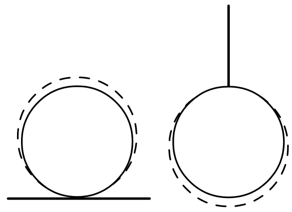
如果网络是无限的，则 A、B 两点间的总电阻应等于 C、D 两点间的总电阻，设为 R_x。

根据它们的串并联关系有：

$$R_x = r + \frac{rR_x}{R_x + r}$$ 图（乙）

解上式可得： $R_x = \frac{1+\sqrt{5}}{2}r$

【题 3】给定两个同样的球，其一放在水平面上，另一个以细线悬挂。供给两球相同的热量，问两球温度是否趋于相同？说明你的理由（忽略各种热量损失）



解答：如右图所示，球体受热，体积增大。放在水平面上的球重心升高，克服重力做功要耗费一部分热量，于是剩下提高球体温度的热量减少了些。以细线悬挂的球与之相反。结果放在水平面上球的温度将稍小于以细线悬挂球的温度。（这一差别是很小的，对于半径为 10cm 的铜球来说，相对差值约为 10^{-7} K）

【实验题】测定石油的比热。可供使用的物品有：天平、量热器、温度计、电源、开关、导线、停表、电热器、容器、水和石油。

解答：把已知温度 t_1 和质量 m_1 的水，与已知温度 t_2 和质量 m_2 的石油在量热器里混合，测出混合物的温度 t_3。从包含一方放热和另一方吸热的方程中可算出石油的比热。这是通常测定石油比热的方法。

也可以先用电热器加热水，再加热等量的石油，并且及时观察温度的改变。两条温度曲线起始点的切线斜率与比热成反比关系，据此可以测定石油的比热。

【替换题】（为在校没有上过电学的学生而设。）密闭容器中装有一个大气压、温度为 0 ℃的干燥空气 10 升，加入 3 克水后将系统加热到 100℃，求容器的压强。

解：在 100℃时，全部水都处于汽相。3 克水是 $\frac{1}{6}$ 摩尔（18÷3＝6），它们在 100℃和 1 atm 下的体积是：$22.4 \times \frac{1}{6} \times \frac{373}{273} = 5.11$（升）kg

由状态方程求出 $\frac{1}{6}$ 摩尔水蒸气的压强：

$$\frac{\frac{1}{6} \times 22.4}{273} = \frac{p_{水气} \times 10}{373}$$

解得： $p_{水气}$＝0.507 atm

由空气的状态方程： $\frac{1}{273} = \frac{p_{空气}}{373}$

解得： $p_{空气}$＝1.366 atm

把两部分压强相加得到总压强为：

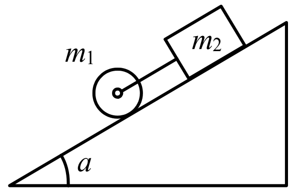
$p = p_{空气} + p_{水气}$＝1.366 atm＋0.507 atm＝1.873 atm

历届国际物理奥林匹克竞赛试题与解答

第 2 届

（1968 年于匈牙利的布达佩斯）

【题 1】 在倾角为 30^0 的斜面上，质量为 m_2＝4 kg 的木块经细绳与质量为 m_1＝8 kg 、半径为 r ＝5 cm 的实心圆柱体相连。求放开物体后的加速度。木块与斜面之间的动摩擦系数 μ＝0.2，忽略轴承的摩擦和滚动摩擦。



解：如果绳子是拉紧，则圆柱体与木块一同加速运动，设加速度为 a，绳子中的张力为 F，圆柱体与斜面之间的摩擦力为 S，则圆柱体的角加速度为 a/r。

对木块有：$m_2a = m_2g\sin\alpha - \mu m_2g\cos\alpha + F$

对圆柱体有：$m_1a = m_1g\sin\alpha - S - F$

$S\,r = Ia/r$

其中 I 是圆柱体的转动惯量，$S\,r$ 是摩擦力矩。

解以上方程组可得

$$a = g\frac{(m_1+m_2)\sin\alpha - \mu m_2\cos\alpha}{m_1+m_2+\frac{I}{r^2}} \qquad (1)$$

$$S = \frac{I}{r^2}g\frac{(m_1+m_2)\sin\alpha - \mu m_2\cos\alpha}{m_1+m_2+\frac{I}{r^2}} \qquad (2)$$

$$F = m_2g\frac{\mu(m_1+\frac{I}{r^2})\cos\alpha - \frac{I}{r^2}\sin\alpha}{m_1+m_2+\frac{I}{r^2}} \qquad (3)$$

均匀圆柱体的转动惯量为 $I = \frac{m_1r^2}{2}$

代入数据可得 a＝0.3317g＝3.25m/s^2

S＝13.01 N

F＝0.196 N

讨论：系统开始运动的条件是 $a>0$。把 $a>0$ 代入（1）式，得出倾角的极限 α_1 为：

$$\tan\alpha_1 = \mu\frac{m_2}{m_1+m_2} = \frac{\mu}{3} = 0.0667$$

$\alpha_1 = 3^0 49'$

单从圆柱体来看，$\alpha_1 = 0$；

单从木块来看，$\alpha_1 = \mathrm{tg}^{-1}\mu = 11^0 19'$

如果绳子没有拉紧，则两物体分开运动，将 F＝0 代入（3）式，得出极限角为：

$$\tan\alpha_2 = \mu(1+\frac{m_1r^2}{I}) = 3\mu = 0.6$$

$\alpha_2 = 30^0 58'$

圆柱体开始打滑的条件是 S 值（由（2）式取同样的动摩擦系数算出）达到 $\mu\ m_1g\cos\alpha$，由此得出的 α_3 值与已得出的 α_2 值相同。

圆柱体与木块两者的中心加速度相同，都为 $g(\sin\alpha-\mu\ g\cos\alpha)$ 圆柱体底部的摩擦力为 $\mu\ m_1g\cos\alpha$，边缘各点的切向加速度为

$$a=\mu\left(\frac{m_1r^2}{I}\right)g\cos\alpha,$$

【题 2】 一个杯里装有体积为 300 cm^3、温度为 0^0C 的甲苯，另一个杯里装有体积为 110 cm^3、温度为 100^0C 的甲苯，两体积之和为 410 cm^3。求两杯甲苯混合以后的最终体积。甲苯的体膨胀系数为 $\beta=0.001\ (^0C)^{-1}$，忽略混合过程中的热量损失。

解：若液体温度为 t_1 时的体积为 V_1，则在 0^0C 时的体积为

$$V_{10}=\frac{V_1}{1+\beta t_1}$$

同理，若液体温度为 t_2 时的体积为 V_2，则在 0^0C 时的体积为

$$V_{20}=\frac{V_2}{1+\beta t_2}$$

如果液体在 0^0C 时的密度为 d，则质量分别为

$$m_1=V_{10}d \qquad m_2=V_{20}d$$

混合后，液体的温度为

$$t=\frac{m_1t_1+m_2t_2}{m_1+m_2}$$

在该温度下的体积分别为 $V_{10}(1+\beta t)$ 和 $V_{20}(1+\beta t)$。所以混合后的体积之和为

$$V_{10}(1+\beta t)+V_{20}(1+\beta t)=V_{10}+V_{20}+\beta(V_{10}+V_{20})t$$

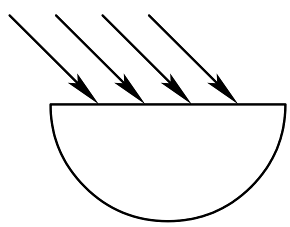
$$=V_{10}+V_{20}+\beta\frac{m_1+m_2}{d}\cdot\frac{m_1t_1+m_2t_2}{m_1+m_2}$$

$$=V_{10}+V_{20}+\beta\left(\frac{m_1t_1}{d}+\frac{m_2t_2}{d}\right)$$

$$=V_{10}+\beta V_{10}t_1+V_{20}+\beta V_{20}t_2=V_{10}(1+\beta t_1)+V_{20}(1+\beta t_2)$$

$$=V_1+V_2$$

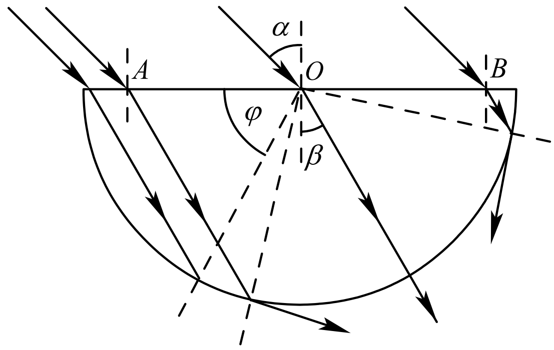
体积之和不变，在本题仍为 410 cm^3。当把多杯甲苯不断地加入进行混合，对任何数量的甲苯这个结果都成立。



【题 3】光线在垂直玻璃半圆柱体轴的平面内，以 45^0 角射在半圆柱体的平面上（如右图），玻璃的折射率为 $\sqrt{2}$。试问光线在何处离开圆柱体表面？

解：用角度 Ψ 描述光线在玻璃半圆柱体内的位置如解图 2.3 所示。按照折射定律：

$$\frac{\sin 45^0}{\sin\beta}=\sqrt{2}$$



得：sin　=　　　，　=30^0

所有折射光线与垂直线的夹角均为30^0，有必要研究一下，当Ψ角从0^0增至180^0的过程中发生了什么现象。

不难看出，Ψ角不可能小于60^0。

光线从玻璃射向空气全反射的临界角由解图 3.2

$$\sin\beta_t=\frac{1}{n}=\frac{\sqrt{2}}{2}$$

求出：　$_t$=45^0，

则：Ψ_t=$180^0-60^0-45^0=75^0$

如果Ψ角大于75^0，光线将离开圆柱体。随着Ψ角的增加，光线将再次发生全反射，此时Ψ_t=$90^0+30^0+45^0=165^0$

故当：$75^0<\Psi<165^0$时光线离开圆柱体。出射光线的圆弧所对应的圆心角为$165^0-75^0=90^0$。

【实验题】参加者每人领取三个封闭的盒子，每个盒上有两个插孔。不许打开盒子，试确定盒中元件的种类，并测定其特性。可供使用的是，内阻和精度已知交流和直流仪器，以及交流电源（频率 50 HZ）和直流电源。

解：在任何一对插孔中都测不到电压，因此，盒子都不含有电源

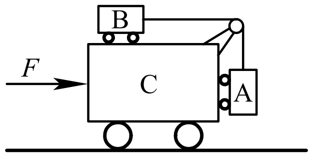
先用交流，再用直流测电阻，有一盒给出相同的结果。结论是：该盒包含一个简单电阻，其阻值由测量确定。

另一盒有极大的直流电阻，但对交流来说是导体。结论是：该盒包含一个电容，其电容值由$C=\dfrac{1}{\omega R}$算得。

第三个盒子对交流和直流都是导体，而交流电阻较大。结论是：该盒包含一个电阻和电感，两者串联。电阻和电感值可从测量中算得。

（1969年于捷克斯洛伐克的布尔诺）

【题1】右图的力学系统由三辆车组成，质量分别为 m_A＝0.3kg，m_B＝0.2kg，m_C＝1.5kg。



（a）沿水平方向作用于 C 车的力 F 很大。使 A、B 两车相对 C 车保持静止。求力 F 及绳子的张力。

（b）C 车静止，求 A、B 两车的加速度及绳子的张力。

（忽略阻力和摩擦力，忽略滑轮和车轮的转动惯量）

解：（a）A、B 两车相对 C 车保持静止，A 车在竖直方向没有加速度，因此它对绳的拉力为 m_Ag。这个力使 B 车得到加速度 $a_B=\dfrac{m_A}{m_B}g$。又三车系统以相同的加速度运动，则：

$$F=(m_A+m_B+m_C)\frac{m_A}{m_B}g$$

由给定的数值得：$a_B=a_C=a_A$＝1.5g＝14.7m/s^2

绳中的张力为：$T=m_A$g＝2.94N

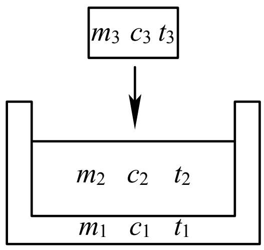
水平推力为：F＝29.4N

（b）如果 C 车静止，则力 m_Ag 使质量 m_A+m_B 加速，加速度为：

$$a_{AB}=\frac{m_A g}{m_A+m_B}=0.6\text{g}=5.88\text{N}$$

绳中的张力为：$T'=m_A\text{g}-m_A\times0.6\text{g}$＝1.176N

【题2】在质量为 m_1 的铜量热器中装有质量为 m_2 的水，共同的温度为 t_{12}；一块质量为 m_3、温度为 t_3 的冰投入量热器中（如右图所示）。试求出在各种可能情形下的最终温度。计算中 t_3 取负值。铜的比热 c_1＝0.1kcal/kg·^{0}C，水的比热 c_2＝1kcal/kg·^{0}C，冰的比热 c_3＝0.5kcal/kg·^{0}C，冰的熔解热 L＝80kcal/kg。



解：可能存在三种不同的终态：（a）只有冰；（b）冰水共存；（c）只有水。

（a）冰温度升高，但没有熔化，达到某一（负）温度 t_a；放出的热量和吸收的热量相等：

$$c_3m_3(t_a-t_3)=(c_1m_1+c_2m_2)(t_{12}-t_a)+m_2L$$

得出最终的温度为 $t_a=\dfrac{(m_1c_1+m_2c_2)t_{12}+m_3c_3t_3+m_2L}{m_1c_1+m_2c_3+m_3c_3}$　　（1）

情况（a）的条件是 $t_a<0$（注：指 0^0C），如果上式的分子为负值，我们得到下列条件：

$$(c_1m_1+c_2m_2)t_{12}<-c_3m_3t_3-m_2L \qquad (2)$$

（c）现在让我们讨论冰块全部熔化的情况。设它们最终的温度为 t_c，冰块吸收的热量等于量热器和水放出的热量：$c_3m_3(0-t_3)+m_3L+c_2m_3t_c=(c_1m_1+c_2m_2)(t_{12}-t_c)$

得出最终的温度为 $t_c=\dfrac{(m_1c_1+m_2c_2)t_{12}+m_3c_3t_3-m_3L}{m_1c_1+m_2c_2+m_3c_2}$　　（3）

这种情况只有在 $t_c>0$ 时才能发生。取上式的分子为正值，得到下列条件：

$$(c_1m_1+c_2m_2)t_{12}>-c_3m_3t_3+m_3L \qquad (4)$$

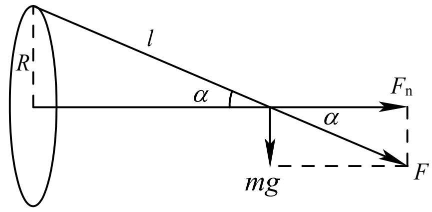
（b）冰水共存这种情况是冰和水混合后都以 0^0C 共存于量热器中。根据（2）式和（4）式，条件为：$-c_3m_3t_3-m_2L<(c_1m_1+c_2m_2)t_{12}<-c_3m_3t_3+m_3L$

如果混合后有 x 克冰熔化了，则 $-c_3\ m_3 t_3+x\ L=(c_1\ m_1+c_2\ m_2)\ t_{12}$

故冰熔化了的质量为 $x=\dfrac{(m_1c_1+m_2c_2)t_{12}+m_3c_3t_3}{L}$

于是混合后，在量热器中有质量为（m_3-x）的冰和质量为（m_2+x）的水。x 为负值意味着有水结为冰，冰的质量增加。对于给定的数值，我们可以从公式容易得到最终的结果。

【题 3】在竖直平面内有半径 R＝5cm 的线圈（如图）。质量 m＝1g 的小球系在长度为 l 的绝缘轻绳上，从线圈的最高点悬挂着。当线圈和小球两者都带有 Q＝9×10^{-8}C 的相同电量时，发现小球在垂直线圈平面的对称轴上处于平衡。求绳的长度。



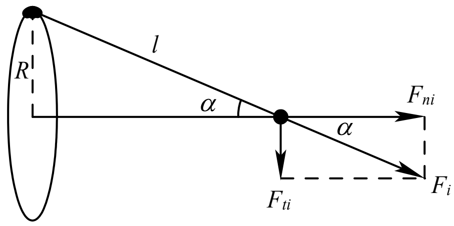
解：如果线圈上的全部电荷集中与一点，则库仑力为 $F=k\dfrac{Q^2}{l^2}$

线圈上各点施于小球的力与对称轴夹角为 ，它们在轴上的投影为 F_n＝Fcos 。小球的重量为 mg。由上图可得：$\sin\alpha=\dfrac{mg}{F}=\dfrac{R}{l}=\dfrac{mg}{k\dfrac{Q^2}{l^2}}$

所以：$l=\sqrt[3]{\dfrac{RkQ^2}{mg}}$＝7.2cm（$k$＝9×10^{9}N m^2/C^2）

（注：以上解答为原解，可能有错）

另解：如解答图 3.3.1，在线圈上取一电荷微元，长为 d ，电荷量为 d ， 为线电荷密度，2πR ＝Q。则微元电荷对小球的作用力为：



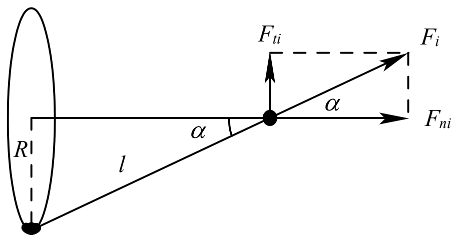
$$F_i=k\frac{\lambda dQ}{l^2}$$

把 F_i 沿平行轴和垂直轴分解：F_{ni}＝F_i cos

解答图

F_{ti}＝F_i sin

在线圈上取与上电荷微元对称的电荷微元，如解答图 3.3.2。对称的电荷微元，长也为 d ，电荷量为 d ，它对小球的作用力为：$F_i^{/}=k\dfrac{\lambda dQ}{l^2}$



把 F_i 沿平行轴和垂直轴分解：

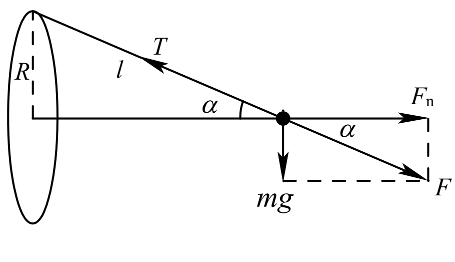
$F_{n\ i}^{/}$＝$F_i^{/}$cos

解答图 3.3.2

$F_{t\ i}^{/}$＝$F_i^{/}$sin

F_{ni} 与 $F_{n\ i}^{/}$ 方向相同，合力为大小相加，F_{ti} 与 $F_{t\ i}^{/}$ 方向相反，合力为大小相减，等于零。所以线圈对小球作用的库仑力为：

$$F_n=\sum F_{ni}=k\frac{2\pi\lambda Q}{l^2}\cos\alpha=k\frac{Q^2}{l^2}\cos\alpha$$



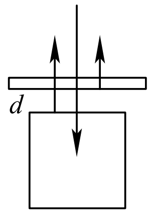
对小球受力分析，小球受三力作用：重力 mg、库仑力 F_n、拉力 T，如解答图 3.3.3。则：

$$\frac{l\cos\alpha}{R}=\frac{F_n}{mg}$$

解答图 3.3.3

把 $F_n = k\dfrac{Q^2}{l^2}\cos\alpha$ 代入上式解得：$l=\sqrt[3]{\dfrac{RkQ^2}{mg}}$ ＝7.2cm　　（k＝9×10^9N m^2/C^2）

【题 4】一块平板玻璃放置在边长为 2cm 的玻璃立方体上，两者之间有一层平行的薄空气隙。波长在 0.4μm 到 1.15μm 之间的电磁波垂直入射到平板上，经空气隙的两边表面反射而发生干涉。在此波段中只有两种波长获得极大的增强，其一是 ${}_1$＝0.4μm。求空气隙的厚度。



解：光在厚度为 d 的空气隙中往返，经过的距离为 2d。光被玻璃反射时，还经受 180^{0}的相位改变。于是对波长为 ${}_1$的光，增强的条件为：

$$2d = k_1\lambda_1 + \frac{\lambda_1}{2} \quad (k_1=0, 1, 2, 3, \cdots\cdots)$$

类似地，对其它波长的光，产生极大增强的条件是：

$$2d = k_2\lambda_2 + \frac{\lambda_2}{2} \quad (k_2=0, 1, 2, 3, \cdots\cdots)$$

比较这两个条件，得到：$\dfrac{2k_1+1}{2k_2+1}=\dfrac{\lambda_2}{\lambda_1}$

根据波长给定的范围，得到：$\dfrac{\lambda_2}{\lambda_1}=\dfrac{1.15}{0.4}=2.875$

这个比值的最小可能值为 1，最大可能值为 2.875。因此我们得到关于 k_1 和 k_2 的下列条件：$1<\dfrac{2k_1+1}{2k_2+1}<2.875$　　　（1）

对不同的 k_1 和 k_2，我们算出上述分数值，得到下表：

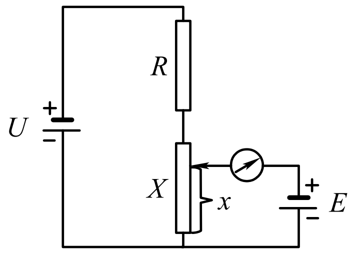
k_2 \ k_1	0	1	2	3	4	5
0	1	3	5	7	9	11
1	0.33	1	1.67	2.33	3	3.67
2	0.2	0.6	1	1.4	1.8	2.2
3	0.14	0.43	0.71	1	1.29	1.57
4	0.11	0.33	0.56	0.78	1	1.22
5	0.09	0.27	0.45	0.64	0.81	1

只有分数值满足条件（1）式的各个 k_1 和 k_2 对才是合格的，我们已在表格中算出。但其中只有一对是允许的。这就是说，我们应当找出这样的一列，其中只能有一对是允许的 k_1 和 k_2。从表中看出，仅有的是 k_1＝2，k_2＝1 这一对，其分数值是 1.67，这就是解答。对于 k_1＝0.4μm 的光，根据 2d＝2×0.4＋0.2＝1μm，得到空气隙的厚度为 d＝0.5μm

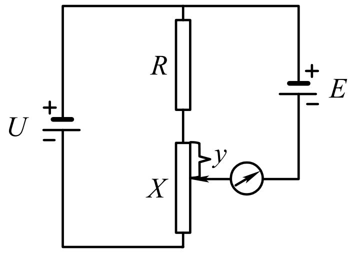
由 2×0.5＝$\lambda_2+\frac{\lambda_2}{2}$

得到第二个波长为 k_2＝0.667μm

【实验题】给定一闭合电路，它是由已知电阻 R、未知电阻 X 以及内阻可以忽略的电源组成的。电阻 X 是可调电阻器，由引线、毫米标尺、滑动接触块组成。另一电路由干电池和零点在中心的电流计组成，它与主电路的连接方式使得没有电流流过电流计。试测定电阻 X 及端电压之比。



解答图 3.5.1



解答图 3.5.2

解答：联接两种补偿电路，如解答图 3.5.1 和解答图 3.5.2。第一次测量不包括 R。滑动接触块的位置在第一次测量中由比率 x 给出，在第二次测量中由 y 给出，在此两中测量下，电阻值之比等于电势差之比，所以有

$$\frac{E}{U}=\frac{xX}{R+X},\qquad \frac{E}{U}=\frac{R+yX}{R+X}$$

解得：$X=R(\frac{1}{x-y})$

把 $X=R(\frac{1}{x-y})$ 代入 $\frac{E}{U}=\frac{xX}{R+X}$ 得：$\frac{E}{U}=\frac{x}{1+x-y}$

历届国际物理奥林匹克竞赛试题与解答

第 4 届

（1970 年于苏联的莫斯科）

【题 1】如图 4.1（a）、（b），在质量 M＝1kg 的木板上有质量 m＝0.1kg 的小雪橇。雪橇

上的马达牵引着一根绳子，使雪橇以速度 v_0＝0.1m/s 运动。忽略桌面与木板之间的摩擦。木板与雪橇之间的摩擦系数 μ＝0.02。把住木板，起动马达。当雪橇达到速度 v_0时，放开木板。在此瞬间，雪橇与木板端面的距离 L＝0.5m。绳子拴在（a）远处的桩子，（b）木板的端面上。

试描述两种情形下木板与雪橇的运动。雪橇何时到达木板端面？

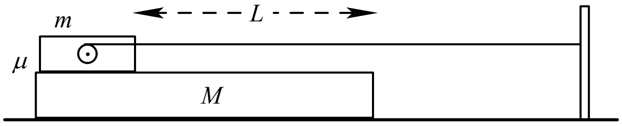


图 4.1（a）

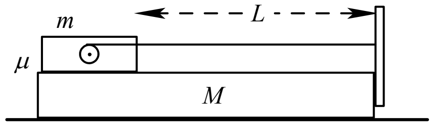


图 4.1（b）

解：（a）在第一种情形中（如图 4.1（a）），雪橇处于匀速运动状态。

雪橇与木板以不同的速度运动。这样引起的<u>最大</u>摩擦力为 mg，它作用在木板上，产生的加速度 $a=\dfrac{\mu mg}{M}$，直至木板达到雪橇的速度 v_0为止。加速时间为 $t_0=\dfrac{v_0}{a}=\dfrac{v_0 M}{\mu mg}=$ 5.1s

在这段时间内，雪橇的位移为 $S_0=\dfrac{v_0^2}{2a}=\dfrac{v_0^2 M}{2\mu mg}$＝0.255m

因此，雪橇离木板右端点的距离为 0.5m－0.255m＝0.245m

雪橇不能达到木板的一端，因为这段时间以后，木板与雪橇以相同的速度 v_0一起运动。在木板加速期间，马达必须用力 mg 牵引绳子，但以后马达不能施加力的作用，它只是卷绳子。

（b）在第二种情形中（如图 4.1（b）），木板与桌面之间无摩擦。木板与雪橇形成一个孤立系统，可以用动量守恒定律。当我们放开木板时，雪橇的动量为 mv_0，释放后的木板具有速度 v_2，它由下式决定：

mv_0＝$M\ v_2$＋m（v_0＋v_2）

此式表明 v_2＝ 0 ，所以木板保持不动，雪橇以同一速度继续前进。

雪橇达到木板右端的时间为 $t=\dfrac{L}{v_0}=\dfrac{0.5}{0.1}$＝5 s

【题 2】NaCl 的晶体点阵由边长为 5.6×10^{-8}cm 的立方晶胞组成，它是面心立方点阵。钠原子量约为 23，氯原子量为 35.5，NaCl 密度为 2.22g/cm^3。试计算氢原子的质量（如图 4.2）。

解：我们先求出一个晶胞的 Na 离子数。在立方晶胞中心有一个离子，在立方晶胞的每一边也有一个离子，但后者仅有四分之一是属于这个晶胞的。

故钠离子数为：$1+\dfrac{12}{4}=4$

氯离子也是这个数。密度可以表示为晶

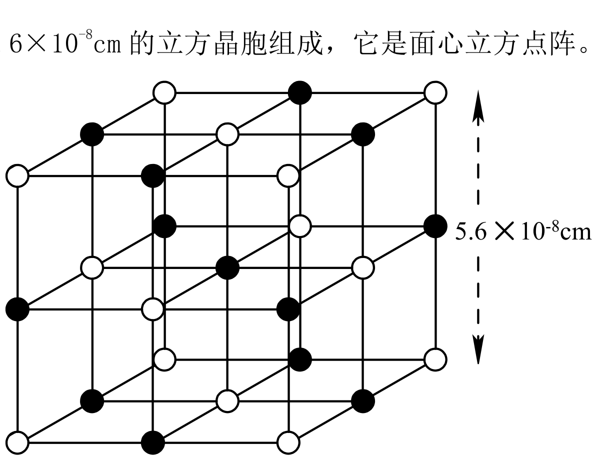


图 4.2

胞的质量与体积之比，故若用 m 表示氢原子的质量，则密度可表示为：

$$\rho = \frac{4\times 23m + 4\times 35.5m}{(5.6\times 10^{-8})^3} = 2.22$$

解上式可求得氢原子的质量为

m＝1.66×10^{-24}g＝1.66×10^{-27}kg

【题 3】半径 r＝10cm 的金属球置于半径 R＝20cm 的薄金属空心球内，两球同心。内球靠一根长导线经过外球的开孔接地。若外球带电量 Q＝10^{-8}C，求外球电势（如图 4.3）。

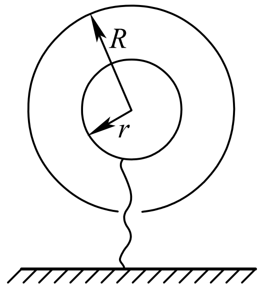


图 4.3

解：这里有两个电容，并联连接。其一由外球和内球组成，另一由地与外球组成。由电容相加便可算出电势。

导体球相对远处地球的电容为 $\frac{R}{k}$，其中 k＝9×10^9 N m²/C²，R 为导体球半径。在空心球情形，如果内球接地①，电容为：

$$\frac{1}{C_a} = k(\frac{1}{r} - \frac{1}{R}),$$

所以：$C_a = \frac{1}{k}\cdot\frac{Rr}{R-r}$

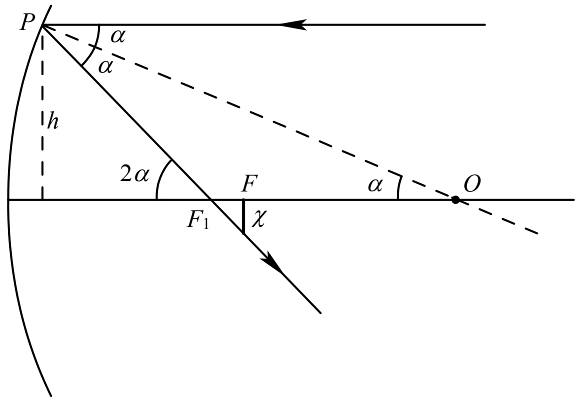
两个电容并联总电容为：$C = \frac{R}{k} + \frac{1}{k}\cdot\frac{Rr}{R-r} = \frac{1}{k}\cdot\frac{R^2}{R-r}$

把 R＝0.2m，r＝0.1m，k＝9×10^9 N m²/C² 代入上式得：C＝44.4×10^{-12}F＝44.4 pF

故外球相对与地球的电势为：$U = \frac{Q}{C}$＝225V

（注：① C_a 是内外球组成的球形电容器的电容，与内球是否接地无关。）

【题 4】在半径 r＝2m、孔径 d＝0.5m 的凹面镜的焦点位置上，放一块圆形屏幕，使平行于轴的所有入射光线经凹面镜反射后都能达到该圆形屏幕。试求圆形屏幕的直径。如果在上述条件下圆形屏幕的直径减少到仅由原来的 1/8，问有多少部分的光能达到在同样位置的屏幕上？



图

解：我们只有采用较精确形式的反射定律，通过利用某些数学近似来求解本题。

按照教科书中通常的理论推导，半径 PO＝R 的凹面镜的焦点位于距离 R 的中点 F 处。我们用 h 表示凹面镜孔径之半。在 P 点的入射光线与半径的夹角为 ，反射后与轴交于 F_1 点。$OP F_1$ 是等腰三角形。

则：$OF_1 = \frac{R}{2\cos\alpha}$

故实际焦点与理论距离的偏差为

$$FF_1 = OF_1 - OF = \frac{R}{2\cos\alpha} - \frac{R}{2} = \frac{R}{2}(\sec\alpha - 1)$$

我们把圆形屏放在点 F 处，要求出屏幕的最小半径值 x。在直角三角形 PFF_1 中，应用通常的小角近似，得：$x = F_1F\tan 2\alpha \approx F_1F\sin 2\alpha = F_1F\dfrac{2h}{R} = \dfrac{R}{2}(\sec\alpha - 1)\dfrac{2h}{R} = h(\sec\alpha - 1)$

对于小角度：$\cos\alpha \approx 1 - \dfrac{\alpha^2}{2}$，故 $\sec\alpha = \dfrac{1}{\cos\alpha} \approx 1 + \dfrac{\alpha^2}{2}$

将 $\alpha \approx \dfrac{h}{R}$ 代入，得焦“斑”的半径为 $x = \dfrac{h^3}{2R^2}$

将数值：h＝50/2＝25cm；R＝200cm，代入

即得：x＝0.195cm＝1.95mm

再看问题的第二部分。如果圆形屏的半径为 x，则入射到凹面镜的光束半径为

$$h = \sqrt[3]{2R^2x}$$

如果我们用半径 kx 的屏代替半径为 x 的屏，则入射光束的半径为：

$$h_k = \sqrt[3]{2R^2kx}$$

入射光的量正比于 h_k^2，因此

$$h_k^2 = (\sqrt{2R^2kx})^2 = h^2\sqrt[3]{k^2}$$

本题情形是 $k = \dfrac{1}{8}$，由此得出，落在圆形屏幕上光的量将是前者的 $\dfrac{1}{4}$

【实验题】桌上有三个装在支架上的透镜，一块有几何图形的屏，一支杆和一把卷尺。仅用所给的工具，以不同的方法测定透镜的焦距。

解答：有几种可能的方法。在凸透镜情形，我们用目视观查虚像的消失，并测定透镜的距离。

我们注视着实像，借助于视差把杆放在实像的位置上，测量物距和像距，从而计算出焦距。

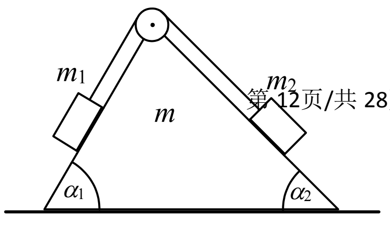
再看凹透镜情形。我们把凹透镜与一个强会聚的凸透镜密接在一起，并用上述方法之一测量系统的焦距，然后算出凹透的焦距。

历届国际物理奥林匹克竞赛试题与解答

第5届

（1971年于保加利亚的索菲亚）

【题1】质量为 m_1 和 m_2 的物体挂在绳子的两端，绳子跨过双斜面顶部的滑轮，如图5.1。斜面质量为 m，与水平面的夹角为 $_1$ 和 $_2$。整个系



统初态静止。求放开后斜面的加速度和物体的加速度。斜面保持静止的条件是什么？摩擦可以忽略。

解：我们用 a 表示双斜面在惯性参照系中的加速度（正号表示向右的方向）。用 a_0 表示物体相对斜面的加速度（正号表示左边物体 m 下降）两个物体在惯性系中的加速度 a_1 和 a_2 可由矢量 a 和 a_0 相加得到（如解　　　　　图 5.1

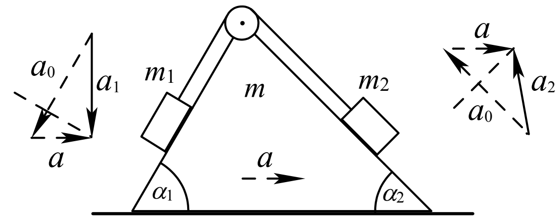
图 5.1）。用 F 表示绳子中的张力。

对沿斜面方向的分量应用牛顿第二定律。使物体 m_1 加速下降的力是

$m_1g\sin{}_1-F$

在惯性系中，沿斜面方向的加速度分量为

$a_0-a\cos{}_1$



解图 5.1

所以，对此斜面分量，牛顿第二定律为：

$m_1（a_0-a\cos{}_1）=m_1g\sin{}_1-F$

同样，对于 m_2 有

$m_2（a_0-a\cos{}_2）=F-m_2g\sin{}_2$

两式相加：$(m_1\cos{}_1+m_2\cos{}_2)a=(m_1+m_2)a_0-(m_1\sin{}_1-m_2\sin{}_2)g$　　（1）

我们用动量守恒原理来研究斜面的运动。

斜面在惯性系中的速度为 V（向右）。物体相对斜面的速度为 V_0。故斜面上两物体在惯性系中的速度的水平分量（向左）分别为：$V_0\cos{}_1-V$　和　$V_0\cos{}_2-V$

利用动量守恒原理：$m_1（V_0\cos{}_1-V）+m_2（V_0\cos{}_2-V）=m\ V$

对匀加速运动，速度与加速度成正比，因此有：$m_1（a_0\cos{}_1-a）+m_2（a_0\cos{}_2-a）=m\ a$

所以 $a=\dfrac{m_1\cos\alpha_1+m_2\cos\alpha_2}{m+m_1+m_2}a_0$　　　（2）

上式给出了有关加速度的信息。很明显，只有当两物体都静止，即两个物体平衡时，斜面才静止，这是动量守恒原理的自然结果。

由方程（1）和（2），可得到加速度为：

$$a_0=\frac{(m+m_1+m_2)(m_1\sin\alpha_1-m_2\sin\alpha_2)}{(m_1+m_2)(m+m_1+m_2)-(m_1\cos\alpha_1+m_2\cos\alpha_2)^2}g$$

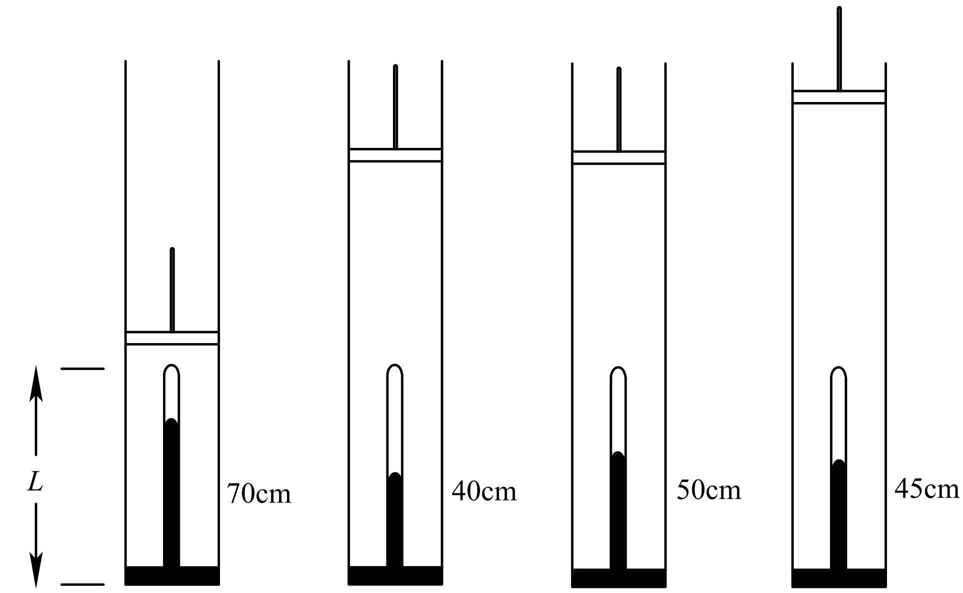
$$a=\frac{(m_1\cos\alpha_1+m_2\cos\alpha_2)(m_1\sin\alpha_1-m_2\sin\alpha_2)}{(m_1+m_2)(m+m_1+m_2)-(m_1\cos\alpha_1+m_2\cos\alpha_2)^2}g$$

如果 $m_1\sin{}_1=m_2\sin{}_2$　　即　　$\dfrac{m_1}{m_2}=\dfrac{\sin\alpha_2}{\sin\alpha_1}$

则两个加速度均为零。

【题 2】在一个带活塞的圆筒内装配着著名的托里拆利装置。在水银柱上方有氢气，在圆筒内有空气。第一步，水银柱高度 h_1＝70cm，空气压强 p_{k1}＝1.314atm＝133.4kPa＝100cmHg，温度为 0^0C＝273K。第二步，向上提升活塞，直至水银柱高度降为 h_2＝40cm，这时空气压强为 p_{k2}＝0.79atm＝80kPa＝60cmHg。第三步，保持体积不变，提高温度到 T_3，此时水银柱的高度为 h_3＝50cm。最后，第四步，温度为 T_4，水银柱的高度为 h_4＝45cm，空气压强没有改变。求出最后一步中氢气的温度和压强。

解：我们将空气和氢气的数据列成表。两者温度是相同的。玻璃管的长度用 L 表示。为了简单起见，我们以装有氢气的管子长度的厘米数来度量氢气的体积。压强全部用 cmHg 为单位给出（见解图 5.2 第一步至第四步）。



次　　数	1	2		3		4
氢气压强	p_{h1}	p_{h2}		p_{h3}		p_{h4}
氢气体积	V_{h1}	V_{h2}		V_{h3}		V_{h4}
空气压强	100cmHg	60cmHg		p_{k3}	$=$	p_{k4}
空气体积	V_{k1}	V_{k2}	$=$	V_{k3}		V_{k4}
两者温度	273K	273K		T_3		T_4

解图 5.2

从第一步到第二步，对氢气应用玻意耳定律：（L－70）（100－70）＝（L－40）（60－40）

由此式求得玻璃管的长度 L＝130cm，

因此，氢气在第一步至第四步中体积分别为：V_{h1}＝60cm，V_{h2}＝90cm，V_{h3}＝80cm，V_{h4}＝85cm

从第二步到第三步，氢气的状态方程为：$\dfrac{(60-40)\times 90}{273}=\dfrac{(p_{h3}-50)\times 80}{T_3}$

对空气应用盖吕萨克定律：$\dfrac{p_{k3}}{T_3}=\dfrac{60}{273}$

从第三步到第四步，我们只有向上提升活塞，以便使空气压强保持不变。氢气的状态方程为：$\dfrac{(p_{k3}-50)\times 80}{T_3}=\dfrac{(p_{k4}-45)\times 85}{T_4}$

解以上方程组，得：p_{k3}＝p_{k4}＝80cmHg，　T_3＝364K，　T_4＝451K，

所以氢气的压强为：p_{h3}＝30cmHg　　　p_{h4}＝35cmHg

算出空气的体积比为：V_{k1}:V_{k2}:V_{k4}＝6:10:12.4

（注：cmHg 为实用单位，应转换成国际单位 Pa）

【题3】四个等值电阻 R、四个 C＝1　F 的电容器以及四个电池分别在立方体的各边连接起来，如图 5.3 所示。各电池的电压为 U_1＝4V，U_2＝8V，U_3＝12V，U_4＝16V，它们的内电阻均可忽略。（a）求每个电容器的电压和电量，（b）若 H 点与 B 点短路，求电容器 C_2 上的电量。

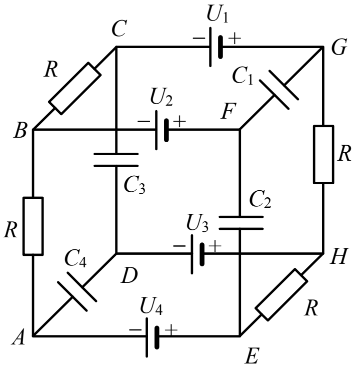
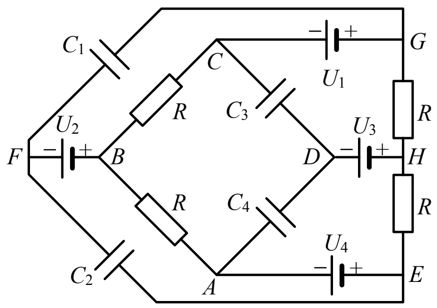


图 5.3



解图 5.3.1

解：（a）将这个网络展开成平面图（如解图 5.3.1）。由于电流不能通过电容器，所以只在图中 A-B-C-G-H-E-A 回路的导线中有电流。在这个回路中，电压为 12V，电阻为 $4R$。

因此电流为：$I=\dfrac{U_4-U_1}{4R}$

于是就知道了电阻和电源两端的电压。设 A 点的电势为零，就能很容易地算出各点的电势。

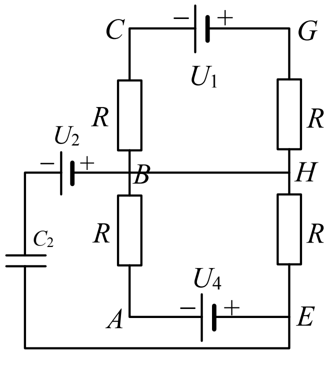
A		0 V
B	$(U_4-U_1)/4$	3 V
C	$(U_4-U_1)/2$	6 V
G	$(U_4-U_1)/2+U_1$	10 V
H	$(U_4-U_1)/2+U_1+(U_4-U_1)/4$	13 V
E	$(U_4-U_1)/2+U_1+(U_4-U_1)/2$	16 V
D	$(U_4-U_1)/2+U_1+(U_4-U_1)/4-U_3$	1 V
F	$(U_4-U_1)/4-U_3+U_2$	11 V

从每个电容器两端的电势差，可以算出其电量如下：

C_1	（11－10）V＝1V，	1×10^{-6}C。
C_2	（16－11）V＝5V，	5×10^{-6}C。
C_3	（6－1）V＝5V，	5×10^{-6}C。
C_4	（1－0）V＝1V，	1×10^{-6}C。

我们可以算出各电容器的储能量 $CU^2/2$。电容器 C_1 和 C_4 各有 0.5×10^{-6} J，电容器 C_2 和 C_3 各有 12.5×10^{-6} J。

（b）H 点与 B 点连接，我们得到两个分电路。如解图 5.3.2。在下方的分电路中，电流为 $\dfrac{U_4}{2R}$，E 点相对 A 点的电势是 U_4＝16 V，H 点与 B 点的电势是 $U_4/2$＝8 V。F 点的电势为 $\dfrac{U_4}{2}+U_2$＝16 V



于是，电容器 C_2 两极板的电势均为 16 V，结果 C_2 上无电量。

解图 5.3.2

【题 4】在直立的平面镜前放置一个半径为 R 的球形玻璃鱼缸，缸壁很薄，其中心距离镜面 $3R$，缸中充满水。远处一观察者通过球心与镜面垂直的方向注视鱼缸。一条小鱼在离镜面最近处以速度 v 沿缸壁游动。求观察者看到的鱼的两个像的相对速度。水的折射率为 $n=\dfrac{4}{3}$。如图 5.4（a），5.4（b）

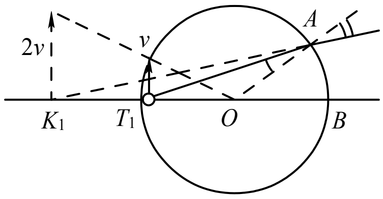


图 5.4（a）

解：鱼在 1 秒钟内游过的距离为 v。我们把这个距离当作物，而必须求出两个不同的像。在计算中，我们只考虑近轴光线和小角度，并将角度的正弦用角度本身去近似。

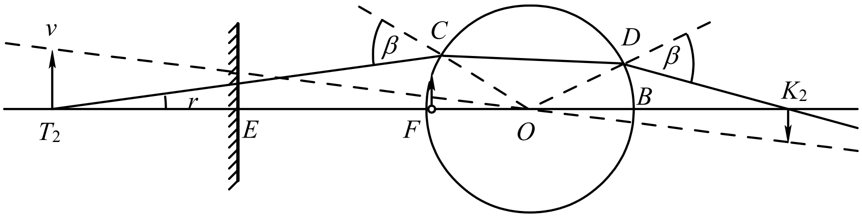


图 5.4（b）

在 T_1 点游动的鱼只经过一个折射面就形成一个像，如图 5.4（a）所示。从 T_1 点以角度 $r=\angle AT_1O$ 发出的光线，在 A 点水中的入射角为 r，在空气中的折射角为 nr。把出射光线向相反方向延长，给出虚像的位置在 K_1，显然 $\angle K_1AT_1=nr-r=(n-1)r$

从三角形 K_1T_1A，有：$\dfrac{K_1T_1}{K_1A}=\dfrac{(n-1)r}{r}=n-1$

利用通常的近似：$K_1A\approx K_1O+R$，　　$K_1AT_1\approx K_1O-R$

于是 $\dfrac{K_1O-R}{K_1O+R}=n-1$

所以这个虚像与球心的距离为 $K_1O=\dfrac{n}{2-n}R$

水的折射率 $n=\dfrac{4}{3}$，从而 $K_1O=2R$。若折射率大于 2，则像是实像。有像距与物距之商得到放大率为 $\dfrac{K_1O}{T_1O}=\dfrac{n}{2-n}$

对水来说，放大率为 2。

以与速度 v 相应的线段为物，它位于在 E 处的平面镜前的距离为 $2R$ 处，它在镜后 $2R$ 远的 T_2 处形成一个与物同样大小的虚像。T_2 离球心的距离为 $5R$。在一般情形下，我们假设 $T_2O=kR$。T_2 处的虚像是我们通过球作为一个透镜观察时的（虚）物。因此，我们只要确定 T_2 的实像而无需再去考虑平面镜。如图 5.4（b）所示。

我们需要求出以 r 角度从 T_2 发出的光线在 C 点的入射角 β，其中 $r=\angle CT_2F$。

在三角形 T_2OC 中，$\dfrac{\beta}{r}=\dfrac{T_2O}{CO}=\dfrac{kR}{R}=k$　　　$\beta=kr$

玻璃中的折射角为：$\dfrac{\beta}{n}=\dfrac{kr}{n}=\angle DCO=\angle CDO$

需要算出∠DOB。 因为：∠COF＝β－r＝$k\ r$－r＝r（k－1）

而且∠COD与C点和D点的两角之和相加，或与∠COF和∠DOB之和相加，两种情况都等于 180^0，因此$\angle DOB + r(k+1) = \frac{2kr}{n}$

即$\angle DOB = r(\frac{2k}{n} - k + 1)$

从三角形 DOK_2，有$\frac{OK_2}{DK_2} = \frac{\beta}{r(\frac{2k}{n} - k + 1)} = \frac{k}{\frac{2k}{n} - k + 1}$

此外$\frac{OK_2}{OK_2 - R} = \frac{k}{\frac{2k}{n} - k + 1}$，

因此像距为：$OK_2 = \frac{nk}{n(2k-1) - 2k} R$

若 k＝5，n＝$\frac{4}{3}$，得$OK_2 = \frac{10}{3} R$

放大率为$\frac{OK_2}{OT_2} = \frac{n}{n(2k-1) - 2k}$

若 k＝5，n＝$\frac{4}{3}$，则放大率为$\frac{2}{3}$

综合以上结果，如鱼以速度 v 向上运动，则鱼的虚像以速度 2 v 向上运动，而鱼的实像以速度$\frac{2}{3}$ v 向下运动。两个像的相对速度为 2 v＋$\frac{2}{3}$ v＝$\frac{8}{3}$ v，

是原有速度的8/3倍。

我们还必须解决的最重要的问题是：从理论上已经知道了像是如何运动的，但是观察者在做此实验时，他将看到什么现象呢？

两个像的速度与鱼的真实速度值，从水中的标尺上的读数来看，是一致的，实际上观察到两个反向的速度，其中一个是另一个的三倍，一个像是另一个像的三倍。我们应当在远处看，因为我们要同时看清楚鱼缸后远处的一个像。两个像的距离 8.33R。用肉眼看实像是可能的，只要我们在比明视距离远得多的地方注视它即可。题目中讲到“在远处的观察者”，是指他观察从两个不同距离的像射来光线的角度变化。只要观察者足够远，尽管有距离差，但所看到的速度将逐渐增加而接近$\frac{8}{3}$。他当然必须具有关于鱼的实际速度（v）的一些信息。

两个像的相对速度与物的原始速度之比的普遍公式为：$\frac{2n}{2-n} \cdot \frac{(k-1)(n-1)}{2k(n-1) - n}$

用一个充满水的圆柱形玻璃缸，一面镜子和一支杆，这个实验很容易做到。沿玻璃缸壁运动的杆代表一条鱼。

【实验题】测量作为电流函数的给定电源的有用功率。确定电源的内阻 R_b和电动势 U_0。

画出作为外电阻 R 函数的有用功率，总功率以及效率　的曲线。

解答：端电压为 $U=\dfrac{U_0R}{R+R_b}$

电流为 $I=\dfrac{U_0}{R+R_b}=\dfrac{U}{R}$

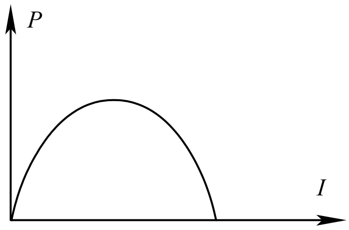
总功率为 $P_0=U_0I$

有用功率为：$P=UI$

效率为 $\eta=\dfrac{P}{P_0}$

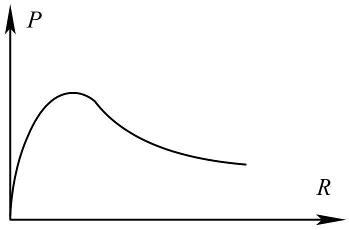
利用以上公式，得到要求的六个函数，如解图 5.4（a）——（f）所示。

（a）



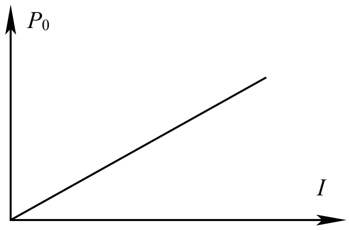
$P=U_0I-R_bI^2$

（b）



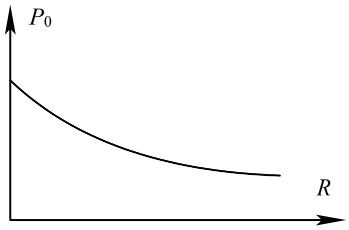
$P=\dfrac{U_0^2R}{(R_b+R)^2}$

（c）



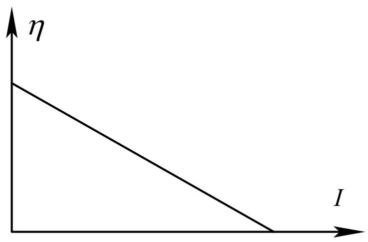
$P_0=U_0I$

（d）



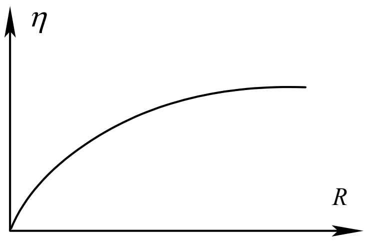
$P_0=\dfrac{U_0^2}{R_b+R}$

（e）



$=1-\dfrac{R_b}{U_0}I$

（f）



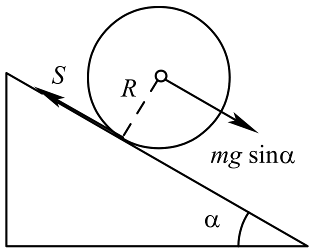
$=\dfrac{R}{R_b+R}$

测出适当选择的两个值，由以上公式便可求出 R_b 和 U_0。这些数据应该是独立于外负载，所以这样的测量并不可靠，大负载时尤其如此。

第 6 届

（1972 年于罗马尼亚的布加勒斯特）

【题 1】给定三个圆柱，它们的长度、外径和质量均相同。第一个是实心圆柱；第二个是空心圆筒，壁有一定厚度；第三个是同样壁厚的圆筒，但两端用薄片封闭，里面充满一种密度与筒壁相同的液体。如将它们放在倾角 α 为的斜面上，如图 6.1 所示，求出并比较这些圆柱的线加速度。研究光滑滚动与又滚又滑两种情况。圆柱与斜面的摩擦系数为 μ，液体与筒壁之间的摩擦可以忽略。



解：沿斜面方向作用在圆柱上的力是：作用于质心重力的分量 mg sin 和作用于接触点的摩擦力 S，如图 6.1 所示。产生的加速度 a ：

$$ma = mg \sin \quad - S$$

纯滚动时的角加速度为：

$$\beta = \frac{a}{R}$$

转动的运动方程为：

$$RS = \frac{a}{R} I$$

以上方程组的解为：

$$a = \frac{g \sin \alpha}{1 + \frac{I}{mR^2}}$$

$$S = \frac{mg \sin \alpha \cdot \frac{I}{mR^2}}{1 + \frac{I}{mR^2}} \qquad (1)$$

当 S 达到最大可能值 μmg cos 时，也就到了纯滚动的极限情形，这时：

$$\mu mg \cos \alpha_h = mg \sin \alpha_h \frac{\frac{I}{mR^2}}{1 + \frac{I}{mR^2}}$$

即维持纯滚动的极限条件为

$$\tan \alpha_h = \mu (1 + \frac{mR^2}{I}) \qquad (2)$$

下面我们来研究三个圆柱体的纯滚动情形。

（Ⅰ）实心圆柱的转动惯量为

$$I = \frac{1}{2} mR^2$$

从（1）式和（2）式分别得到

$$a = \frac{2}{3} g \sin \alpha, \qquad \tan a_h = 3\mu$$

角加速度为：$\beta=\dfrac{a}{R}$

（Ⅱ）设空心圆筒壁的密度是实心圆柱密度的 n 倍。因已知圆柱的质量是相等的，故可以算出圆筒空腔的半径 r：

$$\rho\pi R^2 L = n\rho\pi L(R^2 - r^2)$$

即

$$r^2 = R^2\frac{n-1}{n}$$

转动惯量为：

$$I = 0.5n\rho\pi LR^2\cdot R^2 - 0.5n\rho\pi LR^2\cdot r^2 = 0.5mR^2\frac{2n-1}{n}$$

由（1）式和（2）式分别算出：

$$a = \frac{2n}{4n-1}g\sin\alpha\,,\qquad \tan\alpha_h = \frac{4n-1}{2n-1}\mu$$

角加速度为：$\beta=\dfrac{a}{R}$

（Ⅲ）对充满液体的圆筒，因液体与筒壁之间无摩擦力，故液体不转动。总质量为 m，但转动惯量只需对圆筒壁计算：

$$I = 0.5n\rho\pi LR^2\cdot R^2 - 0.5n\rho\pi LR^2\cdot r^2 = 0.5mR^2\frac{2n-1}{n}$$

由（1）式和（2）式分别算出：

$$a = \frac{2n^2}{2n^2+2n-1}g\sin\alpha\,,\qquad \tan\alpha_h = \frac{2n^2+2n-1}{2n-1}\mu$$

角加速度为：$\beta=\dfrac{a}{R}$

现在比较三个圆柱体的运动特点：线加速度和角加速度之比为：

$$1:\frac{3n}{4n-1}:\frac{3n^2}{2n^2+2n-1}$$

极限角正切之比为：

$$1:\frac{4n-1}{3(2n-2)}:\frac{2n^2+2n-1}{3(2n-1)}$$

如果斜面倾角超过极限角，则圆柱又滑又滚。此时三个圆柱体的摩擦力均为 $\mu mg\cos$ ，故线加速度相同，为：

$a=g$（sin − cos ）

角加速度由 $\beta=\dfrac{R\mu mg\cos\alpha}{I}$ 给出，但转动惯量在三种情况下各不相同。因此，若圆柱体又滚又滑，则三种情况下的角加速度分别为：

$$\beta_1 = \frac{2\mu\cos\alpha}{R}g$$

$$\beta_2 = \frac{2\mu\cos\alpha}{R}\cdot\frac{n}{2n-1}g$$

$$\beta_3 = \frac{2\mu\cos\alpha}{R}\cdot\frac{n^2}{2n-1}g$$

【题 2】有两个底面积为 1dm^{2}的圆筒，如图 6.2 所示，左方圆筒装有一种气体，气体的质量 4g，体积 22.4L，压强 1atm，温度 0^0C。右方圆筒装有同种气体，气体的质量 7.44g，体积 22.4L，压强 1atm，温度 0^0C。左方圆筒筒壁绝热，右方圆筒靠一个大热库维持温度 0^0C。整个系统在真空中。放开活塞，它移动了 5dm 后达到平衡并静止。试问右方圆筒中的气体吸收了多少热量？气体等容比热为 0.75cal/g•K。

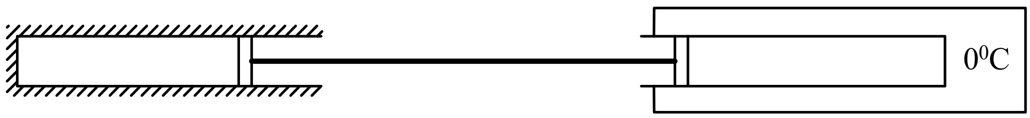


图 6.2

解：放开连杆前，右方气体压强为：

$$7.44/4＝1.86\ (\text{atm})$$

在达到平衡时,左方气体体积为 22.4＋5＝17.4(dm^3),右方气体体积为 22.4＋5＝27.4(dm^3)。左方气体经绝热过程升高温度到 T，压强为 p。右方气体经等温膨胀到同一压强。等温膨胀由下式表示：

$$1.86\times22.4＝\times27.4$$

解得：

$$p＝1.521\ \text{atm}$$

对左方气体应用绝热过程定律，得：

$$1\times22.4^k＝1.521\times17.4^k$$

由此可求得比热之商 k 如下

$$(\frac{22.4}{17.4})^k = 1.521$$

$$1.2874^k＝1.521$$

$$k＝1.66$$

（看来它是一种单原子气体：氦。）

左方气体的温度可从状态方程算出：

$$\frac{1\times22.4}{273} = \frac{1.521\times17.4}{T}$$

解得：

T＝322.5K $\qquad\qquad$ t＝49.5^{0}C

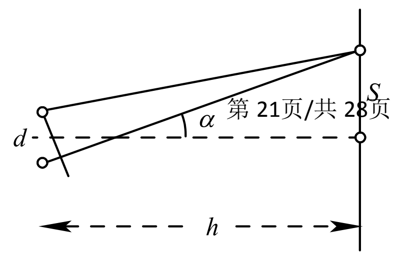
在这个过程中，右方气体的温度没有改变，它吸收了

$$0.75\times4\times49.5＝148.5\ \text{cal}$$

的热量，这些热量表现为气体的内能[注]。

（注：此处是指左方气体的内能。因为右方气体等温膨胀，所吸收的热量等于它对左方气体所作的功。左方气体绝热压缩，右方气体对它所作的功等于左方气体内能的增量。）

【题 3】将焦距为 f 的一个透镜，沿其表面的垂直方向切割成两部分。把两个半透镜移开一段小距离 δ，如果在透镜的一方距离 $t > f$ 处放置一个单



色点光源，问在透镜的另一方距离 H 处的屏幕上将出现多少干涉条纹？

解：由两部分透镜所产生的像是相干光源，所以可以发生干涉。设两个点光源的距离为 d，若光程差等于波长 λ，则在 h 远处的屏幕上将出现第一个极强，如解图 6.1 所示。即：

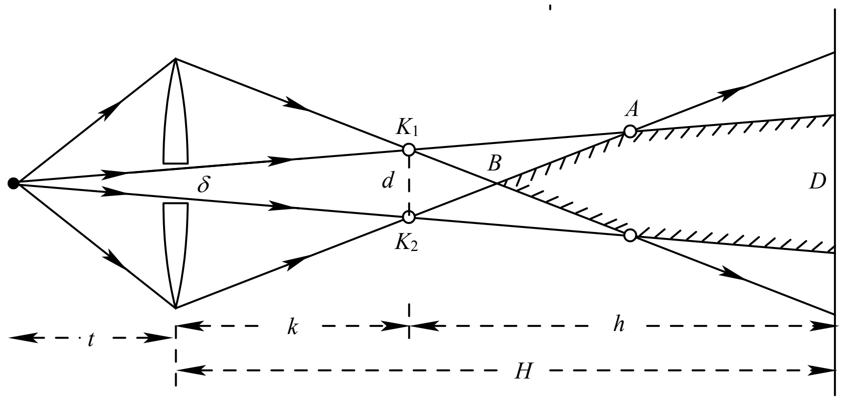
$d\sin\ \ = \lambda$ 解图 6.1

由于 是小角，取近似 $\sin\alpha \approx \dfrac{S}{h}$，各级极强的间距为：

$$\frac{dS}{h} = \lambda, \qquad S = \frac{\lambda h}{d}$$

下面计算两个焦点的位置。

一个点光源位于焦距为 f 的透镜前 t 距离处，它产生的实像位于 $k = \dfrac{tf}{t-f}$，如解图 6.2 所示。



解图 6.2

若切口的宽度为 δ，则两实像点间的距离可从下列比例式中得到：

$$\frac{d}{\delta} = \frac{t+k}{t}$$

因此 $d = \delta\dfrac{t+k}{t} = \dfrac{\delta t}{t-f}$

像点 K_1 和 K_2 是相干光源。它们发射出来的光束的干涉在屏幕上观察到。条纹的间距为 $S = \dfrac{\lambda h}{d}$，其中 d 为已知。屏幕到像点的距离为：$h = H - k = \dfrac{H(t-f) - tf}{t-f}$

在此实验中，条纹间距为：$S = \dfrac{\lambda}{t\delta}\left[H(t-f) - tf\right]$

干涉条纹出现在 K_1 和 K_2 发出的两束光交叠处。由相似三角形求得两束光交叠部分的直径为 $D = \delta\dfrac{h+t}{t}$

用 S 除 D，得条纹数目为 $N = \dfrac{D}{S} = \dfrac{\delta^2}{\lambda}\cdot\dfrac{H+t}{H(t-f)-tf}$

如果 f ＝10cm，t＝20cm，δ＝0.1cm，λ＝0.5μm，H＝50cm，则得 N＝46.6 。

当屏幕比 A 点更近时，对 D 必须另作计算。如屏幕在 B 点以内，则无干涉条纹。

【实验题】给定两个圆柱体，它们的大小、形状、材料均相同，其一是实心体，另一个内部有一个与圆柱轴平行的圆柱形空腔。后者两端用薄片封闭。试确定材料密度，以及空腔轴与圆柱轴之间的距离。

解答：实心圆柱体的密度可由其质量和体积确定。其次我们测量有空腔的圆柱体的质量，根据两个圆柱体质量之差，算出空腔的体积和直径。为求出两轴的距离，可以用几种方法。例如，把圆柱体放在水平面上。确定使它恢复平衡的力矩最大时的位置，这时两轴构成的平面是水平面，由于知道了空腔的大小，便可算出轴间距离。另一种方法在于测定圆柱体对空腔最近或最远的那条母线的转动惯量。

历届国际物理奥林匹克竞赛试题与解答

第 7 届

（1974 年于波兰的华沙）

【题 1】一个处于基态的氢原子与另一个静止基态的氢原子碰撞。问可能发生非弹性碰撞的最小速度为多少？如果速度较大而产生光发射，且在原速度方向可以观察到光。问这种光的频率与简正频率相差多少？氢原子质量是 1.67×10^{-27}kg，电离能 E＝13.6 eV＝2.18×10^{-18} J。

解：处于基态的氢原子能量为 $E_1=-E\cdot\frac{1}{1^2}$，第一激发态能量为 $E_2=-E\cdot\frac{1}{2^2}$。被氢原子吸收的最小能量为 $\Delta E=E_2-E_1=E(\frac{1}{1^2}-\frac{1}{2^2})=\frac{3}{4}E=1.163\times10^{-18}$ J

我们必须求出在碰撞中能量损失为以上数据代最小速度。如果碰撞是完全非弹性的，则碰撞中能量损失最大，碰撞后的速度将是 $v/2$，初动能与末动能之差为：$\frac{mv^2}{2}-\frac{2m(\frac{v}{2})^2}{2}=\frac{mv^2}{4}$

这个值应等于最小的能量子 $\Delta E=\frac{mv^2}{4}$

因此 $v=\sqrt{\frac{4\Delta E}{m}}=6.26\times10^4$ m/s

非弹性碰撞后，两个原子的速度为 $\frac{v}{2}=3.13\times10^4$ m/s

本题第二问的解答与多普勒效应有联系。对于比光速小很多的速度，相对速度之比给出频率相对变化的极好近似：6.26×10^4∶3×10^8＝2.09×10^{-4}＝2.09×10^{-2}％

两束光的频率按此比率稍小于或稍大于简正频率。

【题 2】给定一厚度为 d 的平行平板，其折射率按下式变化 $n(x)=\frac{n_0}{1-\frac{x}{r}}$

束光在 0 点由空气垂直射入平板，并在 A 点以角度　　射出，如图 7.1 所示。求 A 点的折射率 n_A，并确定 A 点的位置及平板的厚度。（设 n_0＝1.2，r＝13cm，$\beta_1=30^0$）

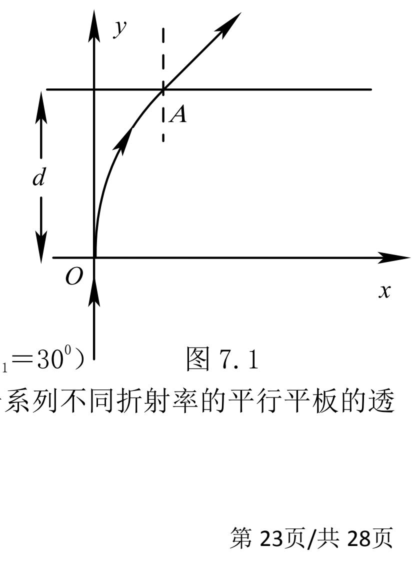


图 7.1

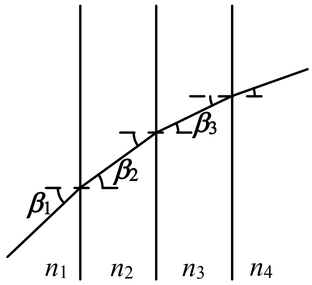
解：首先考虑光的路线，如解图 7.1 所示。对于经过一系列不同折射率的平行平板的透射光，可以应用斯奈尔

定律：$\dfrac{\sin\beta_1}{\sin\beta_2}=\dfrac{n_2}{n_1}$，　　$\dfrac{\sin\beta_2}{\sin\beta_3}=\dfrac{n_3}{n_2}$

更简单的形式是：

$n_1\sin\beta_1=n_2\sin\beta_2=n_3\sin\beta_3=\cdots$

这个公式对任意薄层都是成立的。在我们的情形里，折射率只沿 x 轴变化，即 $n_x\sin\beta_x$＝常数



解图 7.1

在本题中，垂直光束从折射率为 n_0 的点入射，即

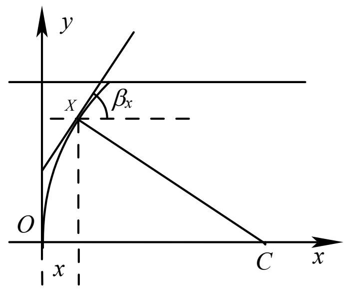
n_x＝n_0　　　　$_x$＝90^0

则常数等于 n_0，于是在平板内任意一点有

$n_x\sin\beta_x$＝n_0

n_x与 x 的关系已知，因此沿平板中的光束为：

$$\sin\beta_x=\frac{n_0}{n_x}=1-\frac{x}{r}=\frac{r-x}{r}$$



解图 7.2

由解图 7.2 表明光束的路径是一个为 XC＝r 的圆，

从而有：$\dfrac{OC-x}{XC}$＝$\sin\beta_x$

现在我们已知光的路径，就有可能找到问题的解答。按照折射定律，当光线在 A 点射出时，有：$n_A=\dfrac{\sin\alpha}{\sin(90^0-\beta_A)}=\dfrac{\sin\alpha}{\cos\beta_A}$

因为 n_A sin　$_A$＝n_0，故有：$\sin\beta_A=\dfrac{n_0}{n_A}$

$$\cos\beta_A=\sqrt{1-(\frac{n_0}{n_A})^2}$$

于是 $n_A=\dfrac{\sin\alpha}{\sqrt{1-(\frac{n_0}{n_A})^2}}$

因此 $n_A=\sqrt{n_0^2+\sin^2\alpha}$

在本题情形 n_A＝1.3

由 $1.3=\dfrac{1.2}{1-\frac{x}{1.3}}$

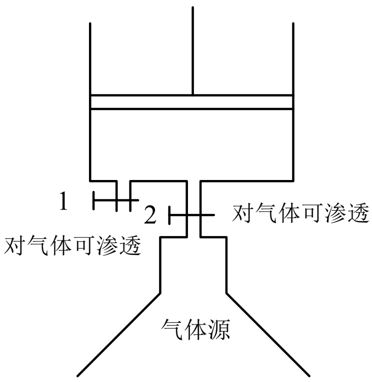
得出 A 点的 x 坐标为 x＝1 cm

光线的轨迹方程为 $y^2+(1-x)^2=r^2$

代入 $x=1$ cm，得到平板厚度为 $y=d=5$ cm

【题 3】一科学探险队因船只失事流落荒岛。他们没有能源，却发现了一种惰性气体源。这种气体比空气重，其压强与温度同周围的大气相等。探险队有两个膜片，其中一个能渗透该气体，另一片只能渗透空气。试设计一个做工的热机。

解：我们要用到两个重要的定律。如果一个容器中装着气体混合物，则每种气体的分压强等于这种气体在同样温度下单独占据相同体积时的压强。压强计在混合气体中读出的是各分压强之和。如果一膜片对某一气体是可渗透的，则在膜片两侧该气体的分压强相等。我们设计这样的热机（见解图 7.3）对惰性气体能渗透的那张膜片装在管子里，这个管子把气源与活塞下面的圆筒连通。对空气能渗透的那张膜片装在圆筒底部。在活塞下部总有同样的一个大气压压强，因而空气对所做的功来说是没有关系的。首先，打开管中的阀门 1，导通可渗透气体的膜片。膜片两侧气体的分压强将相等，于是活塞下部也有这一分压强。结果圆筒内总压强将达到二个大气压，活塞上升做功。关闭阀门 1 可停止活塞的上升运动，然后打开阀门 2，活塞回到初始位置而不做功。



解图 7.3

如果圆筒导热良好，且过程足够缓慢，则上述过程是等温的，做功等于

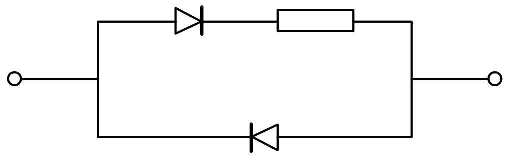
$$RT\ln\frac{V_2}{V_1}$$

这个过程不是循环过程，我们也不在乎它的效率。

有两个膜片就可以实现上述过程，只要有一个周围是真空的气体源。

【实验题】两个同类的半导体二极管和一个欧姆电阻以未知方式联接，并封闭在一个盒里。盒子有两个引出线接线柱，不打开盒子试测量该电阻的欧姆值。

解答：分别在两个方向测定两个不同电压下的电流，我们得到下列结果：两个方向都能观测到电流，但并不相同，且不是电压的线性函数。根据这些结果，不难画出如解图 7.4 所示的网络。


解图 7.4

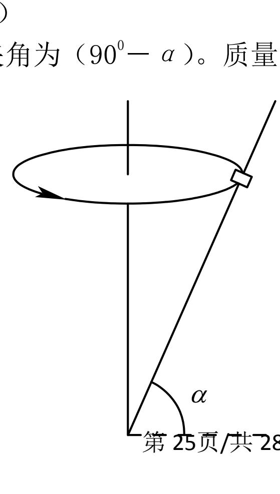
其次，画出两个方向的伏安图，找到在两个方向上电流相同的两个电压。电压之差给出电阻两端的电压，除以电流，得出电阻的欧姆值。

历届国际物理奥林匹克竞赛试题与解答

第 8 届

（1975 年于德意志民主共和国的居斯特罗）

【题 1】一根杆以恒定的角速度 ω 绕竖直轴旋转，杆与轴的夹角为 $(90^0-\alpha)$。质量为 m 的质点可以沿杆滑动，摩擦系数为 μ。求转动过程中，质点保持在同一高度的条件（如图 8.1）。

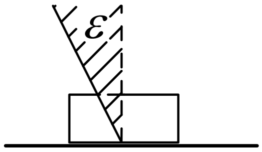


解：我们发现，采用所谓“滑动摩擦角”概念是有用的。如果滑动摩擦系数等于某一角度的正切值，就称这个角 ε 为“滑动摩擦角”（如解图 8.1 所示），即 $\tan\ \varepsilon=\mu$

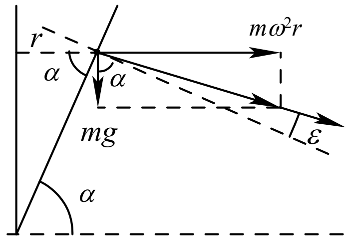
我们必须求出把物体压向平台的合力。如果合力与平面法线之间的夹角在滑动摩擦角之内，则摩擦力大到足以阻止运动。极

限情形是合力与摩擦角的一臂重合。

对于本题，当我们寻找质点在旋转杆上向上滑动的极限情况 时，合力应位于（$\alpha+\varepsilon$）角的双臂内（如解图 8.2 所示）。 图 8.1



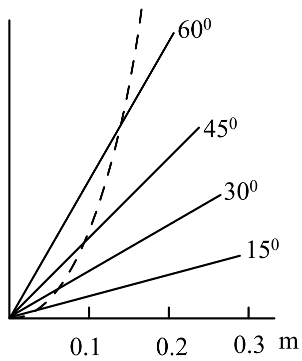
解图 8.1



解图 8.2

把质点压在杆上的力是重力 mg 与 $m\omega^2r=m\omega^2L\cos\alpha$ 的合力。故质点在向上滑动的极限情形下，角（$\alpha+\varepsilon$）的正切为

$$\tan(\alpha+\varepsilon)=\frac{m\omega^2L\cos\alpha}{mg}=\frac{\omega^2L\cos\alpha}{g}$$



解图 8.3

同理，质点向下滑动的极限情形可用角（$\alpha+\varepsilon$）的正切得到。

于是，如果 $\tan(\alpha-\varepsilon)\le\dfrac{\omega^2L\cos\alpha}{g}\le\tan(\alpha+\varepsilon)$

则质点在旋转杆上处于平衡。

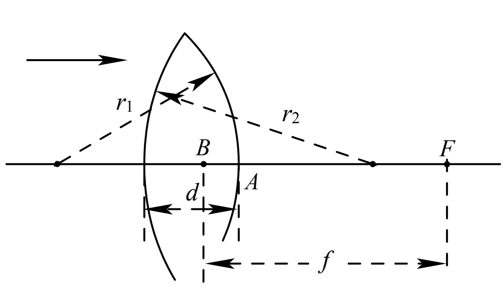
从边界条件可以看出，存在着一个较高位置（L_f）和一个较低位置（L_a），质点在这两位置之间的任何地方将处于随遇平衡状态。在这两边界之外，质点无法平衡，质点将向上或向下滑动。随遇平衡位置 L_f-L_a 可由边界条件导出：

$$L_f-L_a=\frac{2g\tan\varepsilon}{\omega^2\cos^3\alpha(1-\tan^2\alpha\cdot\tan^2\varepsilon)}$$

解图 8.3 对不同的 α 角，画出质点在杆上哪些部分处于随遇平衡，（取 $\omega=10\ \mathrm{s}^{-1}$，$\mu=0.268$，$\varepsilon=15^0$）。虚线表示无摩擦时质点非稳定平衡位置。

【题 2】求出厚透镜对两个不同波长有同一焦距的条件，并就不同类型的透镜讨论可行性。

解答：我们必须知道厚透镜的性质。厚透镜由下述数据表征：球形表面的半径 r_1 和 r_2，厚度 d 和折射率 n（如解图 8.4 所示）。焦距 $f=BF$ 由下式给出



解图 8.4

$$\frac{1}{f}=(n-1)\left[\frac{1}{r_1}+\frac{1}{r_2}-d(\frac{n-1}{n})\frac{1}{r_1r_2}\right]$$

焦距是从主点 B 算起的。B 点离表面的距离为

$$BA=h=\frac{r_2d}{n(r_1+r_2)-d(n-1)}$$

上述公式对任意厚度的厚透镜都成立，但只对近轴光线才给出满意的结果，因为是在一

定的近似下得到的。

光被透镜色散。透镜对波长 λ_a的折射率是 n_a，对波长 λ_b的折射率是 n_b。按折射率的幂次整理焦距公式，得

$f(r_1+r_2-d)n^2+[2fd-f(r_1+r_2)-r_1r_2]n-fd=0$

这是一个二次方程。给定一个 f值，应有两个 n值，因此，我们的问题可望解决。

先后以 n_a和 n_b代入方程，并令其相等：

$$(n_a-1)(\frac{1}{r_1}+\frac{1}{r_2}-d\cdot\frac{n_a-1}{n_ar_1r_2})=(n_b-1)(\frac{1}{r_1}+\frac{1}{r_2}-d\frac{n_b-1}{n_br_1r_2})$$

结果得出$r_1+r_2=d(1-\frac{1}{n_an_b})$

如果半径 r_1、r_2与厚度 d满足这一条件，则对两个不同的波长，即对两个不同的折射率来说，焦距是相同的。有趣的是折射率的乘积 $n_a.n_b$在起作用，而不是色散（n_b-n_a）。因为折射率大于 1，于是括号内的数值小于 1，说明半径之和小于镜厚。这意味着透镜是相当厚的。

结果讨论：首先透镜不能是平凸或平凹的，因为这种透镜有无限大的半径。其次，r_1和 r_2之一为负的发散透镜是许可的，但不能是双凹透镜。

如果要求的不是 f而是（$f-h$）对两个折射率有相同的值（注：即要求消除焦点色差），实现这一点也是可能的，但却是一个复杂得多的问题。

【题 3】质量为 m的一簇离子在 P点以同一速度 v向不同方向散开（如图 8.2）。垂直纸面的均匀磁场 B将这些离子聚焦于 R点，距离 $PR=2a$，离子的轨道应是对称的。试确定磁场区的边界。

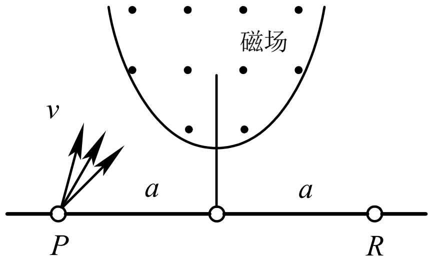


图 8.2

解：在磁场 B中，作用于电量为 Q、速度为 v的质点上的洛仑兹力为 QvB。结果使粒子在半径为 r的圆轨道上运动，即：

$$QvB=\frac{mv^2}{r}$$

质量为 m的所有粒子都在半径为

$r=\frac{mv}{QB}$的相同的圆轨道上运动。离开磁场后，它们将沿最后的切线方向直线飞行。磁场边界应按所有离子都打在同一点 R的要求去寻找。要解决的数学问题是，粒子应从这些半径为 r 的圆的何处离开，才能使它们的切线在 R点相交。这些半径为 r的圆的圆心都位于 y轴上（如解图 8.5 所示）

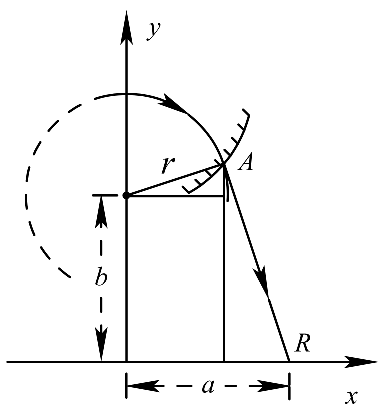
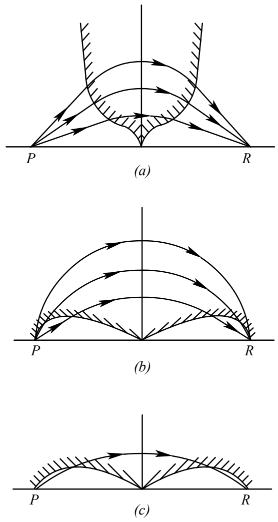


图 8.5

在半径为 r的圆轨道上运动的粒子，在坐标为（x，y）的 A 点离开磁场，沿切线飞向 R。由相似三角形得到：

$$\frac{y-b}{x}=\frac{a-x}{y}$$



圆的方程为 $x^2+(y-b)^2=r^2$

消去（$y-b$），得到满足条件的 A 点的集合，因此，表示磁场边界的函数为：

$$y=\frac{x(a-x)}{\sqrt{r^2-x^2}}$$

这是一个四次函数。只要在第一象限画出这个函数的曲线，把它对 y 轴反演即可。

表示磁场边界的函数的形式取决于给定的距离 a 和半径 r 的相对大小（见解图 8.6（a），（b），（c））。

如果半径 r 小于 a（小速度强磁场），则磁场边界无限延伸，向任何方向出发的离子也都能聚焦①。

如果半径 r 等于 a，所有的离子也都能聚焦②。磁场边界在 P 和 R 点处垂直出发，处在有限的范围内。

如果半径 r 大于 a，边界更为平坦。那些出发方向比 P 点切线更陡的离子不能达到 R 点。

解图 8.6

（注：①原文“向任何方向出的离子都能聚焦”的结论不妥。在 $r<a$ 时，v 与 x 轴夹角大于 90^0 的离子无法聚焦。②在 $r=a$ 时，“所有的离子也都能聚焦” 的结论也不妥。v 与 x 轴夹角大于 90^0 的离子也无法聚焦。）

【实验题】测定有两个接点的某一半导体器件的特性曲线。其最大允许负载为 0.25W，可供使用的是：对所有量程内阻均已知的两个电表，一个 9V 的电池，一个可调电阻器及一个固定电阻器。

解答：通过伏安计测量电压和安培计测量电流所得到的特性曲线，表明该半导体器件是齐纳（ Z e n e r ）二极管。

（注：原文无详细解答，没有给出测量伏安特性的具体线路）

9^{th} IPhO (Budapest, 1976)

Theoretical problems

Problem 1

A hollow sphere of radius R = 0.5 m rotates about a vertical axis through its centre with an angular velocity of $\omega = 5\ s^{-1}$. Inside the sphere a small block is moving together with the sphere at the height of R/2 (Fig. 6). ($g = 10\ m/s^2$.)

a) What should be at least the coefficient of friction to fulfill this condition?

b) Find the minimal coefficient of friction also for the case of $\omega = 8\ s^{-1}$.

c) Investigate the problem of stability in both cases,

α) for a small change of the position of the block,

β) for a small change of the angular velocity of the sphere.

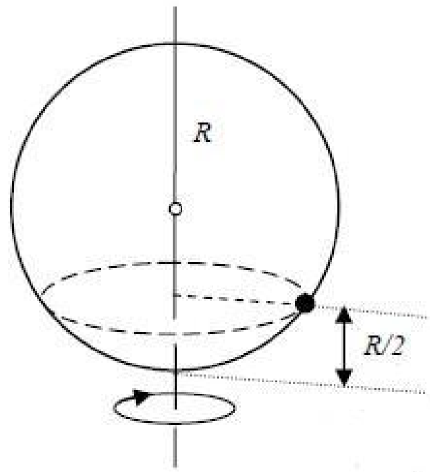


Figure 6

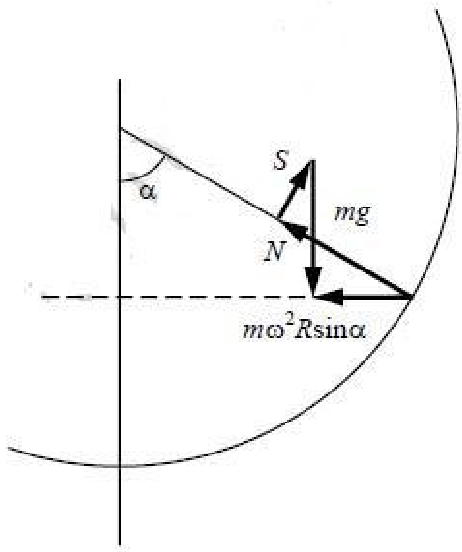


Figure 7

Solution

a) The block moves along a horizontal circle of radius $R\sin\alpha$. The net force acting on the block is pointed to the centre of this circle (Fig. 7). The vector sum of the normal force exerted by the wall N, the frictional force S and the weight mg is equal to the resultant:$m\omega^2 R\sin\alpha$.

The connections between the horizontal and vertical components:

$m\omega^2 R\sin\alpha = N\sin\alpha - S\cos\alpha$,

$mg = N\cos\alpha + S\sin\alpha$.

The solution of the system of equations:

$$S = mg\sin\alpha\left(1 - \frac{\omega^2 R\cos\alpha}{g}\right),$$

$$N = mg\left(\cos\alpha + \frac{\omega^2 R\sin^2\alpha}{g}\right).$$

The block does not slip down if

$$\mu_a \geq \frac{S}{N} = \sin\alpha \cdot \frac{1 - \dfrac{\omega^2 R\cos\alpha}{g}}{\cos\alpha + \dfrac{\omega^2 R\sin^2\alpha}{g}} = \frac{3\sqrt{3}}{23} = \mathbf{0.2259}.$$

In this case there must be at least this friction to prevent slipping, i.e. sliding down.

b) If on the other hand $\dfrac{\omega^2 R\cos\alpha}{g} > 1$ some friction is necessary to prevent the block to slip upwards. $m\omega^2 R\sin\alpha$ must be equal to the resultant of forces S, N and mg. Condition for the minimal coefficient of friction is (Fig. 8):

$$\mu_b \geq \frac{S}{N} = \sin\alpha \cdot \frac{\dfrac{\omega^2 R\cos\alpha}{g} - 1}{\cos\alpha + \dfrac{\omega^2 R\sin^2\alpha}{g}} =$$

$$= \frac{3\sqrt{3}}{29} = \mathbf{0.1792}.$$

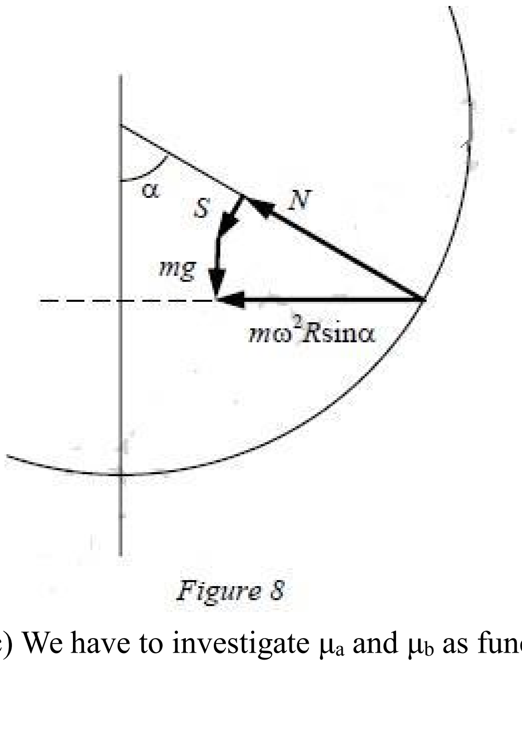


Figure 8

c) We have to investigate μ_a and μ_b as functions of α and ω in the cases a) and b) (see

Fig. 9/a and 9/b):

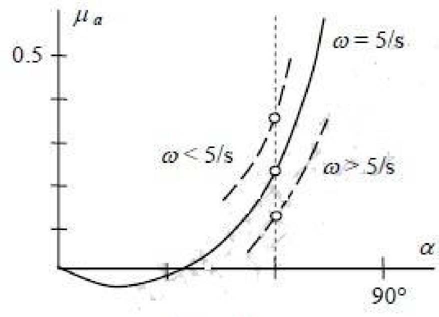


Figure 9/a

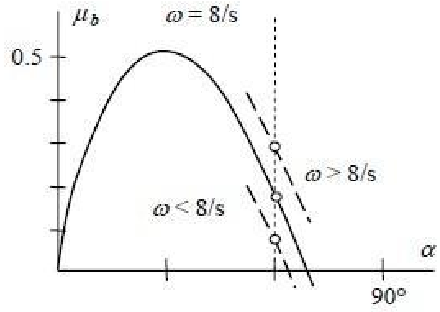


Figure 9/b

In case a): if the block slips upwards, it comes back; if it slips down it does not return. If ω increases, the block remains in equilibrium, if ω decreases it slips downwards. In case b): if the block slips upwards it stays there; if the block slips downwards it returns. If ω increases the block climbs upwards-, if ω decreases the block remains in equilibrium.

Problem 2

The walls of a cylinder of base 1 dm^2, the piston and the inner dividing wall are perfect heat insulators (Fig. 10). The valve in the dividing wall opens if the pressure on the right side is greater than on the left side. Initially there is 12 g helium in the left side and 2 g helium in the right side. The lengths of both sides are 11.2 dm each and the temperature is 0^{o}C.Outside we have a pressure of 100 kPa. The specific heat at constant volume is $c_v = 3.15$ J/gK, at constant pressure it is $c_p = 5.25$ J/gK. The piston is pushed slowly towards the dividing wall. When the valve opens we stop then continue pushing slowly until the wall is reached. Find the work done on the piston by us.

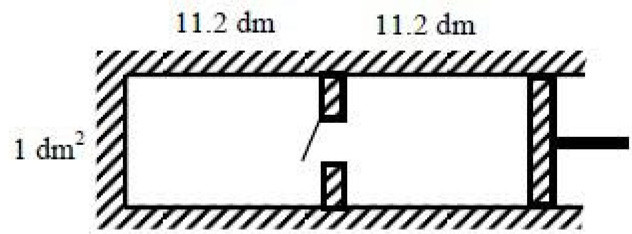


Figure 10

Solution

The volume of 4 g helium at 0^{o}C temperature and a pressure of 100 kPa is 22.4 dm^3 (molar volume). It follows that initially the pressure on the left hand side is 600 kPa, on the
right hand side 100 kPa. Therefore the valve is closed. An adiabatic compression happens until the pressure in the right side reaches 600 kPa (κ = 5/3).

$$100\cdot 11.2^{5/3} = 600\cdot V^{5/3},$$

hence the volume on the right side (when the valve opens):
$V = 3.82\ dm^3$.
From the ideal gas equation the temperature is on the right side at this point

$$T_1 = \frac{pV}{nR} = 552\text{K}.$$

During this phase the whole work performed increases the internal energy of the gas:

$$W_1 = (3.15\ \text{J/gK}) \cdot (2\ \text{g}) \cdot (552\ \text{K} - 273\ \text{K}) = 1760\ \text{J}.$$

Next the valve opens, the piston is arrested. The temperature after the mixing has been completed:

$$T_2 = \frac{12 \cdot 273 + 2 \cdot 552}{14} = 313\text{K}.$$

During this phase there is no change in the energy, no work done on the piston.
An adiabatic compression follows from $11.2 + 3.82 = 15.02\ dm^3$ to $11.2\ dm^3$:

$$313 \cdot 15.02^{2/3} = T_3 \cdot 11.2^{2/3},$$

hence
$T_3 = 381$ K.
The whole work done increases the energy of the gas:

$$W_3 = (3.15\ \text{J/gK}) \cdot (14\ \text{g}) \cdot (381\ \text{K} - 313\ \text{K}) = 3000\ \text{J}.$$

The total work done:

$$W_{\text{total}} = W_1 + W_3 = 4760\ \text{J}.$$

The work done by the outside atmospheric pressure should be subtracted:

$$W_{\text{atm}} = 100\ \text{kPa} \cdot 11.2\ \text{dm}^3 = 1120\ \text{J}.$$

The work done on the piston by us:

$$W = W_{\text{total}} - W_{\text{atm}} = \mathbf{3640\ J}.$$

Problem 3
Somewhere in a glass sphere there is an air bubble. Describe methods how to determine the diameter of the bubble without damaging the sphere.

Solution
We can not rely on any value about the density of the glass. It is quite uncertain. The index of refraction can be determined using a light beam which does not touch the bubble. Another method consists of immersing the sphere into a liquid of same refraction index: its surface becomes invisible.
A great number of methods can be found. We can start by determining the axis, the line which joins the centers of the sphere and the bubble. The easiest way is to use the "tumbler-over" method. If the sphere is placed on a horizontal plane the axis takes up

a vertical position. The image of the bubble, seen from both directions along the axis, is a circle.
If the sphere is immersed in a liquid of same index of refraction the spherical bubble is practically inside a parallel plate (*Fig. 11*). Its boundaries can be determined either by a micrometer or using parallel light beams.

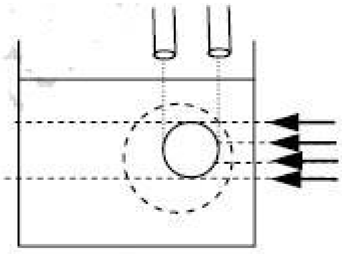


Figure11

Along the axis we have a lens system consisting, of two thick negative lenses. The diameter of the bubble can be determined by several measurements and complicated calculations.
If the index of refraction of the glass is known we can fit a plano-concave lens of same index of refraction to the sphere at the end of the axis (*Fig. 12*). As ABCD forms a parallel plate the diameter of the bubble can be measured using parallel light beams.

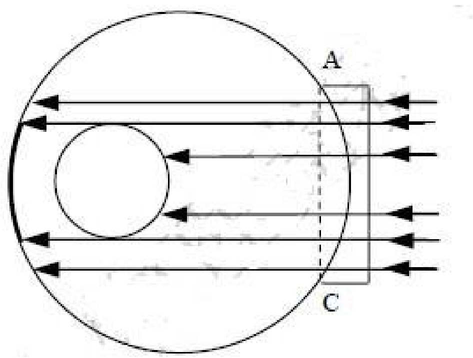


Figure12

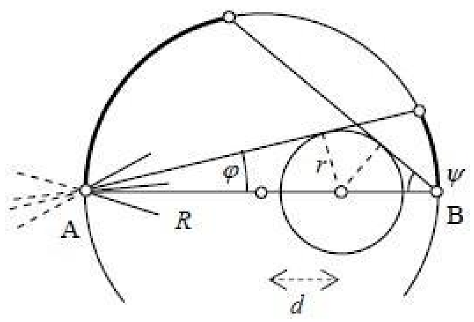


Figure13

Focusing a light beam on point A of the surface of the sphere (*Fig. 13*) we get a diverging beam from point A inside the sphere. The rays strike the surface at the other side and illuminate a cap. Measuring the spherical cap we get angle φ. Angle ψ can be obtained in a similar way at point B. From

$$\sin\varphi = \frac{r}{R+d} \quad \text{and} \quad \sin\psi = \frac{r}{R-d}$$

we have

$$r = 2R\cdot\frac{\sin\psi\sin\varphi}{\sin\psi+\sin\varphi}, \qquad d = R\cdot\frac{\sin\psi-\sin\varphi}{\sin\psi+\sin\varphi}.$$

The diameter of the bubble can be determined also by the help of X-rays. X-rays are not refracted by glass. They will cast shadows indicating the structure of the body, in our case the position and diameter of the bubble.
We can also determine the moment of inertia with respect to the axis and thus the diameter of the bubble.

Experimental problem

The whole text given to the students:

At the workplace there are beyond other devices a test tube with 12 V electrical heating, a liquid with known specific heat (c0 = 2.1 J/g^{o}C) and an X material with unknown thermal properties. The X material is insoluble in the liquid.
Examine the thermal properties of the X crystal material between room temperature and 70 ^{o}C. Determine the thermal data of the X material. Tabulate and plot the measured data.
(You can use only the devices and materials prepared on the table. The damaged devices and the used up materials are not replaceable.)

Solution

Heating first the liquid then the liquid and the crystalline substance together two time-temperature graphs can be plotted. From the graphs specific heat, melting point and heat
of fusion can be easily obtained.

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Problem 1.

The compression ratio of a four-stroke internal combustion engine is $\varepsilon = 9.5.$ The engine draws in air and gaseous fuel at a temperature 27°C at a pressure 1 atm = 100 kPa. Compression follows an adiabatic process from point 1 to point 2, see Fig. 1. The pressure in the cylinder is doubled during the mixture ignition (2–3). The hot exhaust gas expands adiabatically to the volume V_2 pushing the piston downwards (3–4). Then the exhaust valve opens and the pressure gets back to the initial value of 1 atm. All processes in the cylinder are supposed to be ideal. The Poisson constant (i.e. the ratio of specific heats C_p=C_V) for the mixture and exhaust gas is $\kappa = 1.40.$ (The compression ratio is the ratio of the volume of the cylinder when the piston is at the bottom to the volume when the piston is at the top.)

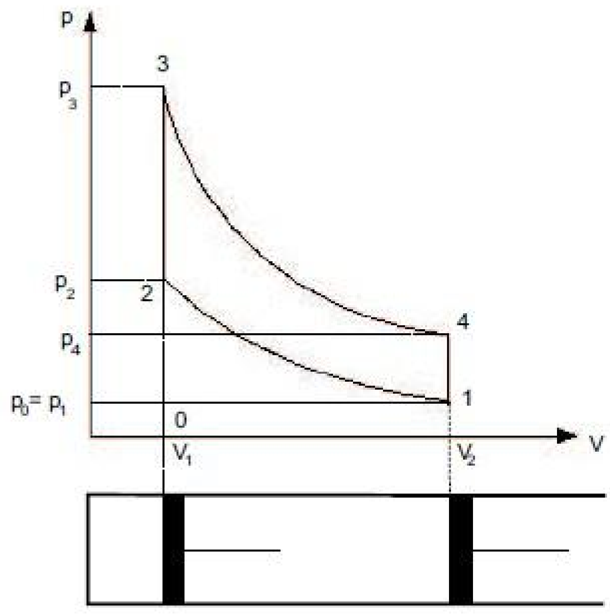

Figure 1:

a) Which processes run between the points 0–1, 2–3, 4–1, 1–0?
b) Determine the pressure and the temperature in the states 1, 2, 3 and 4.
c) Find the thermal efficiency of the cycle.
d) Discuss obtained results. Are they realistic?

Solution: a) The description of the processes between particular points is the following:

0–1 : intake stroke isobaric and isothermal process
1–2 : compression of the mixture adiabatic process

2–3 : mixture ignition isochoric process
3–4 : expansion of the exhaust gas adiabatic process
4–1 : exhaust isochoric process
1–0 : exhaust isobaric process
Let us denote the initial volume of the cylinder before induction at the point 0 by V_1, after induction at the point 1 by V_2 and the temperatures at the particular points by T_0, T_1, T_2, T_3 and T_4.
b) The equations for particular processes are as follows.
0–1 : The fuel-air mixture is drawn into the cylinder at the temperature of $T_0 = T_1 =$ 300 K and a pressure of $p_0 = p_1 =$ 0:10 MPa.
1–2 : Since the compression is very fast, one can suppose the process to be adiabatic. Hence:

$$p_1 V_2^{\kappa} = p_2 V_1^{\kappa} \quad \text{and} \quad \frac{p_1 V_2}{T_1} = \frac{p_2 V_1}{T_2} .$$

From the first equation one obtains

$$p_2 = p_1 \left(\frac{V_2}{V_1} \right)^{\kappa} = p_1 \varepsilon^{\kappa}$$

and by the dividing of both equations we arrive after a straightforward calculation at

$$T_1 V_2^{\kappa - 1} = T_2 V_1^{\kappa - 1} , \quad T_2 = T_1 \left(\frac{V_2}{V_1} \right)^{\kappa - 1} = T_1 \varepsilon^{\kappa - 1} .$$

For given values $\kappa = 1.40.$, $\varepsilon = 9.5.$, $p_1 =$ 0:10 MPa, $T_1 =$ 300 K we have $p_2 =$ 2:34 MPa and $T_2 =$ 738 K($t_2 = 465^{o}$C).
2–3 : Because the process is isochoric and $p_3 = 2p_2$ holds true, we can write

$$\frac{p_3}{p_2} = \frac{T_3}{T_2}\,, \quad \text{which implies} \quad T_3 = T_2\frac{p_3}{p_2} = 2T_2\,.$$

Numerically, $p_3 = 4.68$ MPa, $T_3 = 1476$ K ($t_3 = 1203\,^{\circ}$C).

3–4 : The expansion is adiabatic, therefore

$$p_3V_1^{\kappa} = p_4V_2^{\kappa}\,, \quad \frac{p_3V_1}{T_3} = \frac{p_4V_2}{T_4}\,.$$

The first equation gives

$$p_4 = p_3\left(\frac{V_1}{V_2}\right)^{\kappa} = 2p_2\varepsilon^{-\kappa} = 2p_1$$

and by dividing we get

$$T_3V_1^{\kappa-1} = T_4V_2^{\kappa-1}\,.$$

Consequently,

$$T_4 = T_3\varepsilon^{1-\kappa} = 2T_2\varepsilon^{1-\kappa} = 2T_1\,.$$

Numerical results: $p_4 = 0.20$ MPa, $T_3 = 600$ K ($t_3 = 327\,^{\circ}$C).

4–1 : The process is isochoric. Denoting the temperature by T_1' we can write

$$\frac{p_4}{p_1} = \frac{T_4}{T_1'}\,,$$

which yields

$$T_1' = T_4\frac{p_1}{p_4} = \frac{T_4}{2} = T_1\,.$$

We have thus obtained the correct result $T_1' = T_1$. Numerically, $p_1 = 0.10$ MPa, $T_1' = 300$ K.

c) Thermal efficiency of the engine is defined as the proportion of the heat supplied that is converted to net work. The exhaust gas does work on the piston during the expansion 3–4, on the other hand, the work is done on the mixture during the compression 1–2. No work is done by/on the gas during the processes 2–3 and 4–1. The heat is supplied to the gas during the process 2–3.

The net work done by 1 mol of the gas is

$$W = \frac{R}{\kappa - 1}(T_1 - T_2) + \frac{R}{\kappa - 1}(T_3 - T_4) = \frac{R}{\kappa - 1}(T_1 - T_2 + T_3 - T_4)$$

and the heat supplied to the gas is

$$Q_{23} = C_V(T_3 - T_2)\,.$$

Hence, we have for thermal efficiency

$$\eta = \frac{W}{Q_{23}} = \frac{R}{(\kappa - 1)C_V}\,\frac{T_1 - T_2 + T_3 - T_4}{T_3 - T_2}\,.$$

Since

$$\frac{R}{(\kappa - 1)C_V} = \frac{C_p - C_V}{(\kappa - 1)C_V} = \frac{\kappa - 1}{\kappa - 1} = 1\,,$$

we obtain

$$\eta = 1 - \frac{T_4 - T_1}{T_3 - T_2} = 1 - \frac{T_1}{T_2} = 1 - \varepsilon^{1-\kappa}\,.$$

Numerically, $\eta = 1 - 300/738 = 1 - 0.407$, $\eta = 59,3\%$.

d) Actually, the real pV-diagram of the cycle is smooth, without the sharp angles. Since the gas is not ideal, the real efficiency would be lower than the calculated one.

Problem 2.

Dipping the frame in a soap solution, the soap forms a rectangle film of length b and height h. White light falls on the film at an angle α (measured with respect to the normal direction). The reflected light displays a green color of wavelength λ_0.

a) Find out if it is possible to determine the mass of the soap film using the laboratory scales which has calibration accuracy of 0.1 mg.

b) What color does the thinnest possible soap film display being seen from the perpendicular direction? Derive the related equations.

Constants and given data: relative refractive index $n = 1.33$, the wavelength of the reflected green light $\lambda_0 = 500$ nm, $\alpha = 30^o$, $b = 0.020$ m, $h = 0.030$ m, density $\varrho = 1000$ kg m^{-3}.

Solution: The thin layer reflects the monochromatic light of the wavelength λ in the best way, if the following equation holds true

$$2nd\cos\beta = (2k+1)\frac{\lambda}{2}, \quad k = 0, 1, 2, \dots, \tag{1}$$

where k denotes an integer and β is the angle of refraction satisfying

$$\frac{\sin\alpha}{\sin\beta} = n\,.$$

Hence,

$$\cos\beta = \sqrt{1-\sin^2\beta} = \frac{1}{n}\sqrt{n^2-\sin^2\alpha}\,.$$

Substituting to (1) we obtain

$$2d\sqrt{n^2-\sin^2\alpha} = (2k+1)\frac{\lambda}{2}\,. \tag{2}$$

If the white light falls on a layer, the colors of wavelengths obeying (2) are reinforced in the reflected light. If the wavelength of the reflected light is λ_0, the thickness of the layer satisfies for the kth order interference

$$d_k = \frac{(2k+1)\lambda_0}{4\sqrt{n^2-\sin^2\alpha}} = (2k+1)d_0\,.$$

For given values and $k=0$ we obtain $d_0 = 1.01\cdot 10^{-7}$ m.

a) The mass of the soap film is $m_k = \varrho_k b\,h\,d_k$. Substituting the given values, we get $m_0 = 6.06\cdot 10^{-2}$ mg, $m_1 = 18.2\cdot 10^{-2}$ mg, $m_2 = 30.3\cdot 10^{-8}$ mg, etc. The mass of the thinnest film thus cannot be determined by given laboratory scales.

b) If the light falls at the angle of 30º then the film seen from the perpendicular direction cannot be colored. It would appear dark.

Problem 3.

An electron gun *T* emits electrons accelerated by a potential difference *U* in a vacuum in the direction of the line *a* as shown in Fig. 2. The target *M* is placed at a distance *d* from the electron gun in such a way that the line segment connecting the points *T* and *M* and the line *a* subtend the angle α as shown in Fig. 2. Find the magnetic induction *B* of the uniform magnetic field

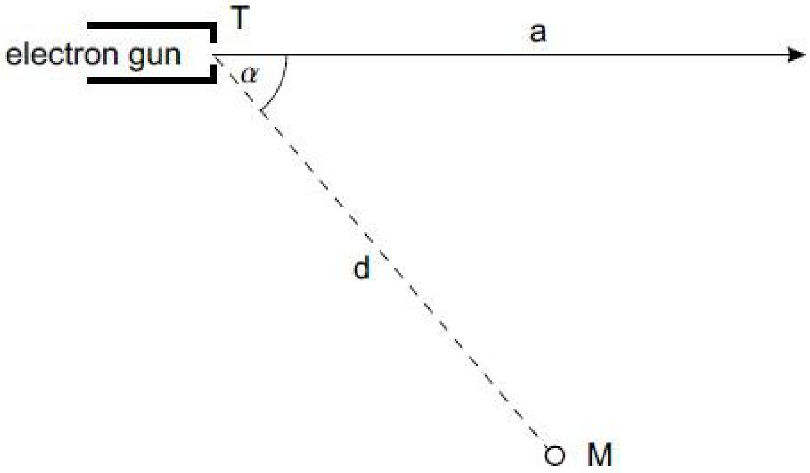

Figure 2:

a) perpendicular to the plane determined by the line a and the point M

b) parallel to the segment TM

in order that the electrons hit the target M. Find first the general solution and then substitute the following values: $U = 1000$ V, $e = 1.60 \cdot 10^{-19}$ C, $m_e = 9.11 \cdot 10^{-31}$ kg, $\alpha = 60^o$, $d = 5.0$ cm, $B < 0.030$ T.

Solution: a) If a uniform magnetic field is perpendicular to the initial direction of motion of an electron beam, the electrons will be deflected by a force that is always perpendicular to their velocity and to the magnetic field. Consequently, the beam will be deflected into a circular trajectory. The origin of the centripetal force is the Lorentz force, so

$$Bev = \frac{m_e v^2}{r} . \tag{3}$$

Geometrical considerations yield that the radius of the trajectory obeys (cf. Fig. 3).

$$r = \frac{d}{2\sin\alpha} . \tag{4}$$

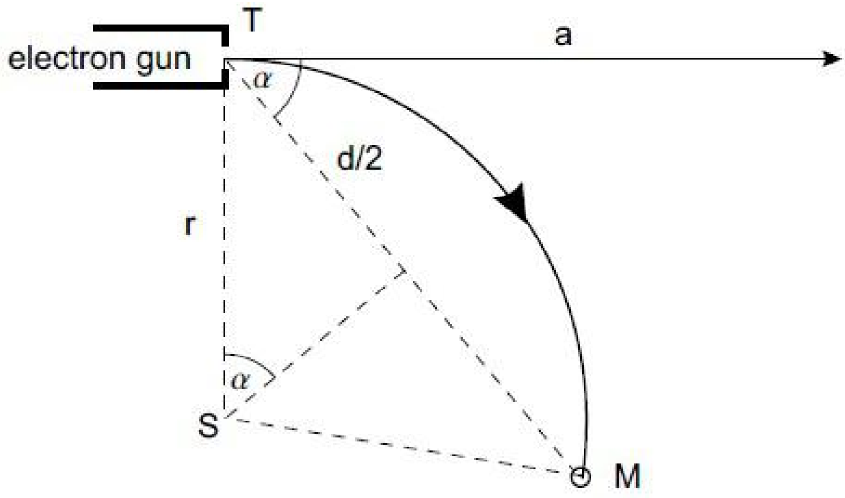

Figure 3:

The velocity of electrons can be determined from the relation between the kinetic energy of an electron and the work done on this electron by the electric field of the voltage U inside the gun,

$$\frac{1}{2}m_e v^2 = eU\,. \tag{5}$$

Using (3), (4) and (5) one obtains

$$B = m_e\sqrt{\frac{2eU}{m_e}}\frac{2\sin\alpha}{ed} = 2\sqrt{\frac{2Um_e}{e}}\frac{\sin\alpha}{d}\,.$$

Substituting the given values we have $B = 3.70\cdot 10^{-3}$ T.

b) If a uniform magnetic field is neither perpendicular nor parallel to the initial direction of motion of an electron beam, the electrons will be deflected into a helical trajectory. Namely, the motion of electrons will be composed of an uniform motion on a circle in the plane perpendicular to the magnetic field and of an uniform rectilinear motion in the direction of the magnetic field. The component $\vec{v}_1$ of the initial velocity $\vec{v}$, which is perpendicular to the magnetic field (see Fig. 4), will manifest itself at the Lorentz force and during the motion will rotate uniformly around the line parallel to the magnetic field. The component $\vec{v}_2$ parallel to the magnetic field will remain

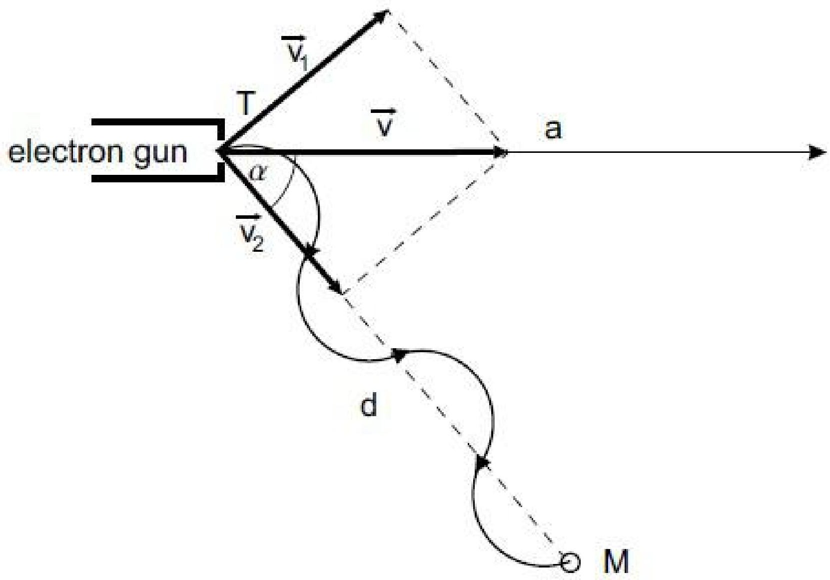

Figure 4:

constant during the motion, it will be the velocity of the uniform rectilinear motion. Magnitudes of the components of the velocity can be expressed as

$$v_1 = v \sin\alpha \quad v_2 = v\cos\alpha\,.$$

Denoting by N the number of screws of the helix we can write for the time of motion of the electron

$$t = \frac{d}{v_2} = \frac{d}{v\cos\alpha} = \frac{2\pi r N}{v_1} = \frac{2\pi r N}{v\sin\alpha}\,.$$

Hence we can calculate the radius of the circular trajectory

$$r = \frac{d\sin\alpha}{2\pi N\cos\alpha}\,.$$

However, the Lorentz force must be equated to the centripetal force

$$Bev\sin\alpha = \frac{m_e v^2\sin^2\alpha}{r} = \frac{m_e v^2\sin^2\alpha}{\frac{d\sin\alpha}{2\pi N\cos\alpha}}\,. \tag{6}$$

Consequently,

$$B = \frac{m_e v^2 \sin^2\alpha \, 2\pi N \cos\alpha}{d \sin\alpha \, ev \sin\alpha} = \frac{2\pi N m_e v \cos\alpha}{de}\,.$$

The magnitude of velocity v again satisfies (5), so

$$v = \sqrt{\frac{2Ue}{m_e}}\,.$$

Substituting into (6) one obtains

$$B = \frac{2\pi N \cos\alpha}{d}\sqrt{\frac{2Um_e}{e}}\,.$$

Numerically we get $B = N \cdot 6.70 \cdot 10^{-3}\ \mathrm{T}$. If $B < 0.030\ \mathrm{T}$ should hold true, we have four possibilities ($N \leq 4$). Namely,

$$\begin{aligned} B_1 &= 6.70 \cdot 10^{-3}\ \mathrm{T}\,,\\ B_2 &= 13.4 \cdot 10^{-3}\ \mathrm{T}\,,\\ B_3 &= 20.1 \cdot 10^{-3}\ \mathrm{T}\,,\\ B_4 &= 26.8 \cdot 10^{-3}\ \mathrm{T}\,. \end{aligned}$$

Problems of the XI International Olympiad, Moscow, 1979
The publication has been prepared by Prof. S. Kozel and Prof. V.Orlov
(Moscow Institute of Physics and Technology)

The XI International Olympiad in Physics for students took place in Moscow, USSR, in July 1979 on the basis of Moscow Institute of Physics and Technology (MIPT). Teams from 11 countries participated in the competition, namely Bulgaria, Finland, Germany, Hungary, Poland, Romania, Sweden, Czechoslovakia, the DDR, the SFR Yugoslavia, the USSR. The problems for the theoretical competition have been prepared by professors of MIPT (V.Belonuchkin, I.Slobodetsky, S.Kozel). The problem for the experimental competition has been worked out by O.Kabardin from the Academy of Pedagogical Sciences.
It is pity that marking schemes were not preserved.

Theoretical Problems

Problem 1.

A space rocket with mass M=12t is moving around the Moon along the circular orbit at the height of h =100 km. The engine is activated for a short time to pass at the lunar landing orbit. The velocity of the ejected gases u = 10^4 m/s. The Moon radius R_M = 1,7•10^3 km, the acceleration of gravity near the Moon surface g_M = 1.7 m/s^2

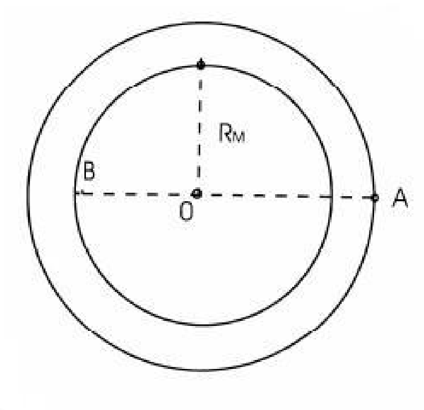

Fig.1

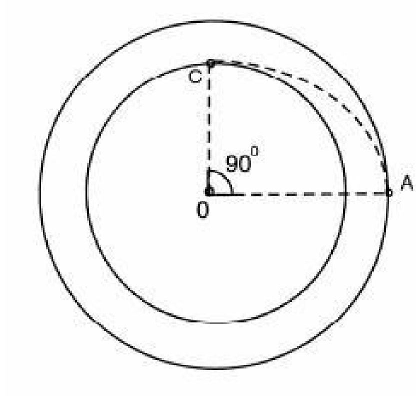

Fig.2

1). What amount of fuel should be spent so that when activating the braking engine at point A of the trajectory, the rocket would land on the Moon at point B (Fig.1)?
2). In the second scenario of landing, at point A the rocket is given an impulse directed towards the center of the Moon, to put the rocket to the orbit meeting the Moon surface at point C (Fig.2). What amount of fuel is needed in this case?

Problem 2.

Brass weights are used to weigh an aluminum-made sample on an analytical balance. The weighing is ones in dry air and another time in humid air with the water vapor pressure P_h =2·10^3 Pa. The total atmospheric pressure (P = 10^5 Pa) and the temperature (t =20°C) are the same in both cases.
What should the mass of the sample be to be able to tell the difference in the balance readings provided their sensitivity is m_0 = 0.1 mg ?

Aluminum density ρ_1= 2700 kg/m^3, brass density ρ_2=.8500 kg/m^3.

Problem 3

.During the Soviet-French experiment on the optical location of the Moon the light pulse of a ruby laser (λ= 0,69 µm) was directed to the Moon's surface by the telescope with a diameter of the mirror D = 2,6 m. The reflector on the Moon's surface reflected the light backward as an ideal mirror with the diameter d = 20 cm. The reflected light was then collected by the same telescope and focused at the photodetector.

1) What must the accuracy to direct the telescope optical axis be in this experiment?

2) What part of emitted laser energy can be detected after reflection on the Moon, if we neglect the light loses in the Earth's atmosphere?

3) Can we see a reflected light pulse with naked eye if the energy of single laser pulse E = 1 J and the threshold sensitivity of eye is equal n =100 light quantum?

4) Suppose the Moon's surface reflects α = 10% of the incident light in the spatial angle 2π steradian, estimate the advantage of a using reflector.

The distance from the Earth to the Moon is L = 380000 km. The diameter of pupil of the eye is d_p = 5mm. Plank constant is h = 6.610 $\cdot^{-34}$ J$\cdot$ s.

Experimental Problem

Define the electrical circuit scheme in a "black box" and determine the parameters of its elements. List of instruments: A DC source with tension 4.5 V, an AC source with 50 Hz frequency and output voltage up to 30 V, two multimeters for measuring AC/DC current and voltage, variable resistor, connection wires.

Solution of Problems of the XI International Olympiad, Moscow, 1979

Solution of Theoretical Problems

Problem 1.

1) During the rocket moving along the circular orbit its centripetal acceleration is created by moon gravity force:

$$G\frac{MM_M}{R^2} = \frac{Mv_0^2}{R},$$

where R = R_M + h is the primary orbit radius, v_0 -the rocket velocity on the circular orbit:

$$v_0 = \sqrt{G\frac{M_M}{R}}$$

Since $g_M = G\frac{M_M}{R_M^2}$ it yields

$$v_0 = \sqrt{\frac{g_M R_M^2}{R}} = R_M\sqrt{\frac{g_M}{R_M + h}} \qquad (1)$$

The rocket velocity will remain perpendicular to the radius-vector OA after the braking engine sends tangential momentum to the rocket (Fig.1). The rocket should then move along the elliptical trajectory with the focus in the Moon's center.

Denoting the rocket velocity at points A and B as v_A and v_B we can write the equations for energy and momentum conservation as follows:

$$\frac{Mv_A^2}{2} - G\frac{MM_M}{R} = \frac{Mv_B^2}{2} - G\frac{MM_M}{R_M} \quad (2)$$

$$Mv_A R = Mv_B R_M \quad (3)$$

Solving equations (2) and (3) jointly we find

$$v_A = \sqrt{2G\frac{M_M R_M}{R(R+R_M)}}$$

Taking (1) into account, we get

$$v_A = v_0\sqrt{\frac{2R_M}{R+R_M}} .$$

Thus the rocket velocity change Δv at point A must be

$$\Delta v = v_0 - v_A = v_0\left(1-\sqrt{\frac{2R_M}{R+R_M}}\right) = v_0\left(1-\sqrt{\frac{2R_M}{2R_M+h}}\right) = 24m/s.$$

Since the engine switches on for a short time the momentum conservation low in the system "rocket-fuel" can be written in the form

$$(M - m_1)\Delta v = m_1 u$$

where m_1 is the burnt fuel mass.

This yields

$$m_1 = \frac{\Delta v}{u + \Delta v}$$

Allow for $\Delta v << u$ we find

$$m_1 \approx \frac{\Delta v}{u} M = 29\text{kg}$$

2) In the second case the vector $\vec{v}_2$ is directed perpendicular to the vector $\vec{v}_0$ thus giving

$$\vec{v}_A = \vec{v}_0 + \Delta\vec{v}_2, \qquad v_A = \sqrt{v_0^2 + \Delta v_2^2} \ .$$

Based on the energy conservation law in this case the equation can be written as

$$\frac{M\left(v_0^2 + \Delta v_2^2\right)}{2} - \frac{GMM_M}{R} = \frac{Mv_C^2}{2} - \frac{GMM_M}{R_M} \qquad (4)$$

and from the momentum conservation law

$$Mv_0R = Mv_CR_M \ . \qquad (5)$$

Solving equations (4) and (5) jointly and taking into account (1) we find

$$\Delta v_2 = \sqrt{g_M \frac{(R - R_M)^2}{R}} = h\sqrt{\frac{g_M}{R_M + h}} \approx 97\text{m/s} \ .$$

Using the momentum conservation law we obtain

$$m_2 = \frac{\Delta v_2}{u} M \approx 116\text{kg} \ .$$

Problem 2.

A sample and weights are affected by the Archimede's buoyancy force of either dry or humid air in the first and second cases, respectively. The difference in the scale indication ΔF is determined by the change of difference of these forces.

The difference of Archimede's buoyancy forces in dry air:

$$\Delta F_1 = \Delta V \rho_a' g$$

Whereas in humid air it is:

$$\Delta F_2 = \Delta V \rho_a'' g$$

where ΔV - the difference in volumes between the sample and the weights, and ρ_a' and ρ_a'' - densities of dry and humid air, respectively.

Then the difference in the scale indications ΔF could be written as follows:

$$\Delta F = \Delta F_1 - \Delta F_2 = \Delta V g\left(\rho_a' - \rho_a''\right) \qquad (1)$$

According to the problem conditions this difference should be distinguished, i.e.
$\Delta F \geq m_0 g$ or $\Delta V g\left(\rho_a' - \rho_a''\right) \geq m_0$, wherefrom

$$\Delta V \geq \frac{m_0}{\rho_a' - \rho_a''} . \qquad (2)$$

The difference in volumes between the aluminum sample and brass weights can be found from the equation

$$\Delta V = \frac{m}{\rho_1} - \frac{m}{\rho_2} = m\left(\frac{\rho_2 - \rho_1}{\rho_1 \rho_2}\right) , \qquad (3)$$

where m is the sought mass of the sample. From expressions (2) and (3) we obtain

$$m = \Delta V\left(\frac{\rho_1 \rho_2}{\rho_2 - \rho_1}\right) \geq \frac{m_0}{\rho_a' - \rho_a''}\left(\frac{\rho_1 \rho_2}{\rho_2 - \rho_1}\right) . \qquad (4)$$

To find the mass m of the sample one has to determine the difference $\left(\rho_a' - \rho_a''\right)$.

With the general pressure being equal, in the second case, some part of dry air is replaced by vapor:

$$\rho_a' - \rho_a'' = \frac{\Delta m_a}{V} - \frac{\Delta m_v}{V} .$$

Changes of mass of air Δm_a and vapor Δm_v can be found from the ideal-gas equation of state

$$\Delta m_a = \frac{P_a V M_a}{RT} , \quad \Delta m_v = \frac{P_v V M_v}{RT} ,$$

wherefrom we obtain

$$\rho_a' - \rho_a'' = \frac{P_a\left(M_a - M_v\right)}{RT} . \qquad (5)$$

From equations (4) and (5) we obtain

$$m \geq \frac{m_0 RT}{P_a\left(M_a - M_v\right)}\left(\frac{\rho_1 \rho_2}{\rho_2 - \rho_1}\right) . \qquad (6)$$

The substitution of numerical values gives the answer: $m \geq 0.0432$ kg ≈ 43 g.

Note. When we wrote down expression (3), we considered the sample mass be equal to the weights' mass, at the same time allowing for a small error.

One may choose another way of solving this problem. Let us calculate the change of Archimede's force by the change of the air average molar mass.

In dry air the condition of the balance between the sample and weights could be written down in the form of

$$\left(\rho_1 - \frac{M_a P}{RT}\right)V_1 = \left(\rho_2 - \frac{M_a P}{RT}\right)V_2 \qquad (7)$$

In humid air its molar mass is equal to

$$M = M_a \frac{P_a}{P} + M_v \frac{P - P_a}{P}, \qquad (8)$$

whereas the condition of finding the scale error could be written in the form

$$\left(\rho_1 - \frac{M_a P}{RT}\right)V_1 - \left(\rho_2 - \frac{M_a P}{RT}\right)V_2 \geq m_0. \qquad (9)$$

From expressions (7) –(9) one can get a more precise answer

$$m \geq \frac{m_0 RT \rho_1 \rho_2 - M_a P_a}{(M_a - M_v)(\rho_2 - \rho_1)P_a}. \qquad (10)$$

Since $M_a P_a << m_0 \rho_1 \rho_2 RT$, then both expressions (6) and (10) lead practically to the same quantitative result, i.e. m $\geq$ 43 g.

Problem 3.

1) The beam divergence angle $\delta\varphi$ caused by diffraction defines the accuracy of the telescope optical axis installation:

$$\delta\varphi \approx \lambda/D \approx 2.6 \cdot 10^{-7} \text{ rad.} \approx 0.05'' .$$

2) The part K1 of the light energy of a laser, directed to a reflector, may be found by the ratio of the area of S_1 reflector ($S_1 = \pi d^2/4$) versus the area S_2 of the light spot on the Moon ($S_2 = \pi r^2$, where $r = L\,\delta\varphi \approx L\lambda/D$, L – the distance from the Earth to the Moon)

$$K_1 = \frac{S_1}{S_2} = \frac{d^2}{(2r)^2} = \frac{d^2 D^2}{4\lambda^2 L^2}$$

The reflected light beam diverges as well and forms a light spot with the radius *R* on the Earth's surface:

$R = \lambda L/d$, as $r << R$

That's why the part K_2 of the reflected energy, which got into the telescope, makes

$$K_2 = \frac{D^2}{(2R)^2} = \frac{D^2 d^2}{4\lambda^2 L^2}$$

The part K_0 of the laser energy, that got into the telescope after having been reflected by the reflector on the Moon, equals

$$K_0 = K_1 K_2 = \left(\frac{dD}{2\lambda L}\right)^4 \approx 10^{-12}$$

3) The pupil of a naked eye receives as less a part of the light flux compared to a telescope, as the area of the pupil S_e is less than the area of the telescope mirror S_t:

$$K_e = K_0 \frac{S_e}{S_t} = K_0 \frac{d_e^2}{D^2} \approx 3.7 \cdot 10^{-18}.$$

So the number of photons N getting into the pupil of the eye is equal

$$N = \frac{E}{h\nu} K_e = 12.$$

Since $N<n$, one can not perceive the reflected pulse with a naked eye.
4) In the absence of a reflector α =10% of the laser energy, that got onto the Moon, are dispersed by the lunar surface within a solid angle $\Omega_1 = 2\pi$ steradian.
The solid angle in which one can see the telescope mirror from the Moon, constitutes $\Omega_2 = S_t / L^2 = \pi D^2/4L^2$
That is why the part K of the energy gets into the telescope and it is equal

$$K = \alpha \frac{\Omega_2}{\Omega_1} = \alpha \frac{D^2}{8L^2} \approx 0.5 \cdot 10^{-18}$$

Thus, the gain β , which is obtained through the use of the reflector is equal
$\beta = K_0/K \approx 2 \cdot 10^6$
Note. The result obtained is only evaluative as the light flux is unevenly distributed inside the angle of diffraction.

Solution of Experimental Problem.
A transformer is built-in in a “black box”. The black box has 4 terminals. To be able to determine the equivalent circuit and the parameters of its elements one may first carry out measurements of the direct current. The most expedient is to mount the circuit according to the layout in Fig.3 and to build volt-ampere characteristics for various terminals of the “box”. This enables one to make sure rightway that there were no e.m.f. sources in the “box” (the plot $I=f(U)$ goes through the origin of the coordinates), no diodes (the current strength does not depend on the polarity of the current’s external source), by the inclination angle of the plot one may define the resistances between different terminals of the “box”. The tests allowed for some estimations of values $R_{1\text{-}2}$ and $R_{3\text{-}4}$. The ammeter did not register any current between the other terminals. This means that between these terminals there might be some other resistors with resistances larger than R_L :

$$R_L = \frac{U_{max}}{I_{min}} = \frac{4{,}5\,\text{V}}{2\cdot 10^{-6}\,\text{A}} = 2.25\cdot 10^{6}\,\text{ohm}$$

where I_{min} - the minimum value of the strength of the current which the instrument would have

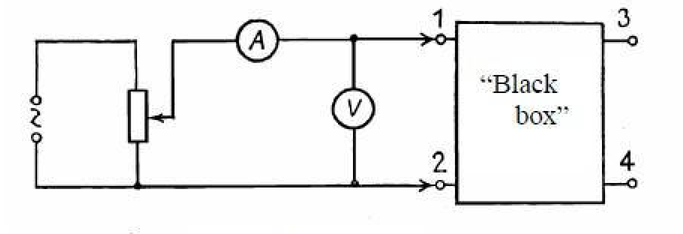

Fig.3

registered. Probably there might be some capacitors between terminals 1-3, 1-4, 2-3, 2-4 (Fig.4).

Then, one can carry out analogous measurements of an alternative current. The taken voltampere characteristics enabled one to find full resistances on the alternative current of sections 1-2 and 3-4: Z_1 and Z_2 and to compare them to the values R_1 and R_2. It turned out, that $Z_1 > R_2$ and $Z_2 > R_2$.

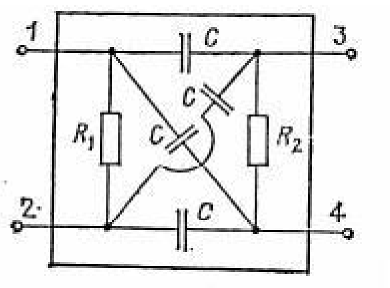

Fig.4

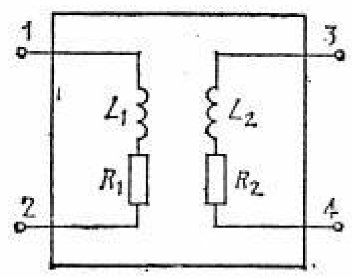

Fig.5

This fact allows one to conclude that in the "black box" the coils are connected to terminals 1-2 and 3-4 (Fig.5). Inductances of coils L_1 and L_2 can be determined by the formulas

$$L_1 = \frac{\sqrt{Z_1^2 - R_1^2}}{2\pi\nu}, \qquad L_2 = \frac{\sqrt{Z_2^2 - R_2^2}}{2\pi\nu}.$$

After that the dependences $Z = f(I)$, $L = f(I)$ are to be investigated. The character of the found dependences enabled one to draw a conclusion about the presence of ferromagnetic cores in the coils. Judging by the results of the measurements on the alternative current one could identify the upper limit of capacitance of the capacitors which could be placed between terminals 1-3, 1-4, 2-3, 2-4:

$$C_{max} = \frac{I_{min}}{2\pi\nu U_{max}} = \frac{5\cdot 10^{-6}\,\text{A}}{2\cdot 3.14\cdot 50\text{s}^{-1}\cdot 3\text{V}} = 5\cdot 10^{-9}\text{F} = 5\text{nF}$$

Then one could check the availability of inductive coupling between circuits 1-2 and 3-4. The plot of dependence of voltage $U_{3\text{-}4}$ versus voltage $U_{1\text{-}2}$ (Fig. 6) allows one to find both the transformation coefficient

$$K = \frac{U_{1-2}}{U_{3-4}} = \frac{1}{2}$$

and the maximum operational voltages on coils L_1 and L_2, when the transformation

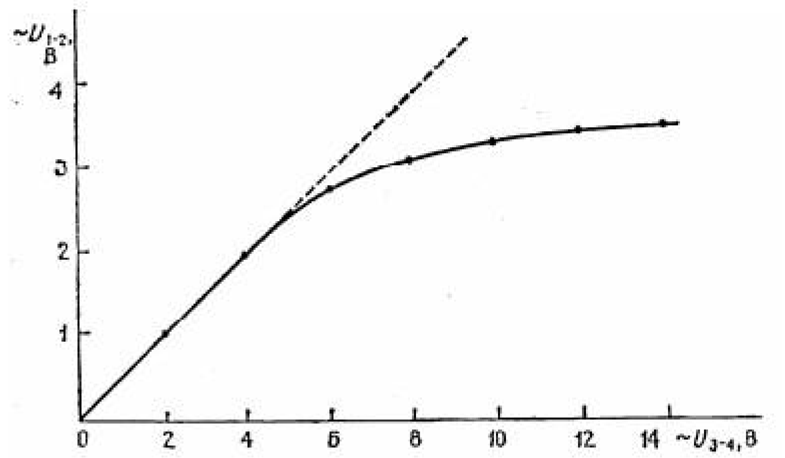

Fig.6

coefficient has not changed yet, i.e. before saturation of the core.

$U_{1\text{-}2(\max)} = 2.5$ V, $U_{3\text{-}4(\max)} = 5$ V.

One could build either plot $K(U_{1\text{-}2})$ or $K(U_{3\text{-}4})$ (Fig. 7).

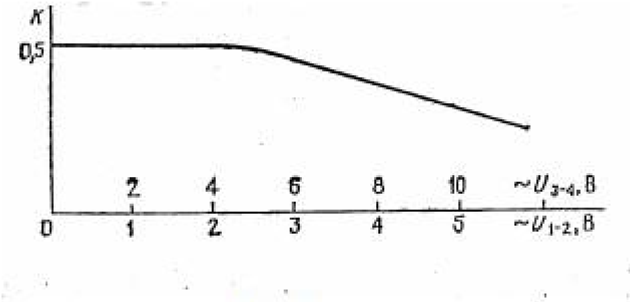

Fig.7

Note: It was also possible to define the “box” circuit after tests of the direct current. To do that one had to find the presence of induction coupling between terminals 1-2 and 3-4, that is the appearance of e.m.f. of induction in circuit 3-4, when closing and breaking circuits 1-2 and vice-versa. When comparing the direction of the pointer’s rejection of the voltmeters connected to terminals 1-2 and 3-4 one could identify directions of the transformer’s windings.

Acknowledgement

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XII International Physics Olympiad

Varna, Bulgaria, July 1981

The problems and the solutions are adapted by Miroslav Abrashev
University of Sofia, Faculty of Physics, 5 James Bourchier Blvd., 1164 Sofia, Bulgaria

Reference: O. F. Kabardin, V. A. Orlov, in "International Physics Olympiads for High School Students", ed. V. G. Razumovski, Moscow, Nauka, 1985. (In Russian).

The Experimental Problem

Materials and Instruments: elastic rubber cord (the length of free cord is $l_0 = 150$ mm), vertically hanged up to a stand, set of weights from 10 g to 100 g, pan for the weights with mass 5 g, chronometer, ruler, millimeter (scaled) paper.

Note: The Earth Acceleration is $g = 10$ m/s^2. The mass of the rubber cord can be neglected.

Make the following study:

1. Load the rubber cord with weights in the range 15 g to 105 g. Put the data obtained into a table. Make a graph (using suitable scale) with the experimentally obtained dependence of the prolongation of the cord on the stress force F.

2. Using the experimental results, obtained in p.1, calculate and put into a table the volume of the cord as a function of the loading in the range 35 g to 95 g. Do the calculations consequently for each two adjacent values of the loading in this range. Write down the formulas you have used for the calculations. Make an analytical proposition about the dependence of the volume on the loading.

Assume that Young's modulus is constant: $E = 2.10^6$ Pa. Take in mind that the Hooke's law is only approximately valid and the deviations from it can be up to 10%.

3. Determine the volume of the rubber cord, using the chronometer, at mass of the weight equal to 60 g. Write the formulas used.

Solution of the Experimental Problem

1. The measurements of the cord length l_n at different loadings m_n must be at least 10. The results are shown in Table I.

Table 1.

m_n, kg	$F_n = m_n.g$, N	l_n, mm	$\Delta l_n = l_n - l_0$, mm
0.005	0.05	153	3
0.015	0.15	158	8
0.025	0.25	164	14
0.035	0.35	172	22
0.045	0.45	181	31
0.055	0.55	191	41
0.065	0.65	202	53
0.075	0.75	215	65
0.085	0.85	228	78

0.095	0.95	243	93
0.105	10.5	261	111

The obtained dependence of the prolongation of the cord on the stress force F can be drawn on graph. It is shown in Fig. 1.

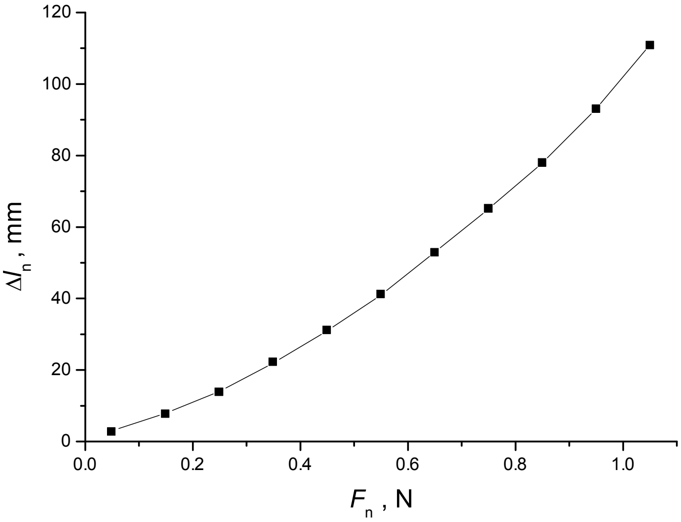

Fig.1

2.For the calculations of the volume the Hooke's law can be used for each measurement:

$$\frac{\Delta l_n^{'}}{l_n} = \frac{1}{E}\frac{\Delta F_n}{S_n},$$

therefore

$$S_n = \frac{l_n \Delta F_n}{E \Delta l_n^{'}},$$

where $\Delta l_n^{'} = l_n - l_{n-1}$, $\Delta F_n = \Delta mg$. (Using the Hooke's law in the form $\frac{\Delta l_n}{l_n} = \frac{1}{E}\frac{F_n}{S_n}$ leads to larger error, because the value of the Δl_n is of the same order as l_n).

As the value of the S_n is determined, it is easy to calculate the volume V_n at each value of F_n:

$$V_n = S_n l_n = \frac{l_n^2 \Delta F_n}{E \Delta l_n^{'}}.$$

Using the data from Table 1, all calculations can be presented in Table 2:

$\Delta m_n = m_n - m_{n-1}$, kg	$\Delta F_n = \Delta m_n g$, N	l_n,m	$\Delta l_n = l_n - l_{n-1}$, m	$S_n = \frac{l_n \Delta F_n}{E \Delta l_n^{'}}$, m^2	$V_n = l_n S_n$, m^3
0.035 – 0.025	0.1	0.172	0.008	$1,07.10^{-6}$	184.10^{-9}
0.045 – 0.035	0.1	0.181	0.009	$1,01.10^{-6}$	183.10^{-9}
0.055 – 0.045	0.1	0.191	0.010	$0,95.10^{-6}$	182.10^{-9}
0.065 – 0.055	0.1	0.203	0.012	$0,92.10^{-6}$	187.10^{-9}
0.075 – 0.065	0.1	0.215	0.012	$0,89.10^{-6}$	191.10^{-9}

0.085 – 0.075	0.1	0.228	0.013	$0,88.10^{-6}$	200.10^{-9}
0.095 – 0.085	0.1	0.243	0.015	$0,81.10^{-6}$	196.10^{-9}
0.105 – 0.095	0.1	0.261	0.018	$0,72.10^{-6}$	188.10^{-9}

The results show that the relative deviation from the averaged value of the calculated values of the volume is:

$$\varepsilon = \frac{\Delta V_{n,aver.}.100\%}{V_{aver.}} \approx \frac{5{,}3.10^{-9}}{189.10^{-9}}.100\% \approx 2.8\%$$

Therefore, the conclusion is that the volume of the rubber cord upon stretching is constant:

$$V_n = const.$$

3. The volume of the rubber cord at fixed loading can be determined investigating the small vibrations of the cord. The reason for these vibrations is the elastic force:

$$F = ES\frac{\Delta l}{l}$$

Using the second law of Newton:

$$-ES\frac{\Delta l}{l} = m\frac{d^2(\Delta l)}{dt^2},$$

the period of the vibrations can be determined:

$$T = 2\pi\sqrt{\frac{ml}{ES}}.$$

Then

$$S = \frac{(2\pi)^2 ml}{ET^2},$$

and the volume of the cord is equal to:

$$V = Sl = \frac{4\pi^2 ml^2}{ET^2}$$

The measurement of the period gives: $T = t/n = 5.25\text{s}/10 = 0.52$ s at used mass $m = 0.065$ kg. The result for the volume $V \approx 195.10^{-9}$ m^3, in agreement with the results obtained in part 2.

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Reference: O. F. Kabardin, V. A. Orlov, in "International Physics Olympiads for High School Students", ed. V. G. Razumovski, Moscow, Nauka, 1985. (In Russian).

Theoretical Problem 1

A static container of mass M and cylindrical shape is placed in vacuum. One of its ends is closed. A fixed piston of mass m and negligible width separates the volume of the container into two equal parts. The closed part contains n moles of monoatomic perfect gas with molar mass M_0 and temperature T. After releasing of the piston, it leaves the container without friction. After that the gas also leaves the container. What is the final velocity of the container?

The gas constant is R. The momentum of the gas up to the leaving of the piston can be neglected. There is no heat exchange between the gas, container and the piston. The change of the temperature of the gas, when it leaves the container, can be neglected. Do not account for the gravitation of the Earth.

Theoretical Problem 2

An electric lamp of resistance $R_0 = 2\ \Omega$ working at nominal voltage $U_0 = 4.5$ V is connected to accumulator of electromotive force E = 6 V and negligible internal resistance.

1. The nominal voltage of the lamp is ensured as the lamp is connected potentiometrically to the accumulator using a rheostat with resistance R. What should be the resistance R and what is the maximal electric current I_{max}, flowing in the rheostat, if the efficiency of the system must not be smaller than $\eta_0 = 0.6$?

2. What is the maximal possible efficiency η of the system and how the lamp can be connected to the rheostat in this case?

Theoretical Problem 3

A detector of radiowaves in a radioastronomical observatory is placed on the sea beach at height $h = 2$ m above the sea level. After the rise of a star, radiating electromagnetic waves of wavelength $\lambda = 21$ cm, above the horizont the detector registers series of alternating maxima and minima. The registered signal is proportional to the intensity of the detected waves. The detector registers waves with electric vector, vibrating in a direction parallel to the sea surface.

1. Determine the angle between the star and the horizont in the moment when the detector registers maxima and minima (in general form).

2. Does the signal decrease or increase just after the rise of the star?

3. Determine the signal ratio of the first maximum to the next minimum. At reflection of the electromagnetic wave on the water surface, the ratio of the intensities of the electric field of the reflected (E_r) and incident (E_i) wave follows the low:

$$\frac{E_r}{E_i} = \frac{n - \cos\varphi}{n + \cos\varphi},$$

where n is the refraction index and φ is the incident angle of the wave. For the surface "air-water" for $\lambda = 21$ cm, the refraction index $n = 9$.

4. Does the ratio of the intensities of consecutive maxima and minima increase or decrease with rising of the star?

Assume that the sea surface is flat.

Solution of the Theoretical Problem 1

Up to the moment when the piston leaves the container, the system can be considered as a closed one. It follows from the laws of the conservation of the momentum and the energy:

$$(M + nM_0)v_1 - mu = 0 \qquad (1)$$

$$\frac{(M + nM_0)v_1^2}{2} + \frac{mu^2}{2} = \Delta U, \qquad (2)$$

where v_1 – velocity of the container when the piston leaves it, u – velocity of the piston in the same moment, ΔU – the change of the internal energy of the gas. The gas is perfect and monoatomic, therefore

$$\Delta U = \frac{3}{2}nR\Delta T = \frac{3}{2}nR(T - T_f); \qquad (3)$$

T_f - the temperature of the gas in the moment when the piston leaves the container. This temperature can be determined by the law of the adiabatic process:

$$pV^\gamma = const.$$

Using the perfect gas equation $pV = nRT$, one obtains

$$TV^{\gamma-1} = const., \qquad TV^{\gamma-1} = T_f V_f^{\gamma-1}$$

Using the relation $V_f = 2V$, and the fact that the adiabatic coefficient for one-atomic gas is $\gamma = \frac{c_p}{c_v} = \frac{\frac{5}{2}R}{\frac{3}{2}R} = \frac{5}{3}$, the result for final temperature is:

$$T_f = T\left(\frac{V}{V_f}\right)^{\gamma-1} = \frac{T}{2^{2/3}} = T2^{-2/3} \qquad (4)$$

Solving the equations (1) – (4) we obtain

$$v_1 = \sqrt{3(1 - 2^{-2/3})\frac{mnRT}{(nM_0 + M)(m + nM_0 + M)}} \qquad (5)$$

If the gas mass nM_0 is much smaller than the masses of the container M and the piston m, then the equation (5) is simplified to:

$$v_1 = \sqrt{3(1 - 2^{-2/3})\frac{mnRT}{M(m + M)}} \qquad (5')$$

When the piston leaves the container, the velocity of the container additionally increases to value v_2 due to the hits of the atoms in the bottom of the container. Each atom gives the container momentum:

$$p = 2m_A\Delta\overline{v_x},$$

where m_A – mass of the atom; $m_A = \frac{M_0}{N_A}$, and $\overline{v_x}$ can be obtained by the averaged quadratic velocity of the atoms $\overline{v^2}$ as follows:

$\overline{v_x^2} + \overline{v_y^2} + \overline{v_z^2} = \overline{v^2}$, and $\overline{v_x^2} = \overline{v_y^2} = \overline{v_z^2}$, therefore $\overline{v_x} = \sqrt{\frac{\overline{v^2}}{3}}$. It appears that due to the elastic impact of one atom the container receives averaged momentum

$$p = 2\frac{M_0}{N_A}\sqrt{\frac{\overline{v^2}}{3}}$$

All calculations are done assuming that the thermal velocities of the atoms are much larger than the velocity of the container and that the movement is described using system connected with the container.

Have in mind that only half of the atoms hit the bottom of the container, the total momentum received by the container is

$$p_t = \frac{1}{2}nN_A p = nM_0\sqrt{\frac{\overline{v^2}}{3}} \tag{6}$$

and additional increase of the velocity of the container is

$$v_2 = \frac{p_t}{M} = n\frac{M_0}{M}\sqrt{\frac{\overline{v^2}}{3}}. \tag{7}$$

Using the formula for the averaged quadratic velocity

$$\sqrt{\overline{v^2}} = \sqrt{\frac{3RT_f}{M_0}}$$

as well eq. (4) for the temperature T_f, the final result for v_2 is

$$v_2 = 2^{-1/3}\frac{n\sqrt{M_0RT}}{M}. \tag{8}$$

Therefore the final velocity of the container is

$$v = v_1 + v_2 = \sqrt{3(1-2^{-2/3})\frac{mnRT}{(nM_0+M)(m+nM_0+M)}} + 2^{-1/3}\frac{n\sqrt{M_0RT}}{M} \approx$$

$$\approx \sqrt{3(1-2^{-2/3})\frac{mnRT}{M(m+M)}} + 2^{-1/3}\frac{n\sqrt{M_0RT}}{M}. \tag{9}$$

Solution of the Theoretical Problem 2

1) The voltage U_0 of the lamp of resistance R_0 is adjusted using the rheostat of resistance R. Using the Kirchhoff laws one obtains:

$$I = \frac{U_0}{R} + \frac{U_0}{R-R_x}, \tag{1}$$

where $R - R_x$ is the resistance of the part of the rheostat, parallel connected to the lamp, R_x is the resistance of the rest part,

$$U_0 = E - IR_x \tag{2}$$

The efficiency η of such a circuit is

$$\eta = \frac{P_{lamp}}{P_{accum.}} = \frac{U_0^2 / r}{IE} = \frac{U_0^2}{RIE}. \tag{3}$$

From eq. (3) it is seen that the maximal current, flowing in the rheostat, is determined by the minimal value of the efficiency:

$$I_{\max} = \frac{U_0^2}{RE\eta_{\min}} = \frac{U_0^2}{RE\eta_0}. \tag{4}$$

The dependence of the resistance of the rheostat R on the efficiency η can determined replacing the value for the current I, obtained by the eq. (3), $I = \frac{U_0^2}{RE\eta}$, in the eqs. (1) and (2):

$$\frac{U_0}{RE\eta} = \frac{1}{R_0} + \frac{1}{R - R_x}, \tag{5}$$

$$R_x = (E - U_0)\frac{RE\eta}{U_0^2}. \tag{6}$$

Then

$$R = R_0\eta\frac{E^2}{U_0^2}\frac{1 + \eta(1 - \frac{E}{U_0})}{1 - \frac{E}{U_0}\eta}. \tag{7}$$

To answer the questions, the dependence $R(\eta)$ must be investigated. By this reason we find the first derivative R'_η:

$$R'_\eta \propto \left(\frac{\eta + \eta^2(1 - \frac{E}{U_0})}{1 - \frac{E}{U_0}\eta}\right)' \propto$$

$$\propto 1 + 2\eta(1 - \frac{E}{U_0})(1 - \frac{E}{U_0}\eta) + \left[\eta + \eta^2(1 - \frac{E}{U_0})\right]\frac{E}{U_0} = \eta(2 - \frac{E}{U_0}\eta)(1 - \frac{E}{U_0}) + 1.$$

$\eta < 1$, therefore the above obtained derivative is positive and the function $R(\eta)$ is increasing. It means that the efficiency will be minimal when the rheostat resistance is minimal. Then

$$R \geq R_{\min} = R_0\eta_0\frac{E^2}{U_0^2}\frac{1 + \eta_0(1 - \frac{E}{U_0})}{1 - \frac{E}{U_0}\eta_0} \approx 8.53\Omega.$$

The maximal current I_{max} can be calculated using eq. (4). The result is: $I_{max} \approx 660$ mA.

2) As the function $R(\eta)$ is increasing one, $\eta \to \eta_{\max}$, when $R \to \infty$. In this case the total current I will be minimal and equal to $\frac{U_0}{R}$. Therefore the maximal efficiency is

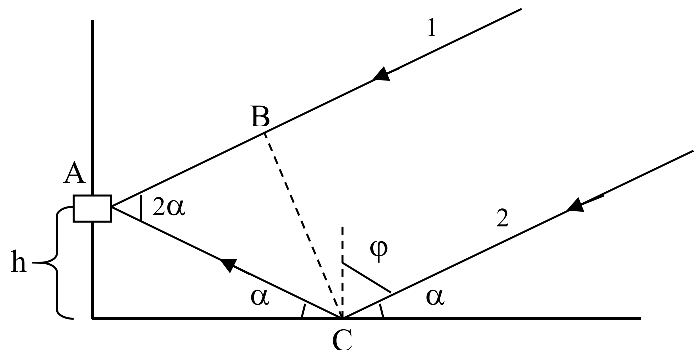
$$\eta_{\max} = \frac{U_0}{E} = 0.75$$

This case can be realized connecting the rheostat in the circuit using only two of its three plugs. The used part of the rheostat is R_1:

$$R_1 = \frac{E - U_0}{I_0} = \frac{E - U_0}{U_0} R_0 \approx 0.67\Omega .$$

Solution of the Theoretical Problem 3

1) The signal, registered by the detector A, is result of the interference of two rays: the ray 1, incident directly from the star and the ray 2, reflected from the sea surface (see the figure).



The phase of the second ray is shifted by π due to the reflection by a medium of larger refractive index. Therefore, the phase difference between the two rays is:

$$\Delta = AC + \frac{\lambda}{2} - AB = \frac{h}{\sin\alpha} + \frac{\lambda}{2} - \left(\frac{h}{\sin\alpha}\right)\cos(2\alpha) =$$

$$= \frac{\lambda}{2} + \frac{h}{\sin\alpha}[1 - \cos(2\alpha)] = \frac{\lambda}{2} + 2h\sin\alpha \qquad (1)$$

The condition for an interference maximum is:

$$\frac{\lambda}{2} + 2h\sin\alpha_{\max} = k\lambda \text{, or}$$

$$\sin\alpha_{\max} = (k - \frac{1}{2})\frac{\lambda}{2h} = (2k-1)\frac{\lambda}{4h}, \qquad (2)$$

where k = 1,2,3,…,19. (the difference of the optical paths cannot exceed $2h$, therefore k cannot exceed 19).

The condition for an interference minimum is:

$$\frac{\lambda}{2} + 2h\sin\alpha_{\max} = (2k+1)\frac{\lambda}{2} \text{, or}$$

$$\sin\alpha_{\min} = \frac{k\lambda}{2h} \qquad (3)$$

where k = 1,2,3,…,19.

2) Just after the rise of the star the angular height α is zero, therefore the condition for an interference minimum is satisfied. By this reason just after the rise of the star, the signal will increase.

3) If the condition for an interference maximum is satisfied, the intensity of the electric field is a sum of the intensities of the direct ray E_i and the reflected ray E_r, respectively: $E_{\max} = E_i + E_r$.

Because $E_r = E_i \dfrac{n-\cos\varphi}{n+\cos\varphi}$, then $E_{\max} = E_i\left(1+\dfrac{n-\cos\varphi_{\max}}{n+\cos\varphi_{\max}}\right)$.

From the figure it is seen that $\varphi_{\max} = \dfrac{\pi}{2} - \alpha_{\max}$, we obtain

$$E_{\max} = E_i\left(1+\frac{n-\sin\alpha_{\max}}{n+\sin\alpha_{\max}}\right) = E_i\frac{2n}{n+\sin(2\alpha_{\max})}. \tag{4}$$

At the interference minimum, the resulting intensity is:

$$E_{\min} = E_i - E_r = E_i\frac{2\sin\alpha_{\min}}{n+\sin\alpha_{\min}}. \tag{5}$$

The intensity I of the signal is proportional to the square of the intensity of the electric field E, therefore the ratio of the intensities of the consecutive maxima and minima is:

$$\frac{I_{\max}}{I_{\min}} = \left(\frac{E_{\max}}{E_{\min}}\right)^2 = \frac{n^2}{\sin^2\alpha_{\min}}\frac{(n+\sin\alpha_{\min})^2}{(n+\sin\alpha_{\max})^2}. \tag{6}$$

Using the eqs. (2) and (3), the eq. (6) can be transformed into the following form:

$$\frac{I_{\max}}{I_{\min}} = \frac{4n^2h^2}{k^2\lambda^2}\left[\frac{n+k\dfrac{\lambda}{2h}}{n+(2k-1)\dfrac{\lambda}{4h}}\right]^2. \tag{7}$$

Using this general formula, we can determine the ratio for the first maximum (k =1) and the next minimum:

$$\frac{I_{\max}}{I_{\min}} = \frac{4n^2h^2}{\lambda^2}\left(\frac{n+\dfrac{\lambda}{2h}}{n+\dfrac{\lambda}{4h}}\right)^2 = 3.10^4$$

4) Using that $n >> \dfrac{\lambda}{2h}$, from the eq. (7) follows :

$$\frac{I_{\max}}{I_{\min}} \approx \frac{4n^2h^2}{k^2\lambda^2}.$$

So, with the rising of the star the ratio of the intensities of the consecutive maxima and minima decreases.

Problems of the 13th International Physics Olympiad

(Malente, 1982)

Gunter Lind & Gunnar Friege [1]

Leibniz-Institute for Science Education (IPN) at the University of Kiel, Germany

Abstract

The 13th International Physics Olympiad took place in 1982 in the Federal Republic of Germany. This article contains the competition problems, their solutions and a grading scheme.

Introduction

In 1982 the Federal Republic of Germany was the first host of the Physics Olympiad outside the so-called Eastern bloc. The 13th International Physics Olympiad took place in Malente, Schleswig-Holstein. The competition was funded by the German Federal Ministry of Science and Education. The organisational guidelines were laid down by the work group "Olympiads for pupils" of the conference of ministers of education of the German federal states. The Institute for Science Education (IPN) at the University of Kiel was responsible for the realisation of the event. A commission of professors, whose chairman was appointed by the German Physical Society, were concerned with the formulation of the competition problems. All other members of the commission came from physics department of the university of Kiel or from the college of education at Kiel.

The problems as usual covered different fields of classical physics. In 1982 the pupils had to deal with three theoretical and two experimental problems, whereas at the previous Olympiads only one experimental task was given. However, it seemed to be reasonable to put more stress on experimental work. The degree of difficulty was well balanced. One of the theoretical problems could be considered as quite simple (problem 3: "hot-air balloon"). Another theoretical problem (problem 1: "fluorescent lamp") had a mean degree of difficulty and the distribution of the points was a normal distribution with only a few

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[1] Contact: Leibniz-Institute for Science Education (IPN) at the University of Kiel
Olshausenstrasse 62, 24098 Kiel, Germany
ipho@ipn.uni-kiel.de

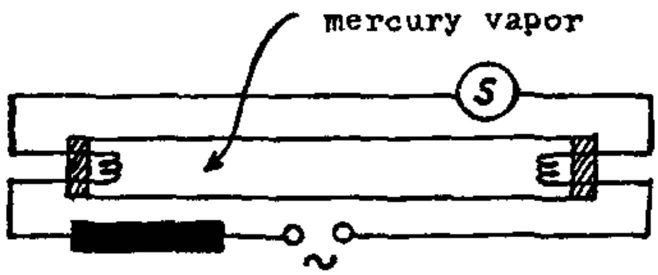
excellent and only a few unsatisfying solutions. The third problem (problem 2: “oscillation coat hanger”) turned out to be the most difficult problem. This problem was generally considered as quite interesting because different ways of solving were possible. About one third of the pupils did not find an adequate start to the problem, but nearly one third of the pupils was able to solve the substantial part of the problem. That means, this problem polarized between the pupils. The two experimental tasks were quite different in respect of the input for the experimental setup and the time required for dealing with the problems, whereas they were quite similar in the degree of difficulty. Both required demandingly theoretical considerations and experimental skills. Both experimental problems turned out to be rather difficult. The tasks were composed in a way that on the one hand almost every pupil had the possibility to come to certain partial results and that there were some difficulties on the other hand which could only be solved by very few pupils. The difficulty in the second experimental problem (problem5: “motion of a rolling cylinder”) was the explanation of the experimental results, which were initially quite surprising. The difficulty in the other task (problem 4: “lens experiment") was the revealing of an observation method with a high accuracy (parallax). The five hours provided for solving the two experimental problems were slightly too short. According to that, in both experiments only a few pupils came up with excellent solutions. In problem 5 nobody got the full points.

The problems presented here are based on the original German and English versions of the competition problems. The solutions are complete but in some parts condensed to the essentials. Almost all of the original hand-made figures are published here.

Theoretical Problems

Problem 1: Fluorescent lamp

An alternating voltage of 50 Hz frequency is applied to the fluorescent lamp shown in the accompanying circuit diagram.



The following quantities are measured:

overall voltage (main voltage)	$U = 228.5$ V
electric current	$I = 0.6$ A
partial voltage across the fluorescent lamp	$U' = 84$ V
ohmic resistance of the series reactor	$R_d = 26.3\Omega$

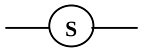
The fluorescent lamp itself may be considered as an ohmic resistor in the calculations.

a) What is the inductance L of the series reactor?

b) What is the phase shift φ between voltage and current?

c) What is the active power P_w transformed by the apparatus?

d) Apart from limiting the current the series reactor has another important function. Name and explain this function!

Hint: The starter  includes a contact which closes shortly after switching on the lamp, opens up again and stays open.

e) In a diagram with a quantitative time scale sketch the time sequence of the luminous flux emitted by the lamp.

f) Why has the lamp to be ignited only once although the applied alternating voltage goes through zero in regular intervals?

g) According to the statement of the manufacturer, for a fluorescent lamp of the described type a capacitor of about 4.7 μF can be switched in series with the series reactor. How does this affect the operation of the lamp and to what intent is this possibility provided for?

h) Examine both halves of the displayed demonstrator lamp with the added spectroscope. Explain the differences between the two spectra. You may walk up to the lamp and you may keep the spectroscope as a souvenir.

Solution of problem 1:

a) The total resistance of the apparatus is $Z = \dfrac{228.5\,\mathrm{V}}{0.6\,\mathrm{A}} = 380.8\ \Omega$,

the ohmic resistance of the tube is $R_R = \dfrac{84\,\mathrm{V}}{0.6\,\mathrm{A}} = 140\ \Omega$.

Hence the total ohmic resistance is $R = 140\,\Omega + 26.3\,\Omega = 166.3\,\Omega$.

Therefore the inductance of the series reactor is: $\omega \cdot L = \sqrt{Z^2 - R^2} = 342.6\,\Omega$.

This yields $L = \dfrac{342.6\,\Omega}{100\,\pi\,\mathrm{s}^{-1}} = 1.09\ \mathrm{H}$.

b) The impedance angle is obtained from $\tan\varphi = \dfrac{\omega \cdot L}{R} = \dfrac{342.6\,\Omega}{166.3\,\Omega} = 2.06$.

Thus $\varphi = 64.1^{\circ}$.

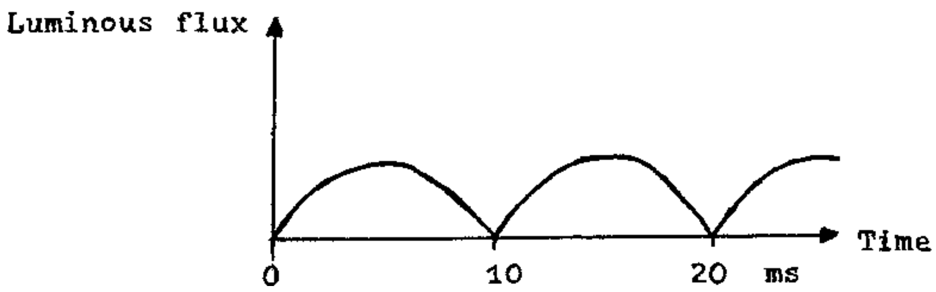
c) The active power can be calculated in different ways:

1) $P_w = U \cdot I \cdot \cos\varphi = 228.5\,\mathrm{V} \cdot 0.6\ \mathrm{A} \cdot \cos 64.1^{\circ} = 59.88\ \mathrm{W}$

2) $P_w = R \cdot I^2 = 166.3\,\Omega \cdot (0.6\ \mathrm{A})^2 = 59.87\ \mathrm{W}$

d) By opening the contact in the starter a high induction voltage is produced across the series reactor (provided the contact does not open exactly the same moment, when the current goes through zero). This voltage is sufficient to ignite the lamp. The main voltage itself, however, is smaller than the ignition voltage of the fluorescent tube.

e)



f) The recombination time of the ions and electrons in the gaseous discharge is sufficiently large.

g) The capacitive resistance of a capacitor of 4.7 μF is

$$\frac{1}{\omega \cdot C} = (100 \cdot \pi \cdot 4.7 \cdot 10^{-6})^{-1}\ \Omega = 677.3\ \Omega .$$

The two reactances subtract and there remains a reactance of 334.7 Ω acting as a capacitor.

The total resistance of the arrangement is now

$$Z' = \sqrt{(334.7)^2 + (166.3)^2}\ \Omega = 373.7\ \Omega ,$$

which is very close to the total resistance without capacitor, if you assume the capacitor to be loss-free (cf. a)). Thus the lamp has the same operating qualities, ignites the same way, and a difference is found only in the impedance angle φ', which is opposite to the angle φ calculated in b):

$$\tan \varphi' = \frac{\omega \cdot L - (\omega \cdot C)^{-1}}{R} = -\frac{334.7}{166.3} = -2.01$$

$$\varphi' = -63.6^{\circ} .$$

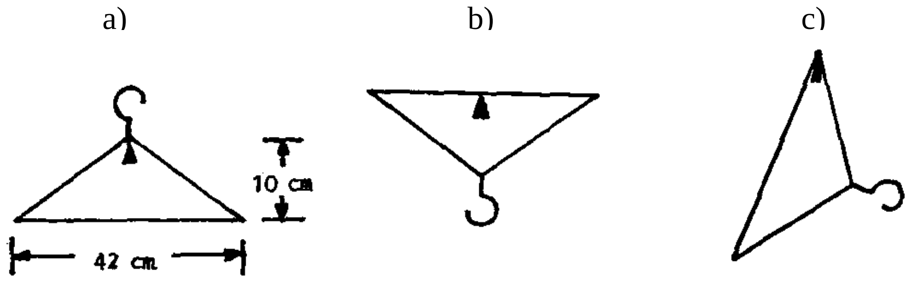
Such additional capacitors are used for compensation of reactive currents in buildings with a high number of fluorescent lamps, frequently they are prescribed by the electricity supply companies. That is, a high portion of reactive current is unwelcome, because the power generators have to be layed out much bigger than would be really necessary and transport losses also have to be added which are not payed for by the customer, if pure active current meters are used.

h) The uncoated part of the demonstrator lamp reveals the line spectrum of mercury, the coated part shows the same line spectrum over a continuous background. The continuous spectrum results from the ultraviolet part of the mercury light, which is absorbed by the fluorescence and re-emitted with smaller frequency (energy loss of the photons) or larger wavelength respectively.

Problem 2: Oscillating coat hanger

A (suitably made) wire coat hanger can perform small amplitude oscillations in the plane of the figure around the equilibrium positions shown. In positions a) and b) the long side is horizontal. The other two sides have equal length. The period of oscillation is the same in all cases.

What is the location of the center of mass, and how long is the period?

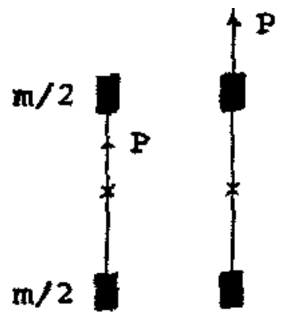


The figure does not contain any information beyond the dimensions given. Nothing is known, e.g., concerning the detailed distribution of mass.

Solution of problem 2

First method:

The motions of a rigid body in a plane correspond to the motion of two equal point masses connected by a rigid massless rod. The moment of inertia then determines their distance. Because of the equilibrium position a) the center of mass is on the perpendicular bipartition of the long side of the coat hanger. If one imagines the equivalent masses and the supporting point P being arranged in a straight line in each case, only two positions of P yield the same period of oscillation (see sketch). One can understand this by considering the limiting cases: 1. both supporting points in the upper mass and 2. one point in the center of mass and the other infinitely high above. Between these extremes the period of oscillation grows continuously. The supporting point placed in the corner of the long side c) has the largest distance from the center of mass, and therefore this point lies outside the two point masses. The two other supporting points a), b) then have to be placed symmetrically to the center of mass between the two point masses, i.e., the center of mass bisects the perpendicular bipartition.



One knows of the reversible pendulum that for every supporting point of the physical pendulum it generally has a second supporting point of the pendulum rotated by 180^o, with the same period of oscillation but at a different distance from the center of mass. The section between the two supporting points equals the length of the corresponding mathematical pendulum. Therefore the period of oscillation is obtained through the corresponding length of the pendulum $s_b + s_c$, where $s_b = 5$ cm and $s_c = \sqrt{5^2 + 21^2}$ cm, to be $T = 1.03$ s .

Second method:

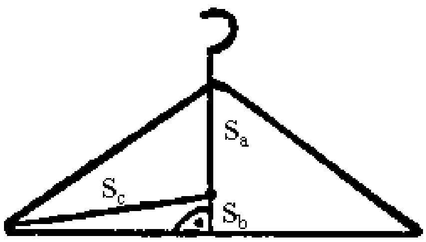
Let s denote the distance between the supporting point and the center of mass, m the mass itself and θ the moment of inertia referring to the supporting point. Then we have the period of oscillation T :

$$T = 2\pi\sqrt{\frac{\theta}{m \cdot g \cdot s}} \quad , \tag{1}$$

where g is the acceleration of gravity, $g = 9.81$ m/s^2. Here θ can be obtained from the moment of inertia θ_o related to the center of mass:

$$\theta = \theta_0 + m \cdot s^2 \tag{2}$$

Because of the symmetrical position in case a) the center of mass is to be found on the perpendicular bisection above the long side. Now (1) and (2) yield



$$\theta_0 + m \cdot s^2 = \left(\frac{T}{2 \cdot \pi}\right)^2 \cdot m \cdot g \cdot s \quad \text{for} \quad s = s_a,\ s_b \ \text{and}\ s_c. \tag{3}$$

because all periods of oscillation are the same. This quadratic equation has only two different solutions at most. Therefore at least two of the three distances are equal. Because of $s_c > 21$ cm $> s_a + s_b$, only s_a and s_b can equal each other. Thus we have

$$s_a = 5 \text{ cm} \tag{4}$$

The moment of inertia θ_0 is eliminated through (3):

$$m \cdot (s_c^{\,2} - s_a^{\,2}) = \left(\frac{T}{2 \cdot \pi}\right)^2 \cdot m \cdot g \cdot (s_c - s_a)$$

and we have $$T = 2 \cdot \pi \sqrt{\frac{s_c + s_a}{g}} \tag{5}$$

with the numerical value $T = 1.03\,\mathrm{s}$,

which has been rounded off after two decimals because of the accuracy of g.

Third method:

This solution is identical to the previous one up to equation (2).

From (1) and (2) we generally have for equal periods of oscillation $T_1 = T_2$:

$$\frac{\theta_0 + m\cdot s_1{}^2}{m\cdot g\cdot s_1} = \frac{\theta_0 + m\cdot s_2{}^2}{m\cdot g\cdot s_2}$$

and therefore $s_2 \cdot\left(\theta_0 + m\cdot s_1{}^2\right) = s_1\cdot\left(\theta_0 + m\cdot s_2{}^2\right)$

or $$\left(s_2 - s_1\right)\cdot\left(\theta_0 - m\cdot s_1\, s_2\right) = 0 \tag{6}$$

The solution of (6) includes two possibilities: $s_1 = s_2$ or $s_1\cdot s_2 = \dfrac{\theta_0}{m}$

Let $2\cdot a$ be the length of the long side and b the height of the coat hanger. Because of $T_b = T_c$ we then have either $s_b = s_c$ or $s_b\cdot s_c = \dfrac{\theta_0}{m}$, where $s_c = \sqrt{s_b{}^2 + a^2}$,

which excludes the first possibility. Thus $$s_b\cdot s_c = \frac{\theta_0}{m}\,. \tag{7}$$

For $T_a = T_b$ the case $s_a \cdot s_b = \dfrac{\theta_0}{m}$ is excluded because of eq. (7), for we have

$$s_a\cdot s_b < s_c\cdot s_b = \frac{\theta_0}{m}\,.$$

Hence $s_a = s_b = \dfrac{1}{2}b$, $s_c = \sqrt{\dfrac{1}{4}b^2 + a^2}$

and $$T = 2\cdot\pi\sqrt{\frac{\dfrac{\theta_0}{m} + s_b{}^2}{g\cdot s_b}} = 2\pi\sqrt{\frac{s_b\cdot s_c + s_b{}^2}{g\cdot s_b}}$$

The numerical calculation yields the value $T = 1.03\,\mathrm{s}$.

Problem 3: Hot-air-balloon

Consider a hot-air balloon with fixed volume $V_B = 1.1\ m^3$. The mass of the balloon-envelope, whose volume is to be neglected in comparison to V_B, is $m_H = 0.187$ kg.

The balloon shall be started, where the external air temperature is $\vartheta_1 = 20\ ^oC$ and the normal external air pressure is $p_o = 1.013 \cdot 10^5$ Pa. Under these conditions the density of air is $\rho_1 = 1.2\ kg/m^3$.

a) What temperature ϑ_2 must the warmed air inside the balloon have to make the balloon just float?

b) First the balloon is held fast to the ground and the internal air is heated to a steady-state temperature of $\vartheta_3 = 110\ ^oC$. The balloon is fastened with a rope.

   Calculate the force on the rope.

c) Consider the balloon being tied up at the bottom (the density of the internal air stays constant). With a steady-state temperature $\vartheta_3 = 110\ ^oC$ of the internal air the balloon rises in an isothermal atmosphere of 20 ^oC and a ground pressure of $p_0 = 1.013 \cdot 10^5$ Pa. Which height h can be gained by the balloon under these conditions?

d) At the height h the balloon (question c)) is pulled out of its equilibrium position by 10 m and then is released again.
   Find out by qualitative reasoning what kind of motion it is going to perform!

Solution of problem 3:

a) Floating condition:

   The total mass of the balloon, consisting of the mass of the envelope m_H and the mass of the air quantity of temperature ϑ_2 must equal the mass of the displaced air quantity with temperature $\vartheta_1 = 20\ ^oC$.

$$V_B \cdot \rho_2 + m_H = V_B \cdot \rho_1$$

$$\rho_2 = \rho_1 - \frac{m_H}{V_B} \qquad (1)$$

Then the temperature may by obtained from

$$\frac{\rho_1}{\rho_2} = \frac{T_2}{T_1},$$

$$T_2 = \frac{\rho_1}{\rho_2} \cdot T_1 = 341.53 \text{ K} = 68.38 \text{ °C} \qquad (2)$$

b) The force F_B acting on the rope is the difference between the buoyant force F_A and the weight force F_G:

$$F_B = V_B \cdot \rho_1 \cdot g \ - \ (V_B \cdot \rho_3 + m_H) \cdot g \qquad (3)$$

It follows with $\rho_3 \cdot T_3 = \rho_1 \cdot T_1$

$$F_B = V_B \cdot \rho_1 \cdot g \cdot \left(1 - \frac{T_1}{T_3}\right) - m_H \cdot g = 1{,}21 \text{ N} \qquad (4)$$

c) The balloon rises to the height h, where the density of the external air ρ_h has the same value as the effective density ρ_{eff}, which is evaluated from the mass of the air of temperature $\vartheta_3 = 110\ ^{o}C$ (inside the balloon) and the mass of the envelope m_H:

$$\rho_{eff} = \frac{m_2}{V_B} = \frac{\rho_3 \cdot V_B + m_H}{V_B} = \rho_h = \rho_1 \cdot e^{-\frac{\rho_1 \cdot g \cdot h}{p_0}} \qquad (5)$$

Resolving eq. (5) for h gives: $h = \frac{p_o}{\rho_1 \cdot g} \cdot \ln \frac{\rho_1}{\rho_{eff}} = 843\,m$ (6).

d) For *small* height differences (10 m in comparison to 843 m) the exponential pressure drop (or density drop respectively) with height can be approximated by a linear function of height. Therefore the driving force is proportional to the elongation out of the equilibrium position.

This is the condition in which harmonic oscillations result, which of course are damped by the air resistance.

Experimental Problems

Problem 4: Lens experiment

The apparatus consists of a symmetric biconvex lens, a plane mirror, water, a meter stick, an optical object (pencil), a supporting base and a right angle clamp. Only these parts may be used in the experiment.

a) Determine the focal length of the lens with a maximum error of $\pm$ 1 %.

b) Determine the index of refraction of the glass from which the lens is made.

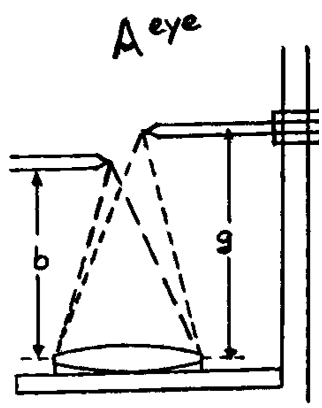
The index of refraction of water is $n_w = 1.33$. The focal length of a thin lens is given by

$$\frac{1}{f} = (n-1)\cdot\left(\frac{1}{r_1} - \frac{1}{r_2}\right),$$

where n is the index of refraction of the lens material and r_1 and r_2 are the curvature radii of the refracting surfaces. For a symmetric biconvex lens we have $r_1 = - r_2 = r$, for a symmetric biconcave lens $r_1 = - r_2 = - r$.

Solution of problem 4:

a) For the determination of f_L , place the lens on the mirror and with the clamp fix the pencil to the supporting base. Lens and mirror are then moved around until the vertically downward looking eye sees the pencil and its image side by side.



In order to have object and image in focus at the same time, they must be placed at an equal distance to the eye. In this case object distance and image distance are the same and the magnification factor is 1 .

It may be proved quite accurately, whether magnification 1 has in fact been obtained, if one concentrates on parallatical shifts between object and image when moving the eye: only when the distances are equal do the pencil-tips point at each other all the time.

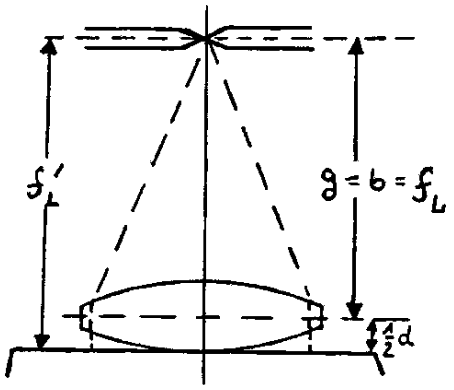
The light rays pass the lens twice because they are reflected by the mirror. Therefore the optical mapping under consideration corresponds to a mapping with two lenses placed directly one after another:

$$\frac{1}{g}+\frac{1}{b}=\frac{1}{f}, \quad \text{where} \quad \frac{1}{f}=\frac{1}{f_L}+\frac{1}{f_L}$$

i.e. the effective focal length f is just half the focal length of the lens. Thus we find for magnification 1:

$$g=b \quad \text{and} \quad \frac{2}{g}=\frac{2}{f_L} \quad \text{i.e.} \quad f_L=g.$$

A different derivation of $f_L = g = b$: For a mapping of magnification 1 the light rays emerging from a point on the optical axis are reflected into themselves. Therefore these rays have to hit the mirror at right angle and so the object distance g equals the focal length f_L of the lens in this case.



The distance between pencil point and mirror has to be determined with an accuracy, which enables one to state f_L with a maximum error of $\pm 1\,\%$. This is accomplished either by averaging several measurements or by stating an uncertainty interval, which is found through the appearance of parallaxe.

Half the thickness of the lens has to be subtracted from the distance between pencil-point and mirror.

$$f_L = f_L' - \frac{1}{2}d, \quad d = 3.0 \pm 0.5\ \text{mm}$$

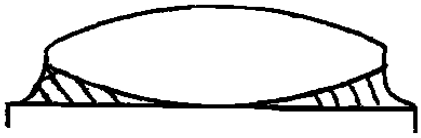
The nominal value of the focal length of the lens is f_L = 30 cm. However, the actual focal length of the single lenses spread considerably. Each lens was measured separately, so the individual result of the student can be compared with the exact value.

b) The refractive index n of the lens material can be evaluated from the equation

$$\frac{1}{f_L} = (n-1)\cdot\frac{2}{r}$$

if the focal length f_L and the curvature radius r of the symmetric biconvex lens are known. f_L was determined in part a) of this problem.

The still unknown curvature radius r of the symmetric biconvex lens is found in the following way: If one pours some water onto the mirror and places the lens in the water, one gets a plane-concave water lens, which has one curvature radius equalling the glass lens' radius and the other radius is ∞.



Because the refractive index of water is known in this case, one can evaluate the curvature radius through the formula above, where $r_1 = -r$ and $r_2 = \infty$:

$$-\frac{1}{f_w} = (n_w - 1) \cdot \frac{1}{r}.$$

Only the focal length f' of the combination of lenses is directly measured, for which we have

$$\frac{1}{f'} = \frac{1}{f_L} + \frac{1}{f_w}.$$

This focal length is to be determined by a mapping of magnification 1 as above.

Then the focal length of the water lens is $\frac{1}{f_w} = \frac{1}{f'} - \frac{1}{f_L}$

and one has the curvature radius $r = -(n_w - 1) \cdot f_w$.

Now the refractive index of the lens is determined by $n = \frac{r}{2 \cdot f_L} + 1$

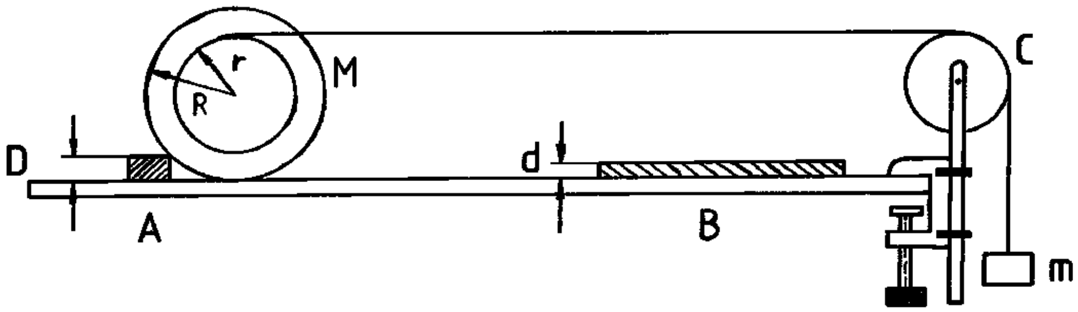
with the known values of f_L and r, or, if one wants to express n explicitly through the measured quantities: $n = \frac{f' \cdot (n_w - 1)}{2 \cdot (f' - f_L)} + 1.$

The nominal values are: $f' =$ 43.9 cm, $f_w =$ -94.5 cm, $r =$ 31.2 cm, $n = 1.52$.

Problem 5: Motion of a rolling cylinder

The rolling motion of a cylinder may be decomposed into rotation about its axis and horizontal translation of the center of gravity. In the present experiment only the translatory acceleration and the forces causing it are determined directly.

Given a cylinder of mass M, radius R, which is placed on a horizontal plane board. At a distance r_i ($i = 1 \ldots 6$) from the cylinder axis a force acts on it (see sketch). After letting the cylinder go, it rolls with constant acceleration.



a) Determine the linear accelerations a_i ($i = 1 \ldots 6$) of the cylinder axis experimentally for several distances r_i ($i = 1 \ldots 6$).

b) From the accelerations a_i and given quantities, compute the forces F_i which act in horizontal direction between cylinder and plane board.

c) Plot the experimental values F_i as functions of r_i. Discuss the results.

Before starting the measurements, adjust the plane board horizontally. For present purposes it suffices to realize the horizontal position with an uncertainty of ± 1 mm of height difference on 1 m of length; this corresponds to the distance between adjacent markings on the level. What would be the result of a not horizontal position of the plane board?

Describe the determination of auxiliary quantities and possible further adjustments; indicate the extent to which misadjustments would influence the results.

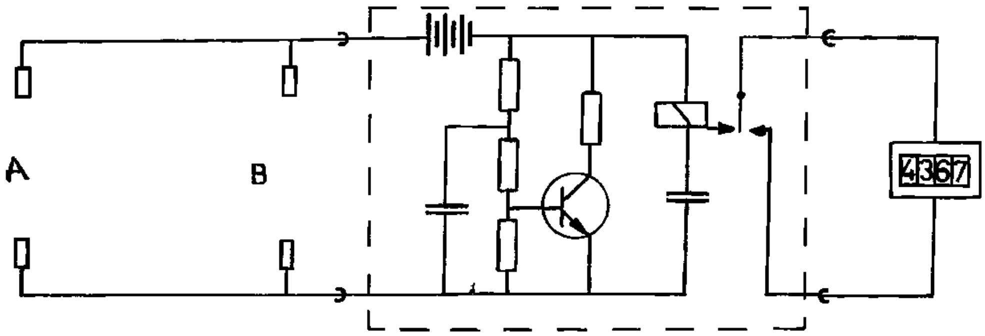
The following quantities are given:

R	=	5 cm	r_1	=	0.75 cm
M	=	3.275 kg	r_2	=	1.50 cm
m	=	2 x 50 g	r_3	=	2.25 cm
D	=	1.50 cm	r_4	=	3.00 cm
d	=	0.1 mm	r_5	=	3.75 cm
			r_6	=	4.50 cm

Mass and friction of the pulleys c may be neglected in the evaluation of the data.

By means of knots, the strings are put into slots at the cylinder. They should be inserted as deeply as possible. You may use the attached paper clip to help in this job.

The stop watch should be connected, as shown in the sketch, with electrical contacts at A and B via an electronic circuit box. The stop watch starts running as soon as the contact at A is opened, and it stops when the contact at B is closed.



The purpose of the transistor circuit is to keep the relay position after closing of the contact at B, even if this contact is opened afterwards for a few milliseconds by a jump or chatter of the cylinder.

Solution of problem 5:

Theoretical considerations:

a) The acceleration of the center of mass of the cylinder is $$a = \frac{2 \cdot s}{t^2} \qquad (1)$$

b) Let a_m be the acceleration of the masses m and T the sum of the tensions in the two strings, then

$$T = m \cdot g - m \cdot a_m \qquad (2)$$

The acceleration a of the center of mass of the cylinder is determined by the resultant force of the string-tension T and the force of interaction F between cylinder and the horizontal plane.

$$M \cdot a = T - F \qquad (3)$$

If the cylinder rotates through an angle θ the mass m moves a distance x_m.
It holds

$$x_m = (R + r) \cdot \theta$$

$$a_m = (R + r) \cdot \frac{a}{R} \qquad (4)$$

From (2), (3) and (4) follows $F = mg - \left[M + m \cdot \left(1 + \frac{r}{R} \right) \right] \cdot a$. (5)

c) From the experimental data we see that for small r_i the forces $M \cdot a$ and T are in opposite direction and that they are in the same direction for large r_i .

For small values of r the torque produced by the string-tensions is not large enough to provide the angular acceleration required to prevent slipping. The interaction force between cylinder and plane acts into the direction opposite to the motion of the center of mass and thereby delivers an additional torque.

For large values of r the torque produced by string-tension is too large and the interaction force has such a direction that an opposed torque is produced.

From the rotary-impulse theorem we find

$$T \cdot r + F \cdot R = I \cdot \ddot{\theta} = I \cdot \frac{a}{R},$$

where I is the moment of inertia of the cylinder.

With (3) and (5) you may eliminate T and a from this equation. If the moment of inertia of the cylinder is taken as $I = \frac{1}{2} \cdot M \cdot R^2$ (neglecting the step-up cones) we find after some arithmetical transformations

$$F = mg \cdot \frac{1 - 2 \cdot \frac{r}{R}}{3 + 2 \cdot \frac{m}{M} \cdot \left(1 + \frac{r}{R} \right)^2}.$$

For $r = 0 \rightarrow \quad F = \frac{m \cdot g}{3 + 2 \cdot \frac{m}{M}} > 0.$

For $r = R \Rightarrow \quad F = \frac{-m \cdot g}{3 + 8 \cdot \frac{m}{M}} < 0.$

Because $\frac{m}{M} \ll 1$ it is approximately $F = \frac{1}{3} m \cdot g - \frac{2}{3} \cdot \frac{r}{R}$.

That means: the dependence of F from r is approximately linear. F will be zero if

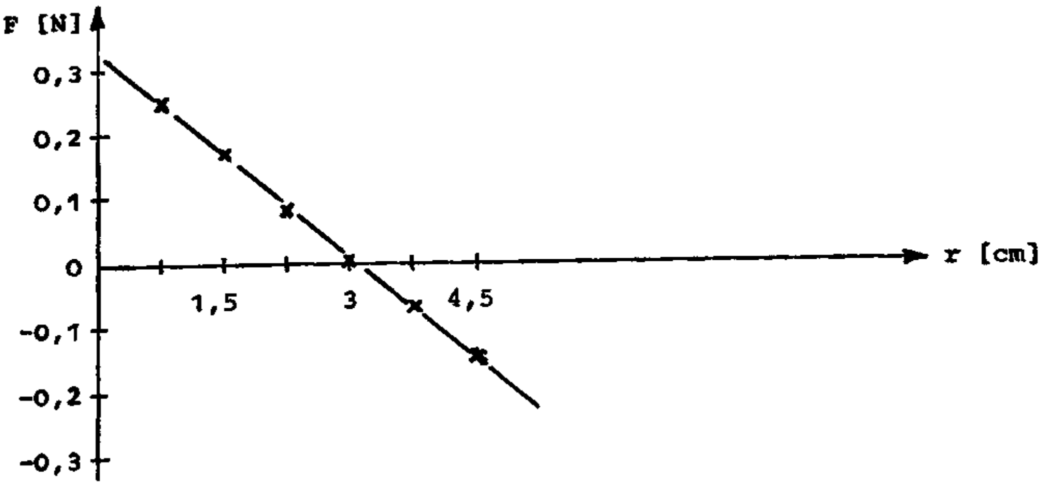
$$\frac{r}{R} = \frac{m \cdot g}{2}.$$

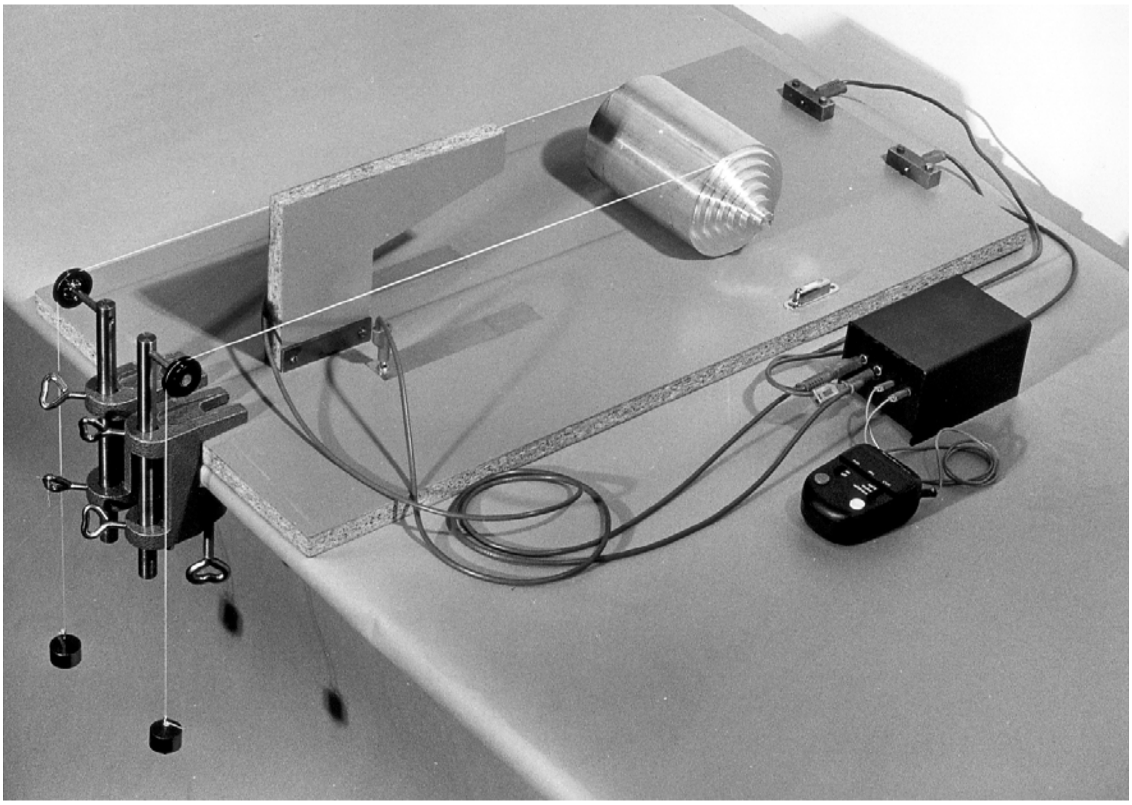
Experimental results:

$$s = L - (2 \cdot R \cdot D + D^2)^{\frac{1}{2}} - (2 \cdot R \cdot d - d^2)^{\frac{1}{2}}$$

$$s = L - 4.5 \text{ cm} = 39.2 \text{ cm} - 4.5 \text{ cm} = 34.7 \text{ cm}$$

r [cm]	t [s]			$\overline{t}$ [s]	a [m/s^2]	F [N]
0.75	1.81	1.82	1.82	1.816	0.211	0.266
1.50	1.71	1.72	1.73	1.720	0.235	0.181
2.25	1.63	1.63	1.64	1.633	0.261	0.090
3.00	1.56	1.56	1.57	1.563	0.284	0.004
3.75	1.51	1.51	1.52	1.513	0.304	- 0.066
4.50	1.46	1.46	1.46	1.456	0.328	- 0.154





Grading schemes

Theoretical problems

Problem 1: Fluorescent lamp	pts.
Part a	2
Part b	1
Part c	1
Part d	1
Part e	1
Part f	1
Part g	2
Part h	1
	10

Problem 2: Oscillating coat hanger	pts.
equation (1)	1,5
equation (2)	1,5
equation (4)	3
equation (5)	2
numerical value for T	1
	10

Problem 3: Hot-air-balloon	pts.
Part a	3
Part b	2
Part c	3
Part d	2
	10

Experimental problems

Problem 4: Lens experiment	pts.
correct description of experimental prodedure	1
selection of magnification one	0.5
parallaxe for verifying his magnification	1
$f_L = g = b$ with derivation	1
several measurements with suitable averaging or other determination of error interval	1
taking into account the lens thickness and computing f_L, including the error	0.5
idea of water lens	0.5
theory of lens combination	1
measurements of f'	0.5
calculation of n and correct result	1
	8

Problem 5: Motion of a rolling cylinder	pts.
Adjustment mentioned of strings a) horizontally and b) in direction of motion	0.5
Indication that angle offset of strings enters the formula for the acting force only quadratically, i.e. by its cosine	0.5
Explanation that with non-horizontal position, the force $m \cdot g$ is to be replaced by $m \cdot g \pm M \cdot g \cdot \sin\alpha$	1.0
Determination of the running length according for formula $s = L - \left(2 \cdot R \cdot D + D^2\right)^{1/2} - \left(2 \cdot R \cdot d + d^2\right)^{1/2}$ including correct numerical result	1.0
Reliable data for rolling time	1.0
accompanied by reasonable error estimate	0.5
Numerical evaluation of the F_i	0.5
Correct plot of $F_i\ (v_i)$	0.5
Qualitative interpretation of the result by intuitive consideration of the limiting cases $r = 0$ and $r = R$	1.0
Indication of a quantitative, theoretical interpretation using the concept of moment of inertia	1.0
Knowledge and application of the formula $a = 2\,s / t^2$	0.5
Force equation for small mass and tension of the string $m \cdot (g - a_m) = T$	1.0
Connection of tension, acceleration of cylinder and reaction force $T - F = M \cdot a$	1.0
Connection between rotary and translatory motion	
$x_m = (R + r) \cdot \theta$	0.5
$a_m = (1 + r/R) \cdot a$	0.5
Final formula for the reaction force $F = m \cdot g - \left(M + m \cdot (1 + r/R)\right) \cdot a$	1.0
If final formulae are given correctly, the knowledge for preceding equations must be assumed and is graded accordingly.	
	12

Mechanics – Problem I (8 points)

A particle moves along the positive axis $0x$ (one-dimensional situation) under a force having a projection $F_x = F_0$ on $0x$, as represented, as function of x, in the figure 1.1. In the origin of the $0x$ axis is placed a perfectly reflecting wall.

A friction force, with a constant modulus $F_f = 1{,}00N$, acts everywhere on the particle.

The particle starts from the point $x = x_0 = 1{,}00m$ having the kinetic energy $E_c = 10{,}0J$.

a. Find the length of the path of the particle until its' final stop

b. Plot the potential energy $U(x)$ of the particle in the force field F_x.

c. Qualitatively plot the dependence of the particle's speed as function of its' x coordinate.

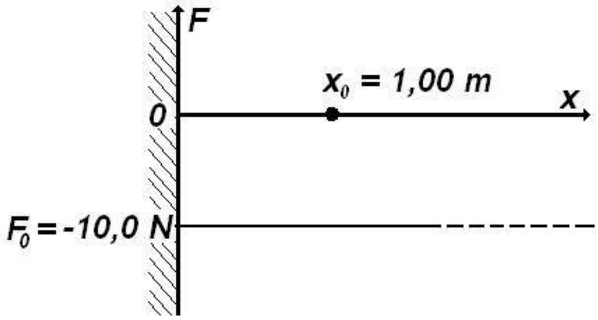


Figure 1.1

Problem I – Solution

a. *It is possible to make a model of the situation in the problem, considering the Ox axis vertically oriented having the wall in its' lower part. The conservative force F_x could be the weight of the particle. One may present the motion of the particle as the vertical motion of a small elastic ball elastically colliding with the ground and moving with constant friction through the medium. The friction force is smaller than the weight.*

The potential energy of the particle can be represented in analogy to the gravitational potential energy of the ball, $m \cdot g \cdot h$, considering $m \cdot g = |F_x|$; $h = x$. As is very well known, in the field of a conservative force, the variation of the potential energy depends only on the initial and final positions of the particle, being independent of the path between those positions.

For the situation in the problem, when the particle moves towards the wall, the force acting on it is directed towards the wall and has the modulus l

$$F_{\leftarrow} = |F_x| - F_f \qquad (1.1)$$

$$F_{\leftarrow} = 9N \qquad (1.2)$$

As a consequence, the motion of the particle towards the wall is a motion with a constant acceleration having the modulus

$$a_{\leftarrow} = \frac{F_{\leftarrow}}{m} = \frac{|F_x| - F_f}{m} \qquad (1.3)$$

During the motion, the speed of the particle increases.
Hitting the wall, the particle starts moving in opposite direction with a speed equal in modulus with the one it had before the collision.
When the particle moves away from the wall, in the positive direction of the *Ox* axis, the acting force is again directed towards to the wall and has the magnitude

$$F_{\rightarrow} = |F_x| + F_f \qquad (1.4)$$

$$F_{\rightarrow} = 11N \qquad (1.5)$$

Correspondingly, the motion of the particle from the wall is slowed down and the magnitude of the acceleration is

$$a_{\rightarrow} = \frac{F_{\rightarrow}}{m} = \frac{|F_x| + F_f}{m} \qquad (1.6)$$

During this motion, the speed of the particle diminishes to zero.
Because during the motion a force acts on the particle, the body cannot have an equilibrium position in any point on axis – the origin making an exception as the potential energy vanishes there. The particle can definitively stop only in this point.
The work of a conservative force from the point having the coordinate $x_0 = 0$ to the point x , $L_{0\rightarrow x}$ is correlated with the variation of the potential energy of the particle $U(x) - U(0)$ as follows

$$\begin{cases} U(x) - U(0) = -L_{0\rightarrow x} \\ U(x) - U(0) = -\int_0^x \overrightarrow{F_x} \cdot \overrightarrow{dx} = \int_0^x |F_x| \cdot dx = |F_x| \cdot x \end{cases} \qquad (1.7)$$

Admitting that the potential energy of the particle vanishes for $x = 0$, the initial potential energy of the particle $U(x_0)$ in the field of conservative force

$$F_x(x) = F_0 \qquad (1.8)$$

can be written

$$U(x_0) = |F_0| \cdot x_0 \qquad (1.9)$$

The initial kinetic energy $E(x_0)$ of the particle is – as given

$$E(x_0) = E_c \qquad (1.10)$$

and, consequently the total energy of the particle $W(x_0)$ is

$$W(x_0)=U(x_0)+E_c \qquad (1.11)$$

The draw up of the particle occurs when the total energy of the particle is entirely exhausted by the work of the friction force. The distance covered by the particle before it stops, D, obeys

$$\begin{cases} W(x_0)=D\cdot F_f \\ U(x_0)+E_c=D\cdot F_f \\ |F_x|\cdot x_0+E_c=D\cdot F_f \end{cases} \qquad (1.12)$$

so that ,

$$D=\frac{|F_x|\cdot x_0+E_c}{F_f} \qquad (1.13)^*$$

and

$$D=20m \qquad (1.14)^*$$

The relations (1.13) and (1.14) represent the answer to the question **I.a**.

b. The relation (1.7) written as

$$U(x)=|F_x|\cdot x \qquad (1.15)$$

gives the linear dependence of the potential energy to the position .
If the motion occurs without friction, the particle can reach a point A situated at the distance δ apart from the origin in which the kinetic energy vanishes. In the point A the energy of the particles is entirely potential.
The energy conservation law for the starting point and point A gives

$$\begin{cases} E_c+|F_x|\cdot x_0=|F_x|\cdot\delta \\ \delta=x_0+\dfrac{E_c}{|F_x|} \end{cases} \qquad (1.16)$$

The numerical value of the position of point A, furthest away from the origin, is
$\delta=2m$
if the motion occurs without friction.
The representation of the dependence of the potential energy on the position in the domain $(0,\delta)$ is represented in the figure 1.2.

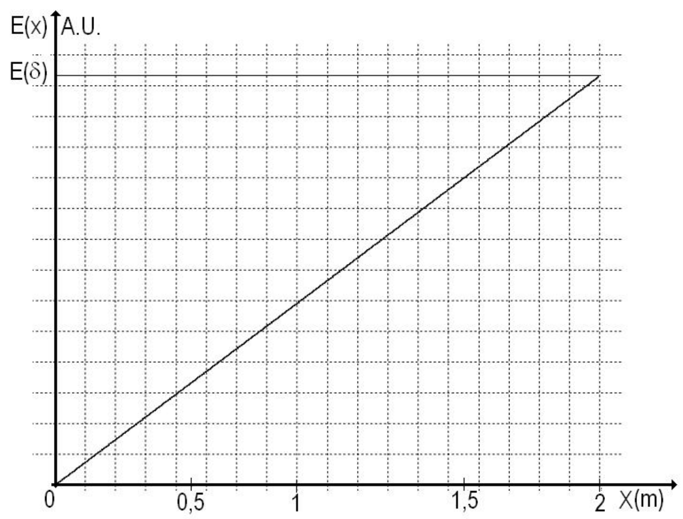


Figure 1.2

During the real motion of the particle (with friction) the extreme positions reached by the particle are smaller than δ (because of the leak of energy due to friction).
The graph in the figure 1.2 is the answer to the question **I.b**.

c. During the motion of the particle its energy decrease because of the dissipation work of the friction force. The speed of the particle has a local maximum near the wall. Denoting v_k the speed of the particle just before its' kth collision with the wall and v_{k+1} the speed just before its' next collision,

$v_k > v_{k+1}$

Among two successive collisions, the particle reaches its' x_k positions in which its' speed vanishes and the energy of the particle is purely potential. These positions are closer and closer to the wall because a part of the energy of the particle is dissipated through friction.

$$x_{k+1} < x_k \qquad (1.17)$$

<u>**Case 1**</u>

When the particle moves towards the wall, both its' speed and its' kinetic energy increases. The potential energy of the particle decreases. During the motion – independent of its' direction- energy is dissipated through the friction force.
The potential energy of the particle, $U(x)$, the kinetic energy $E(x)$ and the total energy of the particle during this part of the motion $W(x)$ obey the relation

$$W(x_0) - W(x) = F_f \cdot (x_0 - x) \qquad (1.18)$$

the position x lying in the domain

$$x \in (0, x_0) \qquad (1.19)$$

covered from x_0 towards origin. The relation (1.18) can be written as

$$\left[E_c + |F_x| \cdot x_0\right] - \left[\frac{m \cdot v^2}{2} + |F_x| \cdot x\right] = F_f \cdot (x_0 - x) \qquad (1.20)$$

so that

$$\begin{cases} v^2 = \dfrac{2}{m}\left[E_c + |F_x| \cdot x_0 - |F_x| \cdot x - F_f \cdot (x_0 - x)\right] \\ v^2 = \dfrac{2}{m}\left[E_c + x_0\left(|F_x| - F_f\right) - x\left(|F_x| - F_f\right)\right] \end{cases} \qquad (1.21)$$

and by consequence

$$v = -\sqrt{\frac{2}{m}\left[E_c + x_0\left(|F_x| - F_f\right) - x\left(|F_x| - F_f\right)\right]} \qquad (1.22)$$

The minus sign in front of the magnitude of the speed indicates that the motion of the particle occurs into the negative direction of the coordinate axis.
Using the problem data

$$\begin{cases} v^2 = \dfrac{2}{m}(19 - 9 \cdot x) \\ v = -\sqrt{\dfrac{2}{m}(19 - 9 \cdot x)} \end{cases} \qquad (1.23)$$

The speed of the particle at the first collision with the wall $v_{1\leftarrow}$ can be written as

$$v_{1\leftarrow} = -\sqrt{\frac{2}{m}\left[E_c + x_0\left(|F_x| - F_f\right)\right]} \qquad (1.24)$$

and has the value

$$v_{1\leftarrow} = -\sqrt{\frac{2}{m}19} \qquad (1.25)$$

The total energy near the wall, purely kinetic $E_{1\leftarrow}$, has the expression

$$E_{1\leftarrow} = E_c + x_0\left(|F_x| - F_f\right) \qquad (1.26)$$

The numerical value of this energy is

$$E_{1\leftarrow} = 19J \qquad (1.27)$$

The graph in the figure (1.3) gives the dependence on position of the square of the speed for the first part of the particle's motion.

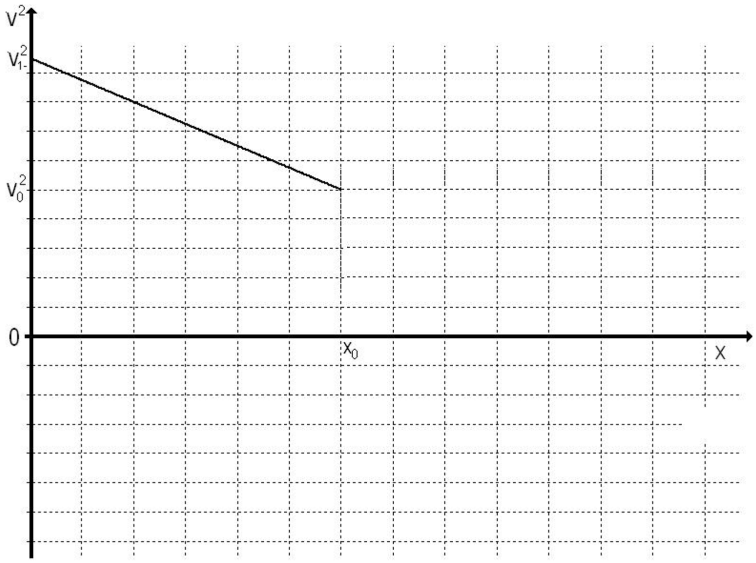


Figure 1.3

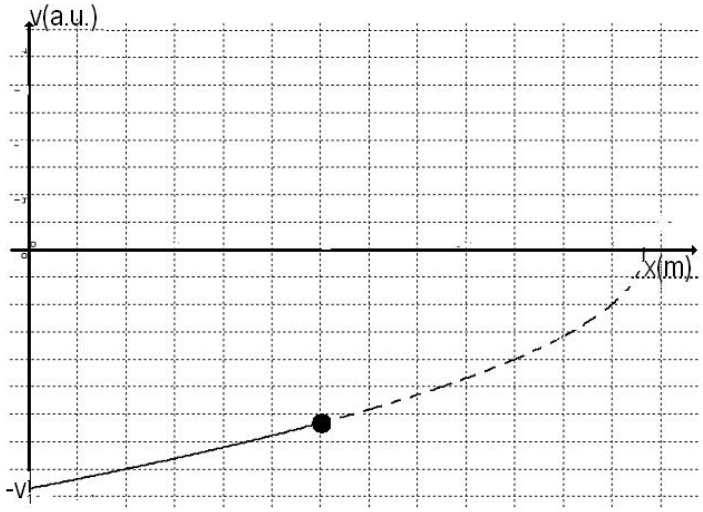


Figure 1.4

The graph in the figure (1.4) presents the speed's dependence on the position in this first part of the particle's motion (towards the wall).

After the collision with the wall, the speed of the particle, $v_{1\rightarrow}$, has the same magnitude as the speed just before the collision but it is directed in the opposite way. In the graphical representation of the speed as a function of position, the collision with the wall is represented as a jump of the speed from a point lying on negative side of the speed axis to a point lying on positive side of the speed axis. The absolute value of the speed just before and immediately after the collision is the same as represented in the figure 1.5.

$$v_{1\rightarrow} = \sqrt{\frac{2}{m}\left[E_c + x_0\left(\left|F_x\right| - F_f\right)\right]} \qquad (1.28)$$

After the first collision, the motion of the particle is slowed down with a constant deceleration $a_{\rightarrow}$ and an initial speed $v_{1\rightarrow}$.

This motion continues to the position x_1 where the speed vanishes.
From Galileo law it can be inferred that

$$\begin{cases} 0 = v_{1\rightarrow}^2 - 2 \cdot a_{\rightarrow} \cdot x_1 \\ x_1 = \dfrac{v_{1\rightarrow}^2}{2 \cdot a_{\rightarrow}} = \dfrac{\dfrac{2}{m}\left[E_c + x_0\left(\left|F_x\right| - F_f\right)\right]}{2 \cdot \dfrac{\left|F_x\right| + F_f}{m}} = \dfrac{\left[E_c + x_0\left(\left|F_x\right| - F_f\right)\right]}{\left|F_x\right| + F_f} \end{cases} \qquad (1.29)$$

The numerical value of the position x_1 is

$$x_1 = \frac{19}{11} m \qquad (1.30)$$

For the positions

$$x \in (0, x_1) \qquad (1.31)$$

covered from the origin towards x_1 the total energy $W(x)$ has the expression

$$W(x) = \frac{m \cdot v^2}{2} + \left|F_x\right| \cdot x \qquad (1.32)$$

From the wall, the energy of the particle diminishes because of the friction – that is

$$\begin{cases} E_{1\leftarrow} - W(x) = F_f \cdot x \\ E_c + x_0\left(\left|F_x\right| - F_f\right) - \dfrac{m \cdot v^2}{2} - \left|F_x\right| \cdot x = F_f \cdot x \end{cases} \qquad (1.33)$$

The square of the magnitude of the speed is

$$\begin{cases} v^2 = \dfrac{2}{m}\left[E_c + x_0\left(\left|F_x\right| - F_f\right) - \left(\left|F_x\right| + F_f\right) \cdot x\right] \\ v^2 = \dfrac{2}{m}\left(\left|F_x\right| + F_f\right) \cdot \left(x_1 - x\right) \end{cases} \qquad (1.34)$$

and the speed is

$$v = \sqrt{\frac{2}{m}\left[E_c + x_0\left(\left|F_x\right| - F_f\right) - \left(\left|F_x\right| + F_f\right) \cdot x\right]} \qquad (1.35)$$

Using the furnished data results

$$v^2 = \frac{2}{m}\left[19 - 11 \cdot x\right] \qquad (1.36)$$

and respectively

$$v = \sqrt{\frac{2}{m}\left[19 - 11 \cdot x\right]} \qquad (1.37)$$

For the positions lying in the domain $x \in (0, x_1)$ - (which correspond to a second part of the motion of particle) the figure 1.5 gives the dependence of the speed on the position.

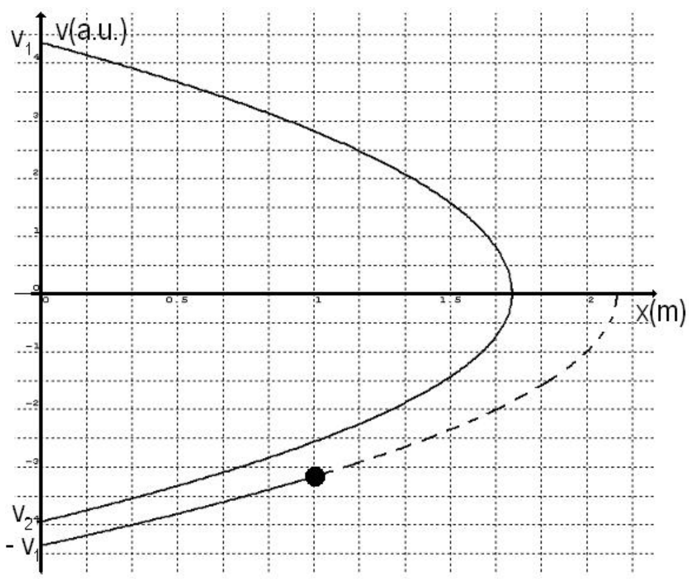


Figure 1.5

As can be observed in the figure, after reaching the furthest away position, x_1, the particle moves towards the origin, without an initial speed, in an accelerated motion having an acceleration with the magnitude of $a_{\leftarrow} = \left(|F_x| - F_f\right)/m$. After the collision with the wall, the particle has a velocity equal in magnitude but opposite in direction with the one it had just before the collision.

When the particle reaches a point in the domain $(0, x_1)$ moving from x_1 towards the origin its' total energy $W(x)$ has the expression (1.32).

Starting from x_1, because of the dissipation determined by the friction force, the energy changes to the value corresponding to the position with coordinate x.

$$\begin{cases} |F_x| \cdot x_1 - W(x) = F_f \cdot (x_1 - x) \\ |F_x| \cdot x_1 - \dfrac{m \cdot v^2}{2} - |F_x| \cdot x = F_f \cdot (x_1 - x) \end{cases} \tag{1.38}$$

The square of the speed has the expression

$$\begin{cases} v^2 = \dfrac{2}{m}\left[\left(|F_x| - F_f\right) \cdot (x_1 - x)\right] \\ v^2 = \dfrac{2}{m}\left[\dfrac{\left[E_c + x_0\left(|F_x| - F_f\right)\right]}{|F_x| + F_f} - x\right] \cdot \left(|F_x| - F_f\right) \end{cases} \tag{1.39}$$

and the speed is

$$v = \sqrt{\frac{2}{m}\left[\frac{\left[E_c + x_0\left(|F_x| - F_f\right)\right]}{|F_x| + F_f} - x\right] \cdot \left(|F_x| - F_f\right)} \tag{1.40}$$

Using the given data, for a position in the domain $(0, x_1)$

$$v^2 = \frac{2}{m}\left[\frac{19}{11} - x\right] \cdot 9 \tag{1.41}$$

respectively

$$v = -\sqrt{\frac{2}{m}\left[\frac{19}{11} - x\right]\cdot 9} \qquad (1.42)$$

The speed of the particle when it reaches for the second time the wall has - using (1.39) - the expression

$$v_{2\leftarrow} = -\sqrt{\frac{2}{m}\left\{\frac{\left[E_c + x_0\left(|F_x| - F_f\right)\right]}{|F_x| + F_f}\cdot\left(|F_x| - F_f\right)\right\}} \qquad (1.43)$$

The resulting numerical value is

$$v_{2\leftarrow} = -\sqrt{\frac{2}{m}\frac{171}{11}} \qquad (1.44)$$

Concluding, after the first collision and first recoil, the particle moves away from the wall, reaches again a position where the speed vanishes and then comes back to the wall. The speed of the particle hitting again the wall is smaller than before – as in the figure 1.5.

As it was denoted before v_k is the speed of the particle just before its' k^{th} run and x_k is the coordinate of the furthest away point reached during the k^{th} run.

The energy of the particle starting from the wall is

$$E_k = \frac{v_k^2 \cdot m}{2} = W_k(0) \qquad (1.45)$$

In the point x_k, the furthest away from the origin after k^{th} collision, the energy verifies the relation

$$U_k = x_k \cdot |F_x| = W_k(x_k) \qquad (1.46)$$

The variation of the energy between starting point and point x_k is

$$\frac{v_k^2 \cdot m}{2} - x_k \cdot |F_x| = F_f \cdot x_k \qquad (1.47)$$

so that

$$x_k = \frac{v_k^2 \cdot m}{2\cdot\left(|F_x| + F_f\right)} \qquad (1.48)$$

After the particle reaches point x_k the direction of the speed changes and, when the particle reaches again the wall

$$\frac{v_{k+1}^2 \cdot m}{2} = E_{k+1} = W_{k+1}(0) \qquad (1.49)$$

The energy conservation law for the x_k point and the state when the particle reaches again the wall gives

$$x_k \cdot |F_x| - \frac{v_{k+1}^2 \cdot m}{2} = F_f \cdot x_k \qquad (1.50)$$

so that

$$v_{k+1}^2 = \frac{2}{m} x_k \left(|F_x| - F_f\right) \qquad (1.51)$$

Considering (1.48), the relation (1.51) becomes

$$v_{k+1}^2 = v_k^2 \cdot \frac{|F_x| - F}{|F_x| + F} \qquad (1.52)$$

Between two consequent collisions the speed diminishes in a geometrical progression having the ratio q.This ratio has the expression

$$q = \sqrt{\frac{|F_x| - F}{|F_x| + F}} \qquad (1.53)$$

and the value

$$q = \sqrt{\frac{9}{11}} \qquad (1.54)$$

For the $k+1$ collision the relation (1.48) becomes

$$x_{k+1} = \frac{v_{k+1}^2 \cdot m}{2 \cdot \left(|F_x| + F_f\right)} \qquad (1.55)$$

Taking into account (1.52), the ratio of the successive extreme positions can be written as

$$\begin{cases} \dfrac{x_{k+1}}{x_k} = \dfrac{|F_x| - F_f}{|F_x| + F_f} = q^2 \\ x_{k+1} = q^2 \cdot x_k \end{cases} \qquad (1.56)$$

From the k run towards origin, (analogous to (1.39)), the dependence of the square of the speed on position can be written as $v_{(k,\leftarrow)}^2$

$$\begin{cases} v_{(k,\leftarrow)}^2 = \dfrac{2}{m}\left[\left(|F_x| - F_f\right) \cdot \left(x_k - x\right)\right] \\ v_{(k,\leftarrow)}^2 = \dfrac{2}{m}\left[\left(|F_x| - F_f\right) \cdot \left(x_1 \cdot q^{2k} - x\right)\right] \end{cases} \qquad (1.57)$$

or, using the data

$$v_{(k,\leftarrow)}^2 = \frac{2}{m}\left[9 \cdot \left(\frac{19}{11} \cdot \left(\frac{9}{11}\right)^k - x\right)\right] \qquad (1.58)$$

For the k th run from the origin (analogous with (1.34)), the dependence on the position of the square of the magnitude of the speed $v^2_{(k,\rightarrow)}$ can be written as

$$\begin{cases} v^2_{(k,\rightarrow)} = \dfrac{2}{m}\left[\left(|F_x| + F_f\right)\cdot\left(x_k - x\right)\right] \\ v^2_{(k,\rightarrow)} = \dfrac{2}{m}\left[\left(|F_x| + F_f\right)\cdot\left(x_1 \cdot q^{2k} - x\right)\right] \end{cases} \qquad (1.59)$$

Using given data

$$v^2_{(k,\rightarrow)} = \frac{2}{m}\left[11\cdot\left(\frac{19}{11}\cdot\left(\frac{9}{11}\right)^k - x\right)\right] \qquad (1.60)$$

The evolution of the square of the speed as function of position is represented in the figure 1.6.

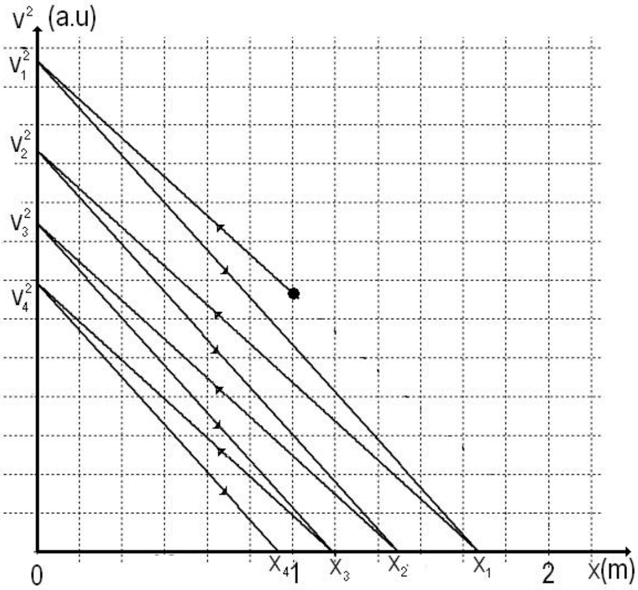


Figure 1.6

And the evolution of the speed as function of position is represented in the figure 1.7.

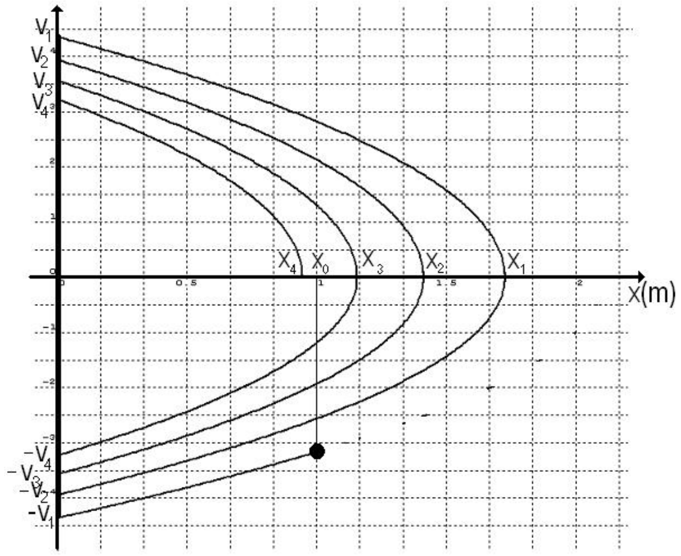


Figure 1.7

The sum of the progression given in (1.56) gives half of the distance covered by the particle after the first collision.

$$\sum_{k=1}^{\infty} x_k = x_1 \frac{1}{1-q^2} \qquad (1.61)$$

Considering (1.53) and (1.29)

$$\sum_{k=1}^{\infty} x_k = \frac{E_c + x_0 \cdot \left(\left|F_x\right| - F_f\right)}{2 \cdot F_f} \qquad (1.62)$$

Numerically,

$$\sum_{k=1}^{\infty} x_k = \frac{19}{2} m \qquad (1.63)$$

The total covered distance is

$$\begin{cases} D = 2 \cdot \sum_{k=1}^{\infty} x_k + x_0 \\ D = 20m \end{cases} \qquad (1.64)$$

which is the same with (1.14).

Case 2

If the particle starts from the x_0 position moving in the positive direction of the coordinate axis *Ox* its' speed diminishes and its' kinetic energy also diminishes while its' potential energy increases to a maximum in the x_1' position where the speed vanishes. During this motion the energy is dissipated due to the friction. The total energy $W(x)$, for the positions x between x_0 and x_1' verify the relation

$$W(x_0) - W(x) = F_f \cdot (x - x_0) \qquad (1.65)$$

the position x lying in the domain

$$x \in \left(x_0, x_1'\right) \qquad (1.66)$$

when the particle moves from x_0 in the positive direction of the axis. The relation (1.65) becomes

$$\left[E_c + |F_x| \cdot x_0\right] - \left[\frac{m \cdot v^2}{2} + |F_x| \cdot x\right] = F_f \cdot \left(x - x_0\right) \qquad (1.67)$$

so that

$$\begin{cases} v^2 = \dfrac{2}{m}\left[E_c + |F_x| \cdot x_0 - |F_x| \cdot x - F_f \cdot \left(x - x_0\right)\right] \\ v^2 = \dfrac{2}{m}\left[E_c + x_0\left(|F_x| + F_f\right) - x\left(|F_x| + F_f\right)\right] \end{cases} \qquad (1.68)$$

and

$$v = \sqrt{\frac{2}{m}\left[E_c + x_0\left(|F_x| + F_f\right) - x\left(|F_x| + F_f\right)\right]} \qquad (1.69)$$

Using provided data

$$\begin{cases} v^2 = \dfrac{2}{m}\left(21 - 11 \cdot x\right) \\ v = \sqrt{\dfrac{2}{m}\left(21 - 11 \cdot x\right)} \end{cases} \qquad (1.70)$$

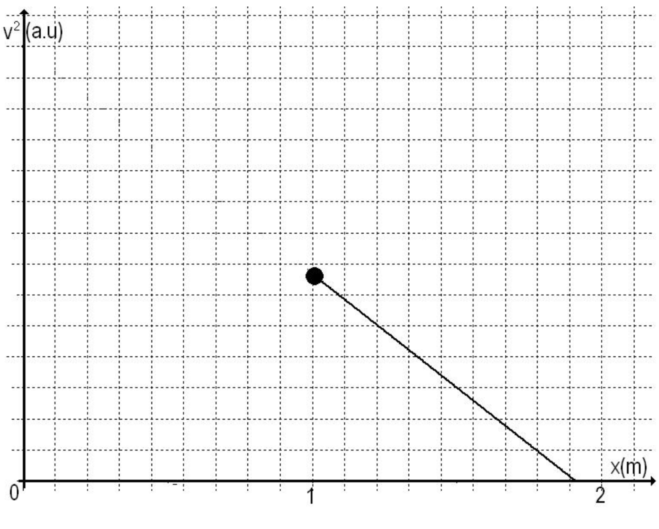


Figure 1.8

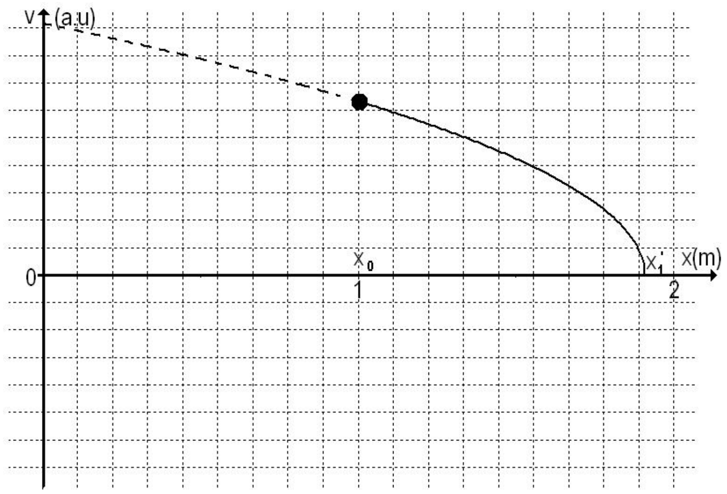


Figure 1.9

The graph in the figure (1.8) presents the dependence of the square speed on the position for the motion in the domain $x \in (x_0, x_1')$. The particle moves in the positive direction of the coordinate axis *Ox.*

This motion occurs until the position x_1' - when the speed vanishes - is reached. From the relation (1.68), in which we take the modulus of the speed zero, results

$$x_1' = x_0 + \frac{E_c}{|F_x| + F_f} \qquad (1.71)$$

the numerical value for x_1' is

$$x_1' = \frac{21}{11} m \qquad (1.72)$$

After furthest away position x_1' is reached, the particle moves again towards the origin, without initial speed, in a speeded up motion having an acceleration of magnitude $a_{\leftarrow} = (|F_x| - F_f)/m$. After the collision with the wall, the particle has a velocity $v_{1\rightarrow}'$ equal in magnitude but opposite direction with the one it had before the collision $v_{1\leftarrow}'$.

When the particle is at a point lying in the domain $(0, x_1')$ running from x_1' to the origin, its' total energy $W(x)$ has the expression

$$W(x) = \frac{m \cdot v^2}{2} + |F_x| \cdot x \qquad (1.73)$$

Because of friction, the value of the energy decreases from the one it had at x_1' to the corresponding to the x position

$$\begin{cases} |F_x| \cdot x_1' - W(x) = F_f \cdot (x_1' - x) \\ |F_x| \cdot x_1' - \frac{m \cdot v^2}{2} - |F_x| \cdot x = F_f \cdot (x_1' - x) \end{cases} \qquad (1.74)$$

The square of the speed has the expression

$$v^2 = \frac{2}{m}\left[\left(\left|F_x\right| - F_f\right)\cdot\left(x_1' - x\right)\right] \qquad (1.75)$$

and the speed is

$$v = -\sqrt{\frac{2}{m}\left[\left(\left|F_x\right| - F_f\right)\cdot\left(x_1' - x\right)\right]} \qquad (1.76)$$

For the given data, in the domain, $(0, x_1')$

$$v^2 = \frac{2}{m}\left[\frac{21}{11} - x\right]\cdot 9 \qquad (1.77)$$

respectively

$$v = -\sqrt{\frac{2}{m}\left[\frac{21}{11} - x\right]\cdot 9} \qquad (1.78)$$

The speed of the particle hitting a second time the wall is – according to (1.78)-

$$v_{1\leftarrow}' = -\sqrt{\frac{2}{m}\left[\left(\left|F_x\right| - F_f\right)\cdot x_1'\right]} \qquad (1.79)$$

and has the value

$$v_{1\leftarrow}' = -\sqrt{\frac{2}{m}\frac{189}{11}} \qquad (1.80)$$

Concluding, after the first collision and first recoil, the particle moves away from the wall, reaches again a position where the speed vanishes and then comes back to the wall. The speed of the particle hitting again the wall is smaller than before – as in the figure 1.11.

Denoting v_k' the speed at the beginning of the k^{th} run and x_k' the coordinate of the furthest away point during the k^{th} run, the energy of the particle leaving the wall is

$$E_k' = \frac{v_k'^2 \cdot m}{2} = W_k'(0) \qquad (1.81)$$

In the position x_k' after the k departure from the wall, the energy is

$$U_k' = x_k' \cdot \left|F_x\right| = W_k'(x_k') \qquad (1.82)$$

The variation of the total energy has the expression

$$\frac{v_k'^2 \cdot m}{2} - x_k' \cdot \left|F_x\right| = F_f \cdot x_k' \qquad (1.83)$$

so that

$$x_k' = \frac{v_k'^2 \cdot m}{2 \cdot \left(|F_x| + F_f \right)} \qquad (1.84)$$

After the particle reaches the position x_k' the direction of the speed changes and, when the particle hits the wall,

$$\frac{v_{k+1}'^2 \cdot m}{2} = E_{k+1}' = W_{k+1}'(0) \qquad (1.85)$$

The energy conservation law for the x_k' position and the point in which the particle hits the wall gives

$$x_k' \cdot |F_x| - \frac{v_{k+1}'^2 \cdot m}{2} = F_f \cdot x_k' \qquad (1.86)$$

so that

$$v_{k+1}'^2 = \frac{2}{m} x_k' \left(|F_x| - F_f \right) \qquad (1.87)$$

Considering (1.84), the relation (1.87) becomes

$$v_{k+1}'^2 = v_k'^2 \cdot \frac{|F_x| - F}{|F_x| + F} \qquad (1.88)$$

Between two successive collisions the speed diminishes in a geometrical progression with the ratio q

$$q = \sqrt{\frac{|F_x| - F}{|F_x| + F}} \qquad (1.89)$$

Using the data provided

$$q = \sqrt{\frac{9}{11}} \qquad (1.90)$$

From ($k+1$)th, collision the relation (1.84) is written as

$$x_{k+1}' = \frac{v_{k+1}'^2 \cdot m}{2 \cdot \left(|F_x| + F_f \right)} \qquad (1.91)$$

Considering (1.84) and (1.91), the ratio of the extreme positions in two successive runs is

$$\begin{cases} \dfrac{x_{k+1}'}{x_k'} = \dfrac{|F_x| - F_f}{|F_x| + F_f} = q^2 \\ x_{k+1}' = q^2 \cdot x_k' \end{cases} \qquad (1.92)$$

For the k th run towards the origin, analogous to (1.57), one may write the dependence of the square speed $v_{(k,\leftarrow)}'^2$ as function of the position as

$$\begin{cases} v'^2_{(k,\leftarrow)} = \dfrac{2}{m}\left[\left(|F_x| - F_f\right)\cdot\left(x_k' - x\right)\right] \\ v'^2_{(k,\leftarrow)} = \dfrac{2}{m}\left[\left(|F_x| - F_f\right)\cdot\left(x_1'\cdot q^{2k} - x\right)\right] \end{cases} \qquad (1.93)$$

Or, using the data

$$v'^2_{(k,\leftarrow)} = \frac{2}{m}\left[9\cdot\left(\frac{21}{11}\cdot\left(\frac{9}{11}\right)^k - x\right)\right] \qquad (1.94)$$

From the k th run from the origin, analogous to (1.59), the dependence on the position of the square speed $v^2_{(k,\rightarrow)}$ can be written as

$$\begin{cases} v'^2_{(k,\rightarrow)} = \dfrac{2}{m}\left[\left(|F_x| + F_f\right)\cdot\left(x_k' - x\right)\right] \\ v'^2_{(k,\rightarrow)} = \dfrac{2}{m}\left[\left(|F_x| + F_f\right)\cdot\left(x_1'\cdot q^{2k} - x\right)\right] \end{cases} \qquad (1.95)$$

Using given data

$$v'^2_{(k,\rightarrow)} = \frac{2}{m}\left[11\cdot\left(\frac{21}{11}\cdot\left(\frac{9}{11}\right)^k - x\right)\right] \qquad (1.96)$$

The evolution of the square of the speed as function on position is presented in the figure 1.10.

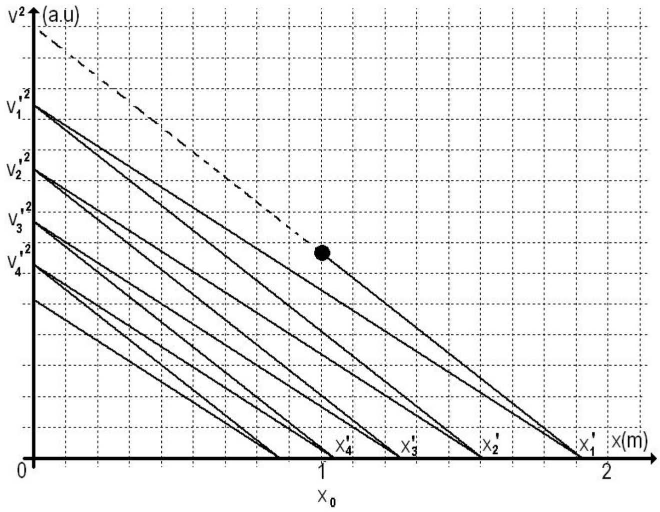


Figure 1.10

And the evolution of the speed as function of the position is presented in the figure 1.11.

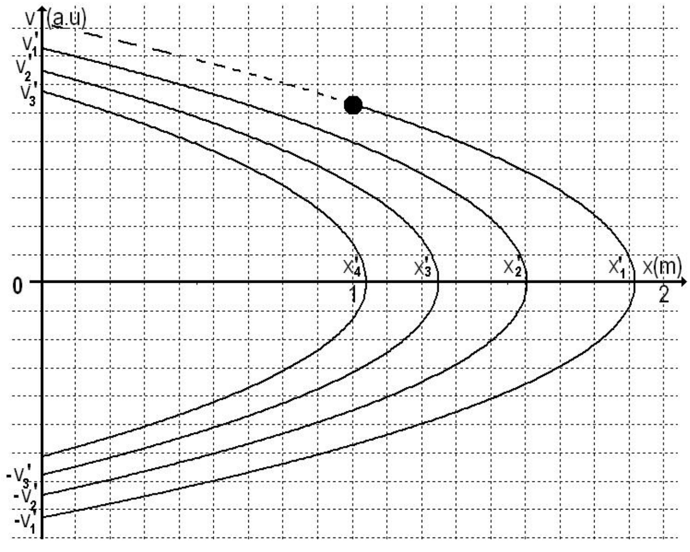


Figure 1.11

The sum of the geometrical progression (1.92) gives (after the doubling and then subtracting of the x_0) the total distance covered by the particle.

$$\sum_{k=1}^{\infty} x_k' = x_1' \frac{1}{1-q^2} \qquad (1.97)$$

Considering (1.97), (1.71) and (1.72) it results

$$\sum_{k=1}^{\infty} x_k' = \frac{21}{2} m \qquad (1.98)$$

The total distance covered by the particle is

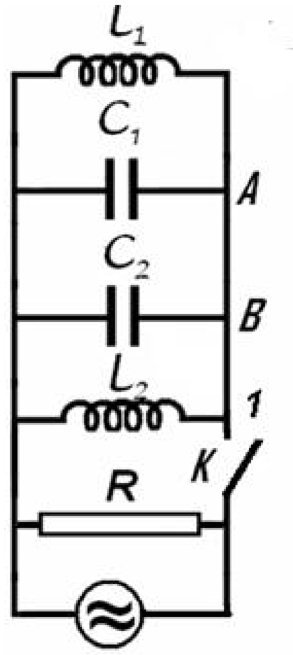
$$\begin{cases} D = 2 \cdot \sum_{k=1}^{\infty} x_k' - x_0 \\ D = 20m \end{cases} \qquad (1.99)$$

which allows us to find again the result (1.14).

Professor Delia DAVIDESCU, National Department of Evaluation and Examination–Ministry of Education and Research-Bucharest, Romania
Professor Adrian S.DAFINEI,PhD, Faculty of Physics – University of Bucharest, Romania

Electricity – Problem II (8 points)

Different kind of oscillation



Let's consider the electric circuit in the figure, for which $L_1 = 10\ mH$, $L_2 = 20\ mH$, $C_1 = 10\ nF$, $C_2 = 5\ nF$ and $R = 100k\Omega$. The switch K being closed the circuit is coupled with a source of alternating current. The current furnished by the source has constant intensity while the frequency of the current may be varied.

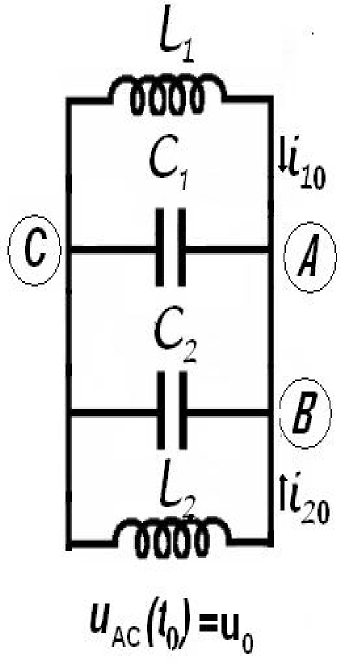
a. Find the ratio of frequency f_m for which the active power in circuit has the maximum value P_m and the frequency difference $\Delta f = f_+ - f_-$ of the frequencies f_+ and f_- for which the active power in the circuit is half of the maximum power P_m .

The switch K is now open. In the moment t_0 immediately after the switch is open the intensities of the currents in the coils L_1 and $i_{01} = 0{,}1\ A$ and $i_{02} = 0{,}2\ A$ L_1 (the currents flow as in the figure); at the same moment, the potential difference on the capacitor with capacity C_1 is $u_0 = 40V$:

b. Calculate the frequency of electromagnetic oscillation in $L_1C_1C_2L_2$ circuit;

c. Determine the intensity of the electric current in the AB conductor;

d. Calculate the amplitude of the oscillation of the intensity of electric current in the coil L_1.



Neglect the mutual induction of the coils, and the electric resistance of the conductors. Neglect the fast transition phenomena occurring when the switch is closed or opened.

Problem II - Solution

a. As is very well known in the study of AC circuits using the formalism of complex numbers, a complex inductive reactance $\overline{X_L} = L \cdot \omega \cdot j$, ($j = \sqrt{-1}$) is attached to the inductance L - part of a circuit supplied with an alternative current having the pulsation ω .

Similar, a complex capacitive reactance $\overline{X_C} = -\frac{j}{C \cdot \omega}$ is attached to the capacity C .

A parallel circuit will be characterized by his complex admittance $\overline{Y}$.

The admittance of the AC circuit represented in the figure is

$$\begin{cases} \overline{Y} = \dfrac{1}{R} + \dfrac{1}{L_1 \cdot \omega \cdot j} + \dfrac{1}{L_2 \cdot \omega \cdot j} - \dfrac{C_1 \cdot \omega}{j} - \dfrac{C_2 \cdot \omega}{j} \\ \overline{Y} = \dfrac{1}{R} + j\left[(C_1 + C_2) - \left(\dfrac{1}{L_1} + \dfrac{1}{L_2} \right) \right] \end{cases} \qquad (2.1)$$

The circuit behave as if has a parallel equivalent capacity C

$$C = C_1 + C_2 \qquad (2.2)$$

and a parallel equivalent inductance L

$$\begin{cases} \dfrac{1}{L} = \dfrac{1}{L_1} + \dfrac{1}{L_2} \\ L = \dfrac{L_1 L_2}{L_1 + L_2} \end{cases} \qquad (2.3)$$

The complex admittance of the circuit may be written as

$$\overline{Y} = \frac{1}{R} + j\left(C \cdot \omega - \frac{1}{L \cdot \omega} \right) \qquad (2.4)$$

and the complex impedance of the circuit will be

$$\begin{cases} \overline{Z} = \dfrac{1}{\overline{Y}} \\ \overline{Z} = \dfrac{\dfrac{1}{R} + j\left(\dfrac{1}{L \cdot \omega} - C \cdot \omega \right)}{\sqrt{\left(\dfrac{1}{R} \right)^2 + \left(C \cdot \omega - \dfrac{1}{L \cdot \omega} \right)^2}} \end{cases} \qquad (2.5)$$

The impedance Z of the circuit, the inverse of the admittance of the circuit Y is the modulus of the complex impedance $\overline{Z}$

$$Z=\left|\overline{Z}\right|=\frac{1}{\sqrt{\left(\frac{1}{R}\right)^{2}+\left(C\cdot\omega-\frac{1}{L\cdot\omega}\right)^{2}}}=\frac{1}{Y} \qquad (2.6)$$

The constant current source supplying the circuit furnish a current having a momentary value $i(t)$

$$i(t)=I\cdot\sqrt{2}\cdot\sin(\omega\cdot t), \qquad (2.7)$$

where I is the effective intensity (constant), of the current and ω is the current pulsation (that can vary) . The potential difference at the jacks of the circuit has the momentary value $u(t)$

$$u(t)=U\cdot\sqrt{2}\cdot\sin(\omega\cdot t+\varphi) \qquad (2.8)$$

where U is the effective value of the tension and φ is the phase difference between tension and current.
The effective values of the current and tension obey the relation

$$U=I\cdot Z \qquad (2.9)$$

The active power in the circuit is

$$P=\frac{U^2}{R}=\frac{Z^2\cdot I^2}{R} \qquad (2.10)$$

Because as in the enounce,

$$\begin{cases} I=\text{constant} \\ R=\text{constant} \end{cases} \qquad (2.11)$$

the maximal active power is realized for the maximum value of the impedance that is the minimal value of the admittance .
The admittance

$$Y=\sqrt{\left(\frac{1}{R}\right)^{2}+\left(C\cdot\omega-\frac{1}{L\cdot\omega}\right)^{2}} \qquad (2.12)$$

has– as function of the pulsation ω - an „the smallest value”

$$Y_{\min}=\frac{1}{R} \qquad (2.13)$$

for the pulsation

$$\omega_m=\frac{1}{\sqrt{L\cdot C}} \qquad (2.14)$$

In this case

$$\left(C\cdot\omega-\frac{1}{L\cdot\omega}\right)=0. \qquad (2.15)$$

So, the minimal active power in the circuit has the value

$$P_m = R \cdot I^2 \qquad (2.16)$$

and occurs in the situation of alternative current furnished by the source at the frequency f_m

$$f_m = \frac{1}{2\pi}\omega_m = \frac{1}{2\pi \cdot \sqrt{C \cdot L}} \qquad (2.17)$$

To ensure that the active power is half of the maximum power it is necessary that

$$\begin{cases} P = \dfrac{1}{2}P_m \\ \dfrac{Z^2 \cdot I^2}{R} = \dfrac{1}{2}R \cdot I^2 \\ \dfrac{2}{R^2} = \dfrac{1}{Z^2} = Y^2 \end{cases} \qquad (2.18)$$

That is

$$\begin{cases} \dfrac{2}{R^2} = \dfrac{1}{R^2} + \left(C \cdot \omega - \dfrac{1}{L \cdot \omega}\right)^2 \\ \pm\dfrac{1}{R} = C \cdot \omega - \dfrac{1}{L \cdot \omega} \end{cases} \qquad (2.19)$$

The pulsation of the current ensuring an active power at half of the maximum power must satisfy one of the equations

$$\omega^2 \pm \frac{1}{R \cdot C}\omega - \frac{1}{L \cdot C} = 0 \qquad (2.20)$$

The two second degree equation may furnish the four solutions

$$\omega = \pm\frac{1}{2R \cdot C} \pm \frac{1}{2}\sqrt{\left(\frac{1}{R \cdot C}\right)^2 + \frac{4}{L \cdot C}} \qquad (2.21)$$

Because the pulsation is every time positive, and because

$$\sqrt{\left(\frac{1}{R \cdot C}\right)^2 + \frac{4}{L \cdot C}} > \frac{1}{R \cdot C} \qquad (2.22)$$

the only two valid solutions are

$$\omega_{\pm} = \frac{1}{2}\sqrt{\left(\frac{1}{R \cdot C}\right)^2 + \frac{4}{L \cdot C}} \pm \frac{1}{2R \cdot C} \qquad (2.23)$$

It exist two frequencies $f_{\pm} = \frac{1}{2\pi}\omega_{\pm}$ allowing to obtain in the circuit an active power representing half of the maximum power.

$$\begin{cases} f_{+} = \dfrac{1}{2\pi}\left(\dfrac{1}{2}\sqrt{\left(\dfrac{1}{R\cdot C}\right)^2 + \dfrac{4}{L\cdot C}} + \dfrac{1}{2R\cdot C}\right) \\ f_{-} = \dfrac{1}{2\pi}\left(\dfrac{1}{2}\sqrt{\left(\dfrac{1}{R\cdot C}\right)^2 + \dfrac{4}{L\cdot C}} - \dfrac{1}{2R\cdot C}\right) \end{cases} \qquad (2.24)$$

The difference of these frequencies is

$$\Delta f = f_{+} - f_{-} = \frac{1}{2\pi}\frac{1}{R\cdot C} \qquad (2.25)$$

the bandwidth of the circuit – the frequency interval around the resonance frequency having at the ends a signal representing $1/\sqrt{2}$ from the resonance signal. At the ends of the bandwidth the active power reduces at the half of his value at the resonance.
The asked ratio is

$$\begin{cases} \dfrac{f_m}{\Delta f} = \dfrac{R\cdot C}{\sqrt{L\cdot C}} = R\sqrt{\dfrac{C}{L}} \\ \dfrac{f_m}{\Delta f} = R\sqrt{\dfrac{(C_1+C_2)\cdot(L_1+L_2)}{L_1\cdot L_2}} \end{cases} \qquad (2.26)^*$$

Because

$$\begin{cases} C = 15\ nF \\ L = \dfrac{20}{3}\ mH \end{cases}$$

it results that

$\omega_m = 10^5\ rad\, s^{-1}$

and

$$\frac{f_m}{\Delta f} = R\sqrt{\frac{C}{L}} = 100\times 10^3 \cdot \sqrt{\frac{3\cdot 15\times 10^{-9}}{20\times 10^{-3}}} = 150 \qquad (2.27)$$

The (2.26) relation is the answer at the question **a**.

b. The fact that immediately after the source is detached it is a current in the coils, allow as to admit that currents dependents on time will continue to flow through the coils.

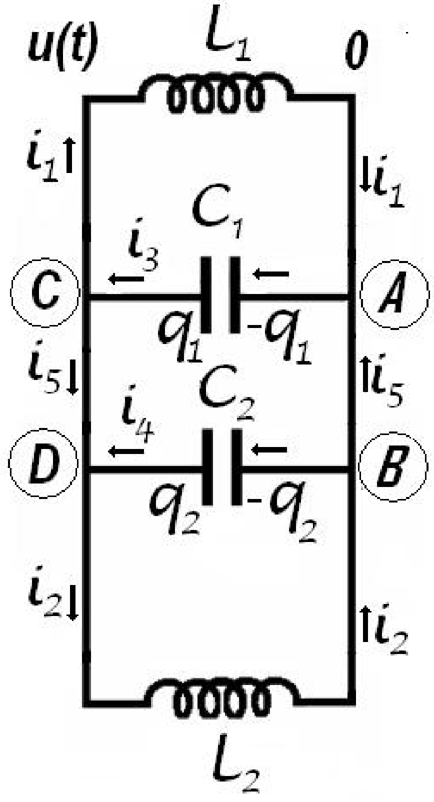


Figure 2.1

The capacitors will be charged with charges variable in time. The variation of the charges of the capacitors will results in currents flowing through the conductors linking the capacitors in the circuit.
The momentary tension on the jacks of the coils and capacitors – identical for all elements in circuit – is also dependent on time. Let's admit that the electrical potential of the points *C* and *D* is $u(t)$ and the potential of the points *A* and *B* is zero. If through the inductance L_1 passes the variable current having the momentary value $i_1(t)$, the relation between the current and potentials is

$$u(t) - L_1 \frac{di_1}{dt} = 0 \qquad (2.28)$$

The current passing through the second inductance $i_2(t)$ has the expression,

$$u(t) - L_2 \frac{di_2}{dt} = 0 \qquad (2.29)$$

If on the positive plate of the capacitor having the capacity C_1 is stocked the charge $q_1(t)$, then at the jacks of the capacitor the electrical tension is $u(t)$ and

$$q_1 = C_1 \cdot u \qquad (2.30)$$

Deriving this relation it results

$$\frac{dq_1}{dt} = C_1 \cdot \frac{du}{dt} \qquad (2.31)$$

But

$$\frac{dq_1}{dt} = -i_3 \qquad (2.32)$$

because the electrical current appears because of the diminishing of the electrical charge on capacitor plate. Consequently

$$i_3 = -C_1 \cdot \frac{du}{dt} \qquad (2.33)$$

Analogous, for the other capacitor,

$$i_4 = -C_4 \cdot \frac{du}{dt} \qquad (2.34)$$

Considering all obtained results

$$\begin{cases} \dfrac{di_1}{dt} = \dfrac{u}{L_1} \\ \dfrac{di_2}{dt} = \dfrac{u}{L_2} \end{cases} \qquad (2.35)$$

respectively

$$\begin{cases} \dfrac{di_3}{dt} = -C_1 \dfrac{d^2u}{dt^2} \\ \dfrac{di_4}{dt} = C_2 \dfrac{d^2u}{dt^2} \end{cases} \qquad (2.36)$$

Denoting $i_5(t)$ the momentary intensity of the current flowing from point B to the point A , then the same momentary intensity has the current through the points C and D. For the point A the Kirchhoff rule of the currents gives

$$i_1 + i_5 = i_3 \qquad (2.37)$$

For B point the same rule produces

$$i_4 + i_5 = i_2 \qquad (2.38)$$

Considering (2.37) and (2.38) results

$$i_1 - i_3 = i_4 - i_2 \qquad (2.39)$$

and deriving

$$\frac{di_1}{dt} - \frac{di_3}{dt} = \frac{di_4}{dt} - \frac{di_2}{dt} \qquad (2.40)$$

that is

$$\begin{cases} -\frac{u}{L_1}-\frac{u}{L_2}=C_1\frac{d^2u}{dt^2}+C_2\frac{d^2u}{dt^2} \\ -u\cdot\left(\frac{1}{L_1}+\frac{1}{L_2}\right)=\frac{d^2u}{dt^2}\cdot(C_1+C_2) \end{cases} \qquad (2.41)$$

Using the symbols defined above

$$\begin{cases} -\frac{u}{L}=\frac{d^2u}{dt^2}\cdot C \\ \ddot{u}+\frac{1}{LC}u=0 \end{cases} \qquad (2.42)$$

Because the tension obeys the relation above, it must have a harmonic dependence on time

$$u(t)=A\cdot\sin(\omega\cdot t+\delta) \qquad (2.43)$$

The pulsation of the tension is

$$\omega=\frac{1}{\sqrt{L\cdot C}} \qquad (2.44)$$

Taking into account the relations (2.43) and (2.36) it results that

$$\begin{cases} i_3=-C_1\frac{d}{dt}(A\cdot\sin(\omega\cdot t+\delta))=-C_1\cdot A\cdot\omega\cdot\cos(\omega\cdot t+\delta) \\ i_4=-C_2\frac{d}{dt}(A\cdot\sin(\omega\cdot t+\delta))=-C_2\cdot A\cdot\omega\cdot\cos(\omega\cdot t+\delta) \end{cases} \qquad (2.45)$$

and

$$\begin{cases} \frac{di_1}{dt}=\frac{u}{L_1}=\frac{1}{L_1}\cdot A\cdot\sin(\omega\cdot t+\delta) \\ \frac{di_2}{dt}=\frac{u}{L_2}=\frac{1}{L_2}\cdot A\cdot\sin(\omega\cdot t+\delta) \end{cases} \qquad (2.46)$$

It results that

$$\begin{cases} i_1=\frac{1}{L_1\cdot\omega}\cdot A\cdot\cos(\omega\cdot t+\delta)+M \\ i_2=\frac{1}{L_2\cdot\omega}\cdot A\cdot\cos(\omega\cdot t+\delta)+N \end{cases} \qquad (2.47)$$

In the expression above, A , M , N and δ are constants that must be determined using initially conditions. It is remarkable that the currents through capacitors are sinusoidal but the currents through the coils are the sum of sinusoidal and constant currents.

In the first moment

$$\begin{cases} u(0) = u_0 = 40V \\ i_1(0) = i_{01} = 0{,}1A \\ i_2(0) = i_{02} = 0{,}2A \end{cases} \qquad (2.48)$$

Because the values of the inductances and capacities are

$$\begin{cases} L_1 = 0{,}01H \\ L_2 = 0{,}02H \\ C_1 = 10nF \\ C_2 = 5nF \end{cases} \qquad (2.49)$$

the equivalent inductance and capacity is

$$\begin{cases} \dfrac{1}{L} = \dfrac{1}{L_1} + \dfrac{1}{L_2} \\ L = \dfrac{L_1 \cdot L_2}{L_1 + L_2} \\ L = \dfrac{2 \times 10^{-4}}{3 \times 10^{-2}} H = \dfrac{1}{150} H \end{cases} \qquad (2.50)$$

respectively

$$\begin{cases} C = C_1 + C_2 \\ C = 15nF \end{cases} \quad . \qquad (2.51)$$

From (2.44) results

$$\omega = \frac{1}{\sqrt{\frac{1}{150} \cdot 15 \times 10^{-9}}} = 10^5 \, rad \, s^{-1} \qquad (2.52)^*$$

The value of the pulsation allows calculating the value of the requested frequency **b**. This frequency has the value f

$$f = \frac{\omega}{2\pi} = \frac{10^5}{2\pi} Hz \qquad (2.53)^*$$

c. If the momentary tension on circuit is like in (2.43), one may write

$$\begin{cases} u(0) = A \cdot \sin(\delta) = u_0 \\ \sin(\delta) = \dfrac{u_0}{A} \end{cases} \qquad (2.54)$$

From the currents (2.47) is possible to write

$$\begin{cases} i_{01} = \dfrac{1}{L_1 \cdot \omega} \cdot A \cdot \cos(\delta) + M \\ i_{02} = \dfrac{1}{L_2 \cdot \omega} \cdot A \cdot \cos(\delta) + N \end{cases} \qquad (2.55)$$

On the other side is possible to express (2.39) as

$$\begin{cases} i_1 - i_3 = i_4 - i_2 \\ \dfrac{1}{L_1 \cdot \omega} \cdot A \cdot \cos(\omega \cdot t + \delta) + M + C_1 \cdot A \cdot \omega \cdot \cos(\omega \cdot t + \delta) = \\ -C_2 \cdot A \cdot \omega \cdot \cos(\omega \cdot t + \delta) - \dfrac{1}{L_2 \cdot \omega} \cdot A \cdot \cos(\omega \cdot t + \delta) - N \end{cases} \qquad (2.56)$$

An identity as

$$A \cdot \cos\alpha + B \equiv C \cdot \cos\alpha + D \qquad (2.57)$$

is valuable for any value of the argument α only if

$$\begin{cases} A = C \\ B = D \end{cases} \qquad (2.58)$$

Considering (2.58), from (2.56) it results

$$\begin{cases} M + N = 0 \\ A \cdot \omega \cdot (C_1 + C_2) = -\dfrac{A}{\omega} \cdot \left(\dfrac{1}{L_1} + \dfrac{1}{L_2} \right) \end{cases} \qquad (2.59)$$

For the last equation it results that the circuit oscillate with the pulsation in the relation (2.44)
Adding relations (2.55) and considering (2.54) and (2.59) results that

$$\begin{cases} i_{01} + i_{02} = A \cdot \cos(\delta) \cdot \dfrac{1}{\omega} \cdot \left(\dfrac{1}{L_1} + \dfrac{1}{L_1} \right) \\ A = \dfrac{i_{01} + i_{02}}{\cos(\delta) \cdot \dfrac{1}{\omega} \cdot \left(\dfrac{1}{L_1} + \dfrac{1}{L_1} \right)} \\ \cos\delta = \dfrac{i_{01} + i_{02}}{A \cdot \dfrac{1}{\omega} \cdot \left(\dfrac{1}{L_1} + \dfrac{1}{L_1} \right)} \\ \cos\delta = \dfrac{(i_{01} + i_{02}) \cdot L \cdot \omega}{A} \end{cases} \qquad (2.60)$$

The numerical value of the amplitude of the electrical tension results by summing the last relations from (2.54) and (2.60)

$$\begin{cases} \sin(\delta) = \dfrac{u_0}{A} \\ \cos\delta = \dfrac{(i_{01} + i_{02}) \cdot L \cdot \omega}{A} \\ (\cos(\delta))^2 + (\sin(\delta))^2 = 1 \\ \left(\dfrac{u_0}{A}\right)^2 + \left(\dfrac{(i_{01} + i_{02}) \cdot L \cdot \omega}{A}\right)^2 = 1 \\ A = \sqrt{(u_0)^2 + ((i_{01} + i_{02}) \cdot L \cdot \omega)^2} \end{cases} \qquad (2.61)$$

The numerical value of the electrical tension on the jacks of the circuit is

$$\begin{cases} A = \sqrt{(40)^2 + \left((0{,}3) \cdot \dfrac{1}{150} \cdot 10^5\right)^2} \\ A = \sqrt{(40)^2 + (200)^2} = 40\sqrt{26}\, V \end{cases} \qquad (2.62)$$

And consequently from (2.54) results

$$\begin{cases} \sin(\delta) = \dfrac{u_0}{A} \\ \sin(\delta) = \dfrac{40}{40\sqrt{26}} = \dfrac{1}{\sqrt{26}} \end{cases} \qquad (2.63)$$

and

$$\cos(\delta) = \frac{5}{\sqrt{26}} \qquad (2.64)$$

Also

$$\begin{cases} tg(\delta) = \dfrac{1}{5} \\ \delta = arc\,tg(1/5) \end{cases} \qquad (2.65)$$

From (2.55)

$$\begin{cases} M = i_{01} - \dfrac{1}{L_1 \cdot \omega} \cdot A \cdot \cos(\delta) \\ N = i_{02} - \dfrac{1}{L_2 \cdot \omega} \cdot A \cdot \cos(\delta) \end{cases} \qquad (2.66)$$

the corresponding numerical values are

$$\begin{cases} M = \left(0{,}1 - \dfrac{1}{0{,}01 \cdot 10^5} \cdot 40\sqrt{26} \cdot \dfrac{5}{\sqrt{26}}\right) A = -0{,}1 A \\ N = \left(0{,}2 - \dfrac{1}{0{,}02 \cdot 10^5} \cdot 40\sqrt{26} \cdot \dfrac{5}{\sqrt{26}}\right) A = 0{,}1 A \end{cases} \qquad (2.67)^*$$

The relations (2.47) becomes

$$\begin{cases} i_1 = \left(\dfrac{4\sqrt{26}}{100} \cdot \cos\left(10^5 \cdot t + arctg(1/5)\right) - 0{,}1\right) A = \tilde{i}_1 - I_0 \\ i_2 = \left(\dfrac{2\sqrt{26}}{100} \cdot \cos\left(10^5 \cdot t + arctg(1/5)\right) + 0{,}1\right) A = \tilde{i}_2 + I_0 \end{cases} \qquad (2.68)$$

The currents through the coils are the superposition of sinusoidal currents having different amplitudes and a direct current passing only through the coils. This direct current has the constant value

$$I_0 = 0{,}1 A \qquad (2.69)^*$$

as in the figure 2.2.

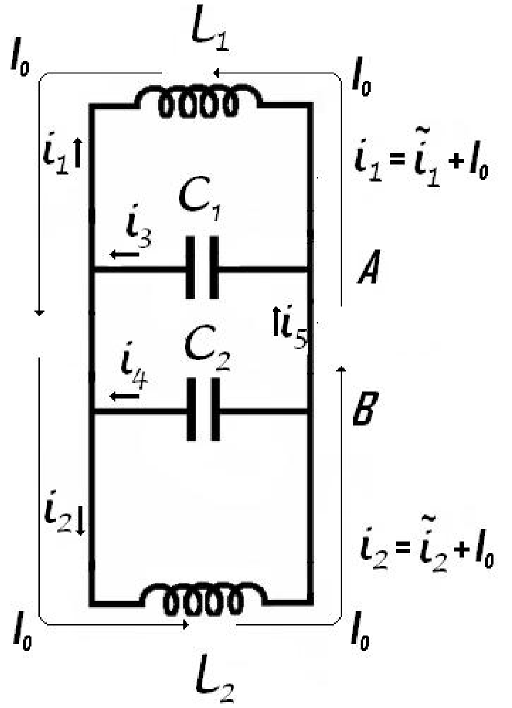


Figure 2.2

The alternative currents through the coils has the expressions

$$\begin{cases} \tilde{i}_1 = \left(\dfrac{4\sqrt{26}}{100} \cdot \cos\left(10^5 \cdot t + arctg(1/5)\right)\right) A \\ \tilde{i}_2 = \left(\dfrac{2\sqrt{26}}{100} \cdot \cos\left(10^5 \cdot t + arctg(1/5)\right)\right) A \end{cases} \qquad (2.70)$$

The currents through the capacitors has the forms

$$\begin{cases} i_3 = \left(-10\times10^{-4}\cdot 40\sqrt{26}\cdot\cos\left(10^5\cdot t + arctg(1/5)\right)\right)A \\ i_3 = \left(-\dfrac{4\sqrt{26}}{100}\cos\left(10^5\cdot t + arctg(1/5)\right)\right)A \\ i_4 = \left(-5\times10^{-4}\cdot 40\sqrt{26}\cdot\cos\left(10^5\cdot t + arctg(1/5)\right)\right)A \\ i_4 = \left(-\dfrac{2\sqrt{26}}{100}\cos\left(10^5\cdot t + arctg(1/5)\right)\right)A \end{cases} \qquad (2.71)$$

The current i_5 has the expression

$$\begin{cases} i_5 = i_3 - i_1 \\ i_5 = \left(-\dfrac{8\sqrt{26}}{100}\cos\left(10^5\cdot t + arctg(1/5)\right) + 0{,}1\right)A \end{cases} \qquad (2.72)$$

The value of the intensity of i_5 current is the answer from the question **c**.
The initial value of this current is

$$i_5 = \left(-\frac{8\sqrt{26}}{100}\frac{5}{\sqrt{26}} + 0{,}1\right)A = -0{,}3\,A \qquad (2.73)^*$$

d. The amplitude of the current through the inductance L_1 is

$$\max\left(\tilde{i}_1\right) = \max\left(\frac{4\sqrt{26}}{100}\cdot\cos\left(10^5\cdot t + arctg(1/5)\right)A\right) = \frac{4\sqrt{26}}{100}A \approx 0{,}2\,A \qquad (2.74)^*$$

representing the answer at the question **d**.

Professor Delia DAVIDESCU, National Department of Evaluation and Examination–Ministry of Education and Research-Bucharest, Romania
Professor Adrian S.DAFINEI,PhD, Faculty of Physics – University of Bucharest, Romania

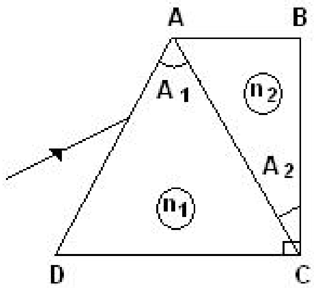
Optics – Problem III (7points)

Prisms

Two dispersive prisms having apex angles $\hat{A}_1 = 60°$ and $\hat{A}_2 = 30°$ are glued as in the figure ($\hat{C} = 90°$). The dependences of refraction indexes of the prisms on the wavelength are given by the relations

$$n_1(\lambda) = a_1 + \frac{b_1}{\lambda^2};$$

$$n_2(\lambda) = a_2 + \frac{b_2}{\lambda^2}$$



were

$a_1 = 1{,}1$, $b_1 = 1 \cdot 10^5\ nm^2$, $a_2 = 1{,}3$, $b_2 = 5 \cdot 10^4\ nm^2$.

- **a.** Determine the wavelength λ_0 of the incident radiation that pass through the prisms without refraction on AC face at any incident angle; determine the corresponding refraction indexes of the prisms.
- ***b.*** Draw the ray path in the system of prisms for three different radiations λ_{red}, λ_0, λ_{violet} incident on the system at the same angle.
- ***c.*** Determine the minimum deviation angle in the system for a ray having the wavelength λ_0.
- ***d.*** Calculate the wavelength of the ray that penetrates and exits the system along directions parallel to DC.

Problem III - Solution

a. The ray with the wavelength λ_0 pass trough the prisms system without refraction on AC face at any angle of incidence if :

$$n_1(\lambda_0) = n_2(\lambda_0)$$

Because the dependence of refraction indexes of prisms on wavelength has the form :

$$n_1(\lambda) = a_1 + \frac{b_1}{\lambda^2} \qquad (3.1)$$

$$n_2(\lambda) = a_2 + \frac{b_2}{\lambda^2} \qquad \textbf{\textit{(3.2)}}$$

The relation (3.1) becomes:

$$a_1 + \frac{b_1}{\lambda_0{}^2} = a_2 + \frac{b_2}{\lambda_0{}^2} \qquad \textbf{\textit{(3.3)}}$$

The wavelength λ_0 has correspondingly the form:

$$\lambda_0 = \sqrt{\frac{b_1 - b_2}{a_2 - a_1}} \qquad \textbf{\textit{(3.4)}}$$

Substituting the furnished numerical values

$$\lambda_0 = 500\,nm \qquad \textbf{\textit{(3.5)}}$$

The corresponding common value of indexes of refraction of prisms for the radiation with the wavelength λ_0 is:

$$n_1(\lambda_0) = n_2(\lambda_0) = 1{,}5 \qquad \textbf{\textit{(3.6)}}$$

The relations (3.6) and (3.7) represent the answers of question **a.**

b. For the rays with different wavelength (λ_{red}, λ_0, λ_{viole}) having the same incidence angle on first prism, the paths are illustrated in the figure 1.1.

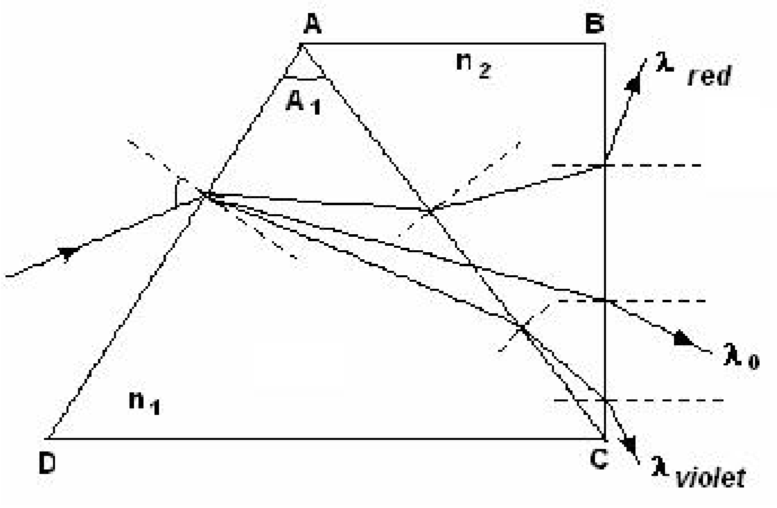


Figure 3.1

The draw illustrated in the figure 1.1 represents the answer of question **b.**

c. In the figure 1.2 is presented the path of ray with wavelength λ_0 at minimum deviation (the angle between the direction of incidence of ray and the direction of emerging ray is minimal).

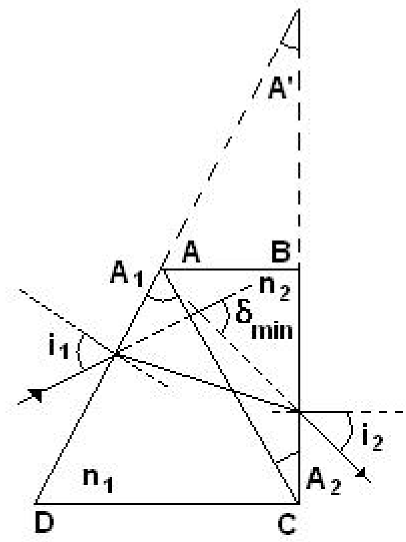


Figure 3.2

In this situation

$$n_1(\lambda_0) = n_2(\lambda_0) = \frac{\sin\dfrac{\delta_{\min} + A'}{2}}{\sin\dfrac{A'}{2}} \qquad \textbf{\textit{(3.7)}}$$

where
$m(\hat{A}') = 30°$,
as in the figure 1.1
Substituting in (3.8) the values of refraction indexes the result is

$$\sin\frac{\delta_{\min} + A'}{2} = \frac{3}{2}\cdot\sin\frac{A'}{2} \qquad \textbf{\textit{(3.8)}}$$

or

$$\delta_{\min} = 2\arcsin\left(\frac{3}{2}\cdot\sin\frac{A'}{2}\right) - \frac{A'}{2} \qquad \textbf{\textit{(3.9)}}$$

Numerically

$$\delta_{\min} \cong 30{,}7° \qquad \textbf{\textit{(3.10)}}$$

The relation (3.11) represents the answer of question **c**.

d. Using the figure 1.3 the refraction law on the AD face is

$$\sin i_1 = n_1 \cdot \sin r_1 \qquad (3.11)$$

The refraction law on the AC face is

$$n_1 \cdot \sin r_1' = n_2 \cdot \sin r_2 \qquad (3.12)$$

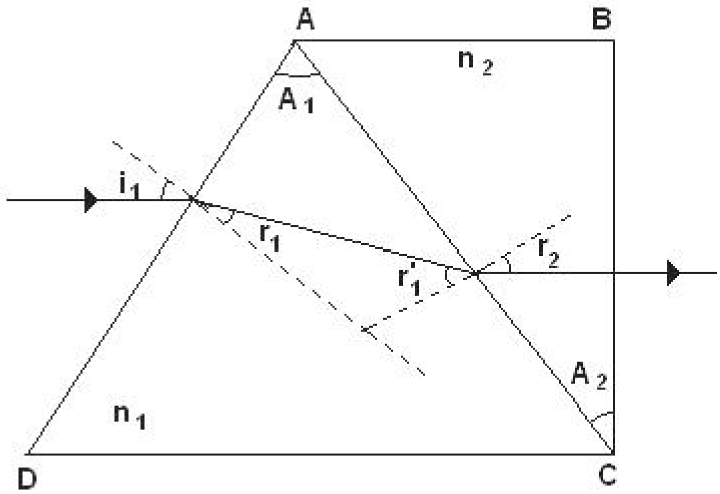


Figure 3.3

As it can be seen in the figure 1.3

$$r_2 = A_2 \qquad (3.13)$$

and

$$i_1 = 30° \qquad (3.14)$$

Also,

$$r_1 + r_1' = A_1 \qquad (3.15)$$

Substituting (3.16) and (3.14) in (3.13) it results

$$n_1 \cdot \sin(A_1 - r_1) = n_2 \cdot \sin A_2 \qquad (3.16)$$

or

$$n_1 \cdot (\sin A_1 \cdot \cos r_1 - \sin r_1 \cdot \cos A_1) = n_2 \cdot \sin A_2 \qquad (3.17)$$

Because of (3.12) and (3.15) it results that

$$\sin r_1 = \frac{1}{2 n_1} \qquad (3.18)$$

and

$$\cos r_1 = \frac{1}{2n_1}\sqrt{4{n_1}^2 - 1} \qquad \textbf{\textit{(3.19)}}$$

Putting together the last three relations it results

$$\sqrt{4{n_1}^2 - 1} = \frac{2n_2 \cdot \sin A_2 + \cos A_1}{\sin A_1} \qquad \textbf{\textit{(3.20)}}$$

Because
$\hat{A}_1 = 60°$
and
$\hat{A}_2 = 30°$
relation (3.21) can be written as

$$\sqrt{4{n_1}^2 - 1} = \frac{2n_2 + 1}{\sqrt{3}} \qquad \textbf{\textit{(3.21)}}$$

or

$$3 \cdot {n_1}^2 = 1 + n_2 + {n_2}^2 \qquad \textbf{\textit{(3.22)}}$$

Considering the relations (3.1), (3.2) and (3.23) and operating all calculus it results:

$$\lambda^4 \cdot \left(3{a_1}^2 - {a_2}^2 - a_2 - 1\right) + \left(6a_1 b_1 - b_2 - 2a_2 b_2\right) \cdot \lambda^2 + 3{b_1}^2 - {b_2}^2 = 0 \qquad \textbf{\textit{(3.23)}}$$

Solving the equation (3.24) one determine the wavelength λ of the ray that enter the prisms system having the direction parallel with DC and emerges the prism system having the direction again parallel with DC. That is

$$\lambda = 1194\, nm \qquad \textbf{\textit{(3.24)}}$$

or

$$\lambda \cong 1{,}2\, \mu m \qquad \textbf{\textit{(3.25)}}$$

The relation (3.26) represents the answer of question **d**.

Professor Delia DAVIDESCU, National Department of Evaluation and Examination–Ministry of Education and Research-Bucharest, Romania
Professor Adrian S.DAFINEI,PhD, Faculty of Physics – University of Bucharest, Romania

Atomics - Problem IV (7 points)

Compton scattering

A photon of wavelength λ_i is scattered by a moving, free electron. As a result the electron stops and the resulting photon of wavelength λ_0 scattered at an angle $\theta = 60°$ with respect to the direction of the incident photon, is again scattered by a second free electron at rest. In this second scattering process a photon with wavelength of $\lambda_f = 1{,}25 \times 10^{-10}\,m$ emerges at an angle $\theta = 60°$ with respect to the direction of the photon of wavelength λ_0. Find the de Broglie wavelength for the first electron before the interaction. The following constants are known:

$h = 6{,}6 \times 10^{-34}\,J \cdot s$ - Planck's constant
$m = 9{,}1 \times 10^{-31}\,kg$ - mass oh the electron
$c = 3{,}0 \times 10^{8}\,m/s$ - speed of light in vacuum

Problem III - Solution

The purpose of the problem is to calculate the values of the speed, momentum and wavelength of the first electron.

To characterize the photons the following notation are used:

Table 4.1

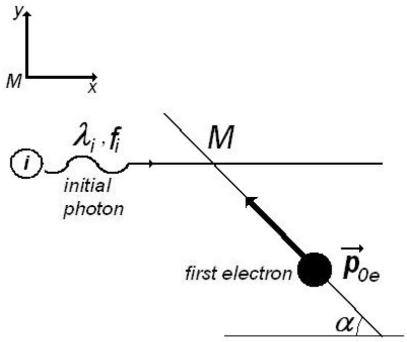
	initial photon	photon – after the first scattering	final photon
momentum	$\vec{p}_i$	$\vec{p}_0$	$\vec{p}_f$
energy	E_i	E_0	E_f
wavelength	λ_i	λ_i	λ_f

To characterize the electrons one uses

Table 4.2

	first electron before collision	first electron after collision	second electron before collision	Second electron after collision
momentum	$\vec{p}_{1e}$	0	0	$\vec{p}_{2e}$
energy	E_{1e}	E_{0e}	E_{0e}	E_{2e}
speed	$\vec{v}_{1e}$	0	0	$\vec{v}_{2e}$

The image in figure 4.1 presents the situation before the first scattering of photon.



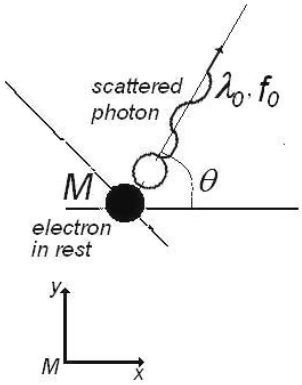
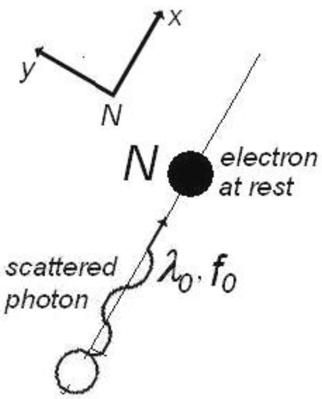


Figure 4.1 *Figure 4.2*



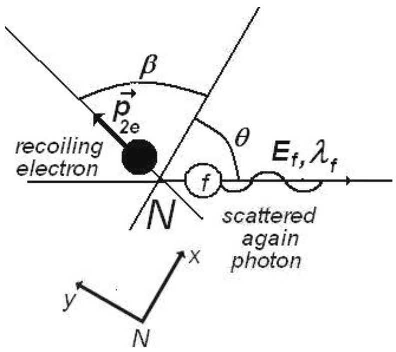


Figure 4.3 *Figure 4.4*

To characterize the initial photon we will use his momentum $\vec{p}_i$ and his energy E_i

$$\begin{cases} \vec{P}_i = \dfrac{h}{\lambda_i} = \dfrac{h \cdot f_i}{c} \\ E_i = h \cdot f_i \end{cases} \qquad (4.1)$$

$$f_i = \frac{c}{\lambda_i} \qquad (4.2)$$

is the frequency of initial photon.

For initial, free electron in motion the momentum $\vec{p}_{oe}$ and the energy E_{oe} are

$$\begin{cases} \vec{P}_{oe} = m \cdot \vec{v}_{1e} = \dfrac{m_0 \cdot \vec{v}_{1e}}{\sqrt{1-\beta^2}} \\ E_{oe} = m \cdot c^2 = \dfrac{m_0 \cdot c^2}{\sqrt{1-\beta^2}} \end{cases} \qquad (4.3)$$

where m_0 is the rest mass of electron and m is the mass of moving electron. As usual, $\beta = \frac{V_{1e}}{c}$. De Broglie wavelength of the first electron is

$$\lambda_{oe} = \frac{h}{p_{0e}} = \frac{h\cdot}{m_0 \cdot v_{1e}} \sqrt{1-\beta^2}$$

The situation after the scattering of photon is described in the figure 4.2.
To characterize the scattered photon we will use his momentum $\vec{p}_0$ and his energy E_0

$$\begin{cases} \vec{P}_o = \dfrac{h}{\lambda_o} = \dfrac{h \cdot f_o}{c} \\ E_o = h \cdot f_o \end{cases} \qquad (4.4).$$

where

$$f_o = \frac{c}{\lambda_0} \qquad (4.5)$$

is the frequency of scattered photon.

The magnitude of momentum of the electron (that remains in rest) after the scattering is zero; his energy is E_{1e}. The mass of electron after collision is m_0 - the rest mass of electron at rest. So,

$$E_{1e} = m_0 \cdot c^2$$

To determine the moment of the first moving electron, one can write the principles of conservation of moments and energy. That is

$$\vec{P}_i + \vec{p}_{oe} = \vec{p}_0 \qquad (4.6)$$

and

$$E_i + E_{0e} = E_0 + E_{1e} \qquad (4.7)$$

The conservation of moment on $0x$ direction is written as

$$\frac{h \cdot f_i}{c} + m \cdot v_{1e} \cdot \cos\alpha = \frac{h \cdot f_0}{c} \cos\theta \qquad (4.8)$$

and the conservation of moment on $0y$ is

$$m \cdot v_{1e} \cdot \sin\alpha = \frac{h \cdot f_0}{c} \sin\theta \qquad (4.9)$$

To eliminate α, the last two equation must be written again as

$$\begin{cases} (m \cdot v_{1e} \cdot \cos\alpha)^2 = \frac{h^2 \cdot}{c^2}(f_0 \cdot \cos\theta - f_i)^2 \\ (m \cdot v_{1e} \cdot \sin\alpha)^2 = \left(\frac{h \cdot f_0}{c}\sin\theta\right)^2 \end{cases} \qquad (4.10)$$

and then added.
The result is

$$m^2 \cdot v_{1e}^2 = \frac{h^2 \cdot}{c^2}\left(f_0^2 + f_1^2 - 2f_0 \cdot f_i \cdot \cos\theta\right) \qquad (4.11)$$

or

$$\frac{m_0^2 \cdot c^2}{1-\left(\frac{v_{1e}}{c}\right)^2} \cdot v_{1e}^2 = h^2 \cdot \left(f_0^2 + f_1^2 - 2f_0 \cdot f_i \cdot \cos\theta\right) \qquad (4.12)$$

The conservation of energy (4.7) can be written again as

$$m \cdot c^2 + h \cdot f_1 = m_0 \cdot c^2 + h \cdot f_0 \qquad (4.13)$$

or

$$\frac{m_0 \cdot c^2}{\sqrt{1-\left(\frac{v_{1e}}{c}\right)^2}} = m_0 \cdot c^2 + h \cdot (f_0 - f_1) \qquad (4.14)$$

Squaring the last relation results

$$\frac{m_0^2 \cdot c^4}{1-\left(\frac{v_{1e}}{c}\right)^2} = m_0^2 \cdot c^4 + h^2 \cdot (f_0 - f_1)^2 + m_0 \cdot h \cdot c^2 \cdot (f_0 - f_1) \qquad (4.15)$$

Subtracting (4.12) from (4.15) the result is

$$2m_0 \cdot c^2 \cdot h \cdot (f_0 - f_1) + 2h^2 \cdot f_1 \cdot f_0 \cdot \cos\theta - 2h^2 \cdot f_1 \cdot f = 0 \qquad (4.16)$$

or

$$\frac{h}{m_0 \cdot c}(1-\cos\theta) = \frac{c}{f_1} - \frac{c}{f_0} \qquad (4.17)$$

Using

$$\Lambda = \frac{h}{m_0 \cdot c} \qquad (4.18)$$

the relation (4.17) becomes

$$\Lambda \cdot (1-\cos\theta) = \lambda_i - \lambda_0 \qquad (4.19)$$

The wavelength of scattered photon is

$$\lambda_0 = \lambda_i - \Lambda \cdot (1-\cos\theta) \qquad (4.20)$$

shorter than the wavelength of initial photon and consequently the energy of scattered photon is greater that the energy of initial photon.

$$\begin{cases} \lambda_i < \lambda_0 \\ E_i > E_0 \end{cases} \qquad (4.21)$$

Let's analyze now the second collision process that occurs in point N. To study that, let's consider a new referential having $0x$ direction on the direction of the photon scattered after the first collision.

The figure 4.3 presents the situation before the second collision and the figure 4.4 presents the situation after this scattering process. The conservation principle for moment in the scattering process gives

$$\begin{cases} \dfrac{h}{\lambda_0} = \dfrac{h}{\lambda_f}\cos\theta + m \cdot v_{2e} \cdot \cos\beta \\ \dfrac{h}{\lambda_f}\sin\theta - m \cdot v_{2e} \cdot \sin\beta = 0 \end{cases} \qquad (4.22)$$

To eliminate the unknown angle β must square and then add the equations (4.22)

That is

$$\begin{cases} \left(\dfrac{h}{\lambda_0} - \dfrac{h}{\lambda_f}\cos\theta\right)^2 = (m \cdot v_{2e} \cdot \cos\beta)^2 \\ \left(\dfrac{h}{\lambda_f}\sin\theta\right)^2 = (m \cdot v_{2e} \cdot \sin\beta)^2 \end{cases} \qquad (4.23)$$

or

$$\left(\frac{h}{\lambda_f}\right)^2 + \left(\frac{h}{\lambda_0}\right)^2 - \frac{2 \cdot h^2}{\lambda_0 \cdot \lambda_f}\cos\theta = (m \cdot v_{2e})^2 \qquad (4.24)$$

The conservation principle of energy in the second scattering process gives

$$\frac{h \cdot c}{\lambda_0} + m_0 \cdot c^2 = \frac{h \cdot c}{\lambda_f} + m \cdot c^2 \qquad (4.25)$$

(4.24) and (4.25) gives

$$\frac{h^2 \cdot c^2}{\lambda_f^2} + \frac{h^2 \cdot c^2}{\lambda_0^2} - \frac{2 \cdot h^2 \cdot c^2}{\lambda_0 \cdot \lambda_f}\cos\theta = m^2 \cdot c^2 \cdot v_{2e}^2 \qquad (4.26)$$

and

$$h^2 \cdot c^2 \cdot \left(\frac{1}{\lambda_f} - \frac{1}{\lambda_0}\right)^2 + m_0^2 \cdot c^4 + 2h \cdot c^3 \cdot m_0 \cdot \left(\frac{1}{\lambda_f} - \frac{1}{\lambda_0}\right) = m^2 \cdot c^4 \qquad (4.27)$$

Subtracting (4.26) from (1.27), one obtain

$$\begin{cases} \dfrac{h}{m_0 \cdot c} \cdot (1 - \cos\theta) = \lambda_f - \lambda_0 \\ \lambda_f - \lambda_0 = \Lambda \cdot (1 - \cos\theta) \end{cases} \qquad (4.28)$$

That is

$$\begin{cases} \lambda_f > \lambda_0 \\ E_f < E_0 \end{cases} \qquad (4.29)$$

Because the value of λ_f is know and Λ can be calculate as

$$\begin{cases} \lambda_f = 1{,}25 \times 10^{-10}\, m \\ \Lambda = \dfrac{6{,}6 \times 10^{-34}}{9{,}1 \times 10^{-31} \cdot 3 \times 10^8}\, m = 2{,}41 \times 10^{-12}\, m = 0{,}02 \times 10^{-10}\, m \end{cases} \qquad (4.30)$$

the value of wavelength of photon before the second scattering is

$$\lambda_0 = 1{,}23 \times 10^{-10}\, m \qquad (4.31)$$

Comparing (4.28) written as:

$$\lambda_f = \lambda_0 + \Lambda \cdot (1 - \cos\theta) \qquad (4.32)$$

and (4.20) written as

$$\lambda_i = \lambda_0 + \Lambda \cdot (1 - \cos\theta) \qquad (4.33)$$

clearly results

$$\lambda_i = \lambda_f \qquad (4.34)$$

The energy of the double scattered photon is the same as the energy of initial photon. The direction of "final photon" is the same as the direction of "initial" photon. Concluding, the final photon is identical with the initial photon. The result is expected because of the symmetry of the processes.
Extending the symmetry analyze on electrons, the first moving electron that collides the initial photon and after that remains at rest, must have the same momentum and energy as the second electron after the collision – because this second electron is at rest before the collision.
That is

$$\begin{cases} \vec{p}_{1e} = \vec{p}_{2e} \\ E_{1e} = E_{2e} \end{cases} \qquad (4.35)$$

Taking into account (4.24), the moment of final electron is

$$p_{2e} = h\sqrt{\frac{1}{\lambda_f^2} + \frac{1}{\left(\lambda_f - \Lambda(1-\cos\theta)\right)^2} - \frac{2\cdot\cos\theta}{\lambda_f\cdot\left(\lambda_f - \Lambda(1-\cos\theta)\right)}} \qquad (4.36)$$

The de Broglie wavelength of second electron after scattering (and of first electron before scattering) is

$$\lambda_{1e} = \lambda_{2e} = 1\bigg/\left(\sqrt{\frac{1}{\lambda_f^2} + \frac{1}{\left(\lambda_f - \Lambda(1-\cos\theta)\right)^2} - \frac{2\cdot\cos\theta}{\lambda_f\cdot\left(\lambda_f - \Lambda(1-\cos\theta)\right)}}\right) \qquad (4.37)$$

Numerical value of this wavelength is

$$\lambda_{1e} = \lambda_{2e} = 1{,}24\times 10^{-10}\,m \qquad (4.38)$$

Professor Delia DAVIDESCU, National Department of Evaluation and Examination–Ministry of Education and Research- Bucharest, Romania
Professor Adrian S.DAFINEI,PhD, Faculty of Physics – University of Bucharest, Romania

IPhO's LOGO – Problem V

The Logo of the International Physics Olympiad is represented in the figure below.
The figure presents the phenomenon of the curving of the trajectory of a jet of fluid around the shape of a cylindrical surface. The trajectory of fluid is not like the expected dashed line but as the circular solid line.
Qualitatively explain this phenomenon (first observed by Romanian engineer Henry Coanda in 1936).

This problem will be not considered in the general score of the Olympiad. The best solution will be awarded a special prize.

Figure 5.1

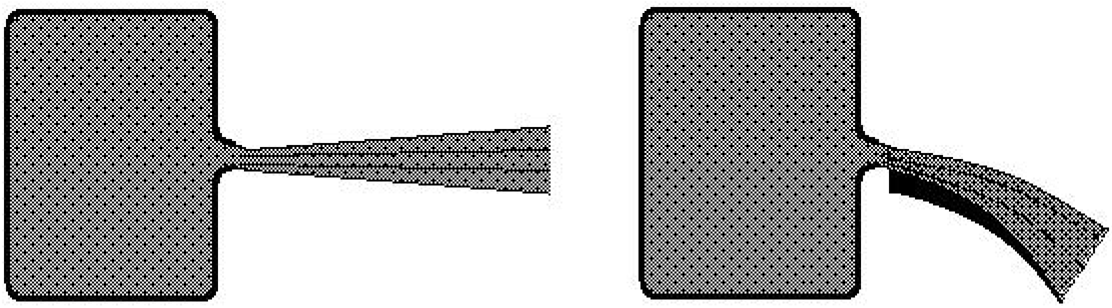
Problem V -Solution

Suppose a fluid is in a recipient at a constant pressure. If a thin jet of fluid (gas or liquid) having a small circular or rectangular cross section leaves the recipient through a nozzle entering the medium, the particles belonging to the medium will be carried out by the jet. Other particles belonging to the medium will be attracted to the jet.

If the jet flows over a large surface, the particles belonging to the medium over the jet and the particles leaving between the jet and the surface will be carried out by the jet. The density of particles over the jet remains constant because of newly arriving particles, but the particles between the surface and the jet cannot be replaced. A pressure difference appears between the upper and lower side of the jet, pushing the jet to the surface. If the surface is curved, the jet will follow its shape.

The left image in the figure below presents the normal flow of a fluid jet leaving through a nozzle of a recipient with a high, constant pressure. The final pressure of the fluid is of medium pressure.

The right image in the figure below presents the flow of a fluid over the large surface. The jet is “stuck”



against the surface.

The process of deflection of the jet increases the speed of the jet without any variation of the pressure and temperature of the jet.

During the tests of the first jet plane in Paris, December 1936, the Romanian engineer Henry Coanda was the first to observe this phenomenon, occurring when the flames of the engine passed through a flap.

The logo of the Olympiad illustrates the Coanda flow of a fluid.

Professor Delia DAVIDESCU, National Department of Evaluation and Examination–Ministry of Education and Research-Bucharest, Romania
Professor Adrian S.DAFINEI,PhD, Faculty of Physics – University of Bucharest, Romania

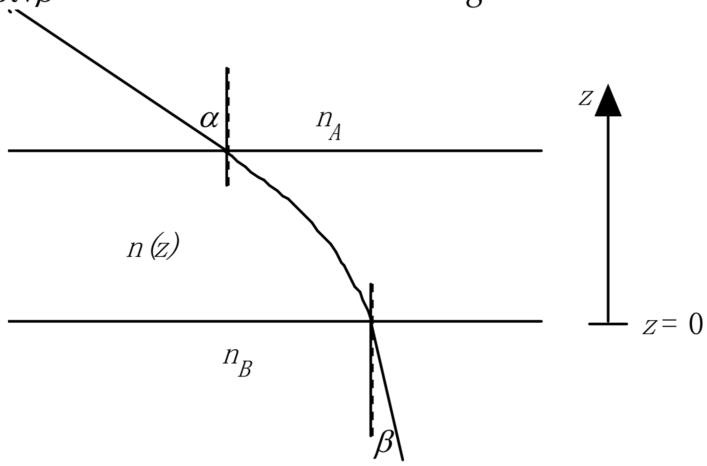
Problems of the XV International Physics Olympiad
(Sigtuna, 1984)

Lars Gislén
Department of Theoretical Physics, University of Lund, Sweden

Theoretical problems

Problem 1

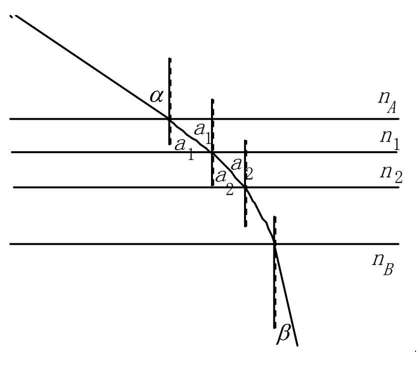
a) Consider a plane-parallel transparent plate, where the refractive index, *n*, varies with distance, *z*, from the lower surface (see figure). Show that $\nu_A \sigma\iota\nu\alpha = \nu_B \sigma\iota\nu\beta$. The notation is that of the figure.



b) Assume that you are standing in a large flat desert. At some distance you see what appears to be a water surface. When you approach the "water" is seems to move away such that the distance to the "water" is always constant. Explain the phenomenon.

c) Compute the temperature of the air close to the ground in b) assuming that your eyes are located 1.60 m above the ground and that the distance to the "water" is 250 m. The refractive index of the air at 15 ˚C and at normal air pressure (101.3 kPa) is 1.000276. The temperature of the air more than 1 m above the ground is assumed to be constant and equal to 30 ˚C. The atmospheric pressure is assumed to be normal. The refractive index, *n*, is such that *n* – 1 is proportional to the density of the air. Discuss the accuracy of your result.

Solution:



a) From the figure we get

$$\nu_A \sigma\iota\nu\alpha = \nu_1 \sigma\iota\nu\alpha_1 = \nu_2 \sigma\iota\nu\alpha_2 = \ldots = \nu_B \sigma\iota\nu\beta$$

b) The phenomenon is due to total reflexion in a warm layer of air when β = 90˚. This gives

$$n_A \sin\alpha = n_B$$

c) As the density, ρ, of the air is inversely proportional to the absolute temperature, T, for fixed pressure we have

$$n(T) = 1 + k \cdot \rho = 1 + k / T$$

The value given at 15 ˚C determines the value of k = 0.0795.
In order to have total reflexion we have $n_{30} \sin\alpha = n_T$ or

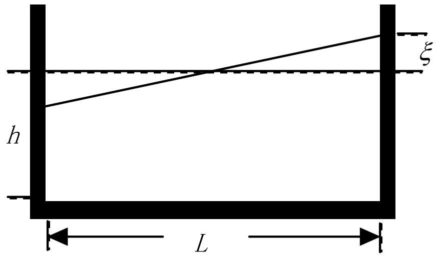
$$\left(1+\frac{k}{303}\right)\cdot\frac{L}{\sqrt{h^2+L^2}} = \left(1+\frac{k}{T}\right)$$ with h = 1.6 m and L = 250 m

As $h \ll L$ we can use a power expansion in h/L:

$$T = \frac{303}{\left(\frac{303}{k}+1\right)\frac{1}{\sqrt{1+h^2/L^2}} - \frac{303}{k}} \approx 303\left(1+\frac{303h^2}{2kL^2}\right) = 328\,\mathrm{K} = 56\,^\circ\mathrm{C}$$

Problem 2

In certain lakes there is a strange phenomenon called "seiching" which is an oscillation of the water. Lakes in which you can see this phenomenon are normally long compared with the depth and also narrow. It is natural to see waves in a lake but not something like the seiching, where the entire water volume oscillates, like the coffee in a cup that you carry to a waiting guest.



In order to create a model of the seiching we look at water in a rectangular container. The length of the container is L and the depth of the water is h. Assume that the surface of the water to begin with makes a small angle with the horizontal. The seiching will then start, and we assume that the water surface continues to be plane but oscillates around an axis in the horizontal plane and located in the middle of the container.

Create a model of the movement of the water and derive a formula for the oscillation period T. The starting conditions are given in figure above. Assume that $\xi << \eta$. The table below shows experimental oscillation periods for different water depths in two containers of different lengths. Check in some reasonable way how well the formula that you have derived agrees with the experimental data. Give your opinion on the quality of your model.

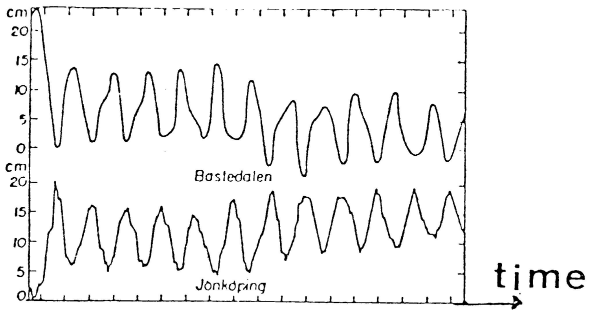
Table 1. L = 479 mm

$\eta/\ \mu\mu$	30	50	69	88	107	124	142
$T/\ \sigma$	1.78	1.40	1.18	1.08	1.00	0.91	0.82

Table 2. L = 143 mm

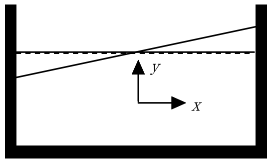
$\eta/\ \mu\mu$	31	38	58	67	124
$T/\ \sigma$	0.52	0.52	0.43	0.35	0.28

The graph below shows results from measurements in lake Vättern in Sweden. This lake has a length of 123 km and a mean depth of 50 m. What is the time scale in the graph?



The water surface level in Bastudalen (northern end of lake Vättern) and Jönköping (southern end).

Solution:



In the coordinate system of the figure, we have for the centre of mass coordinates of the two triangular parts of the water

$$(\xi_1, \psi_1) = (\Lambda/3, \eta/2 + \xi/3) \quad (\xi_2, \psi_2) = (-\Lambda/3, \eta/2 - \xi/3).$$

For the entire water mass the centre of mass coordinates will then be

$$(\xi_{XoM}, \psi_{XoM}) = \left(\frac{\xi\Lambda}{6\eta}, \frac{\xi^2}{6\eta}\right)$$

Due to that the y component is quadratic in ξ will be much much smaller than the x component.

The velocities of the water mass are

$$(\varpi_\xi, \varpi_\psi) = \left(\frac{\dot{\xi}\Lambda}{6\eta}, \frac{\xi\dot{\xi}}{3\eta}\right),$$

and again the vertical component is much smaller the the horizontal one. We now in our model neglect the vertical components. The total energy (kinetic + potential) will then be

$$\Omega = \Omega_K + \Omega_\Pi = \tfrac{1}{2} M \frac{\dot{\xi}^2 \Lambda^2}{36\eta^2} + M\gamma \frac{\xi^2}{6\eta}$$

For a harmonic oscillator we have

$$\Omega = \Omega_K + \Omega_\Pi = \tfrac{1}{2}\mu\dot{\xi}^2 + \tfrac{1}{2}\mu\omega^2\xi^2$$

Identifying gives

$$\omega = \sqrt{\frac{12\gamma\eta}{\Lambda}} \quad \text{or} \quad T_{\mu o\delta\varepsilon\lambda} = \frac{\pi\Lambda}{\sqrt{3\eta}}.$$

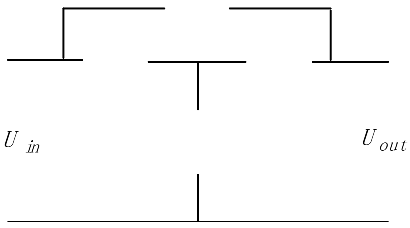
Comparing with the experimental data we find $T_{\varepsilon\xi\pi\varepsilon\rho\iota\mu\varepsilon\nu\tau} \approx 1.1 \cdot T_{\mu o\delta\varepsilon\lambda}$, our model gives a slight underestimation of the oscillation period.

Applying our corrected model on the Vättern data we have that the oscillation period of the seiching is about 3 hours.

Many other models are possible and give equivalent results.

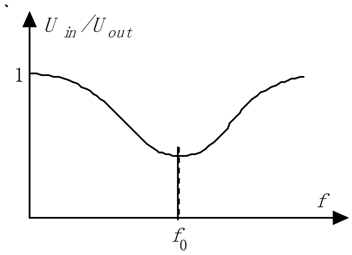
Problem 3

An electronic frequency filter consists of four components coupled as in the upper figure. The impedance of the source can be neglected and the impedance of the load can be taken as infinite. The filter should be such that the voltage ratio Y_{out}/Y_{in} has a frequency dependence shown in the lower where Y_{in} is the input voltage and Y_{on} is the output voltage. At frequency ϕ_0 the phase lag between the two voltages is zero.



In order to build the filter you can choose from the following components:

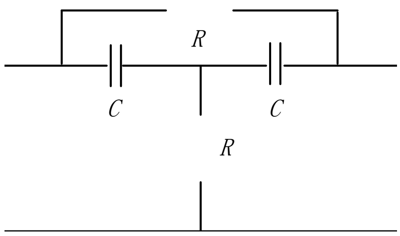
2 resistors, 10 kΩ
2 capacitors, 10 nF
2 solenoids, 160 mH (iron-free and with neglible resistance)



Construct, by combining four of these components, a filter that fulfils the stated conditions. Determine the frequency ϕ_0 and the ratio Y_{out}/Y_{in} at this frequency for as many component combinations as possible.

Solution:

The conditions at very high and very low frequencies can be satisfied with for example the following circuit



Using either the graphic vector method or the analytic jω method we can show that the minimum occurs för a frequency $\phi_0 = \frac{1}{2\pi PX}$ when the ratio between the output and input voltages is 2/3. Switching the resistors and the capacitors gives a new circuit with the same frequency ϕ_0. Another two possibilities is to exchange the capacitors for solenoids where we get $\phi_0 = \frac{P}{2\pi \Lambda}$.

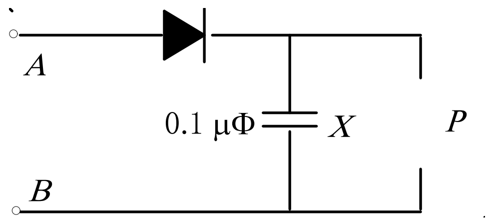
There are further eight solutions with unsymmetric patterns of the electronic components.

Experimental problems

Problem 1

You have at your disposal the following material:

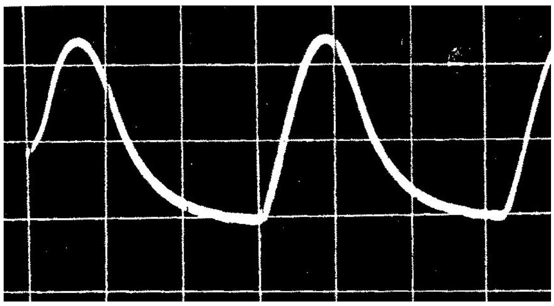
(1) A sine wave voltage generator set to a frequency of 0,20 kHz.
(2) A dual ray oscilloscope.
(3) Millimeter graph paper.
(4) A diod.
(5) A capacitor of 0.10 μF (square and black).
(6) An unknown resistor *R* (red).
(7) A coupling plate.
(8) Coupling wires.



Build the circuit shown in the figure.

Connect the terminals *A* and *B* to the sine wave generator set to a frequency of 0.20 kHz. Determine experimentally the mean power developed in the resistor *R* when the amplitude of the generator voltage is 2.0 V (that is the peak-to-peak voltage is 4.0 V).

Solution:



The picture to the right shows the oscilloscope voltage over the resistor. The period of the sine wave is 5 ms and this gives the relation 1 horizontal division = 1.5 ms. The actual vertical scale was 0.85 V / division. The first rising part of the curve is a section of a sine wave, the second falling part is an exponential decay determined by the time constant of the resistor and capacitor. Reading from the display the "half-life" $\tau_{1/2} = PX \cdot \lambda\nu 2$ turns out to be 0.5 ms. This gives R = 7.2 kΩ. The mean power developed in the resistor is

$\langle \Pi \rangle = \frac{1}{T}\int_0^T \frac{Y^2(\tau)}{P}\delta\tau$. Numerical integration (counting squares) gives

$\int_0^T Y^2(\tau)\delta\tau = 4,5 \cdot 10^{-3}\ \varsigma^2 c$ from which $\langle \Pi \rangle \approx 0.1$ mW.

Problem 2

Material:
(1) A glow discharge lamp connected to 220 V, alternating current.
(2) A laser producing light of unknown wavelength.
(3) A grating.
(4) A transparent "micro-ruler", 1 mm long with 100 subdivisions, the ruler is situated exactly in the centre of the circle.
(5) A 1 m long ruler
(6) Writing material.

The spectrum of the glow discharge lamp has a number of spectral lines in the region yellow-orange-red. On of the yellow lines in the short wavelength part of this spectrum is very strong. Determine the wavelength of this spectral line. Estimate the accuracy of your measurement.

Note: If you happen to know the wavelength of the laser light beforehand you are not allowed to use that value in your computation.

Warning. Do not look into the laser beam. Do not touch the surface of the grating or the surface of the transparent micro-ruler.

Solution:

Using the micro-ruler with we can determine the wavelength of the laser light. Knowing this wavelength we can calibrate the grating and then use it to determine the unknown wavelength from the glow discharge lamp. We cannot use the micro-ruler to determine this wavelength because the intensity of the light from the lamp is too weak.

第十六届国际中学生物理奥林匹克竞赛试题(理论部分)

(1985 南斯拉夫 波尔托罗日)

题 1　一位年青的业余无线电爱好者用无线电与住在两个镇上的两位女孩保持联系。他放置两根竖直的天线棒，使得当住在 *A* 镇的女孩接收到最大信号时，住在 *B* 镇的女孩接收不到信号，反之也一样。这个天线阵由两根竖直的天线棒构成，它们在水平面内均匀地向各个方向发射同等强度的信号。

(a)求此天线阵的参数，即两棒间距离及它们的方位和馈入两棒电信号之间的位相差，使得两棒间距离为最小。

(b)求上述数值解。如果男孩的无线电台发射 27MH$_Z$ 的电磁波，该天线阵位于波尔托罗日，利用地图，他发现正北方与 *A* 方向(科佩尔)和 *B* 方向(位于 伊斯特拉半岛上的小镇布热)的夹角分别为 158°和 72°。

〔解〕a)如图 16-1 所示，设 A 方向和 B 方向的夹角为φ，两棒间距为 r，棒间连线与 A 方向夹角为 a。

A 方向最小位相差为：

$\Delta A=2\pi\quad\cos\alpha+\Delta\varphi$

B 方向的最小位相差为：

$\Delta B=2\pi\quad\cos(\psi-\alpha)+\Delta\varphi$

Δφ为两根天线之间的相位差。当 A 方向强度最小，B 方向强度最大时，

$\Delta A=(2n+1)\pi$，$\Delta B=2\kappa\pi$。

则

$\Delta B-\Delta A=(2(\kappa-n)-1)\pi=2\pi\quad$。〔$\cos(\psi-\alpha)-\cos\alpha$〕

得到

$r=$　　　　.

当ψ一定时，只有 k=n，α-　=-　时，r 为最小，或者 k=n+1，

α-　=　时，r 也为最小。

此时，

r 最小=

把上述结果代入含有Δφ的方程中，可得

Δφ=π/2(k=n 时)，或Δφ=-　　时，(k=n+1 时)

当Δφ从　　变为-　　时，产生的效应正好相反，即 A 方向强度最大，B 方向强度为 0。

b)如图 16-2 所示，A 方向和 B 方向夹角为

ψ=157°-72°=85°

则棒间距最小为

r 最小=　　　　　=

=　　　　　　　　　　=4.1(米)

两棒连线与 A 方向夹角为

α=　+90°=132.5°

题 2　一根边长为 a、b、c(a>>b>>c)的矩形截面长棒，是由半导体锑化铟制成的。棒中有平行于 a 边的电流 I 流过。该棒放在平行于 c 边的外磁场 B 中，电流 I 所产生的磁场可以忽略。该电流的载流子为电子。在只有电场存在时，电子在半导体中的平均速度是 $v=\mu E$，其中μ为迁移率。如果磁场也存在的话，则总电场不再与电流平行，这个现象叫做霍尔效应。

（a）确定在棒中产生上述电流的总电场的大小和方向。

（b）计算夹 b 边两表面上相对两点间的电势差。

（c）如果电流和磁场都是交变的，且分别为 $I=I_0\sin\omega t$，$B=B_0\sin(\omega t+\varphi)$。写出 b)情形中电势差的直流分量解析表达式。

（d）利用 c）的结果，设计一个电子线路，使其能测量连接于交流电网的电子设备所消耗的功率，并给出解释。

利用下列数据：

锑化铟中的电子迁移率为 $7.8\mathrm{m^2/V\cdot s}$　锑化铟中的电子密度为 $2.5\times10^{22}\mathrm{m^{-3}}$

I=1.0A　　B=1.0T　　b=1.0cm　　c=1.0mm　　$e=1.6\times10^{-19}$C

〔解〕a)

如图 16-3 所示，电子沿 a 边的运动，将使其受到洛仑兹力 evB 的作用,这样电子将具有沿 b 方向运动分量，并在样品两侧有电荷积累，形成与洛仑 兹力相抵的电场力，这样有

$E_{\perp}e=evB$，即 $E_{\perp}=vB$

而沿 a 方向的电场分量可由下式求出

$v=\mu E_{/\!/}$,

即

$E_{/\!/}=v/\mu$

电子的速度可由电流求出

$I=s\cdot j=cbnev$，

故 $v=I/nebc=25$ 米/秒， 则 $E_{/\!/}=3.2$ 伏/米，$E_{\perp}=2.5$ 伏/米。 因此总电场大小为

E= =4.06(伏/米)

电场方向如图 16-3 所示，

tga= =3.2/2.5=1.28。

b)b 边相应两表面间电势差为

$V_H=E_{\perp}b=25$ 毫伏

c)$V_H=\dfrac{IBb}{enbc}=(\dfrac{I_0B_0}{2enc})\sin\omega t\sin(\omega t+\varphi)$，直流分量 $V=(\dfrac{I_0B_0}{2enc})\cos\varphi$。

d)电子线路如图 16-4 所示。

题 3 现在讨论和研究的是关于某空间研究规划，把宇宙飞船发射到太阳系外去的两种发射方案。第一种方案是以足够大的速度发射飞船，使其 直接逃逸出太阳系。第二种方案是使飞

船接近某一颗外行星并依靠它的帮助，改变飞船的运动方向以达到逃逸出太阳系所必需的速度。假定飞船仅仅在太阳或行星的引力场中运动。那么究竟是在太阳的引力场中运动还是 在行星的引力场中运动，这要由该点是哪一个场较强而定。

(a)按照方案 1 确定发射飞船所必需的相对于地球运动的最小速度 v_a 和它的方向。

(b)假定飞船已按(a)中确定的方向发射，但具有另一个相对于地球的速度 v_b。求飞船穿过火星轨道时的速度，亦即相对于此轨道的平行分量和垂直分量。当飞船穿过火星轨道时，火星不在此交点附近。

(c)设飞船进入火星的引力场，试求从地球发射飞船使其逃逸出太阳系所需的最小速度。

提示：从结果(a)可以知道飞船在脱离火星引力场后逃逸出太阳系所需的最佳速度的大小和方向(不必考虑在穿越火星轨道时火星的精确位置)。求这个最佳速度与飞船进入火星引力场以前的速度分量，即你在(b)中确定的速度分量之间的关系。飞船的能量守恒情况又是怎样？

（d）估算第二种方案比第一种方案所可能节省能量的最大百分比。

注：设所有行星在同一平面内以同一方向绕着太阳在圆轨道上运转。

忽略空气阻力，地球的自转以及从地球引力场逸出所消耗的能量。

数据： 地球绕太阳旋转的速度为 30km/s，地球到太阳与火星到太阳的距离之比为 2/3。

〔解〕 如图 16-5，16-6 所示，设 V_a 为相对于地球的发射速度，V_E 为地球速度，θ是 V_E 与 V_a 间的夹角。宇宙飞船在太阳系中的总能量为

E= mV²-

其中，m 为飞船质量，V 为相对于太阳系的速度，M 是太阳的质量，M_E 为地球的质量。宇宙飞船逃逸出太阳系的必要条件是：

E≥0

另一方面，地球的速度由下式给出：

，V_E=

故由 E≥0 得

$V^2 \geq$ $=2V_E^2$

由图 16-5，16-6 有

$v_a \geq v_E[\sqrt{1+\cos^2\theta} - \cos\theta]$

最小速度的大小为

$v_a = v_E(\sqrt{2} - 1) = 12.3$km/s

此时，方向为θ=0。

b)设 V_b 和 V 分别为以地球为参照系和以太阳为参照系时的发射速度(见图 16-7)，从 a)得到

$V=V_b+V_E$

由角动量守恒给出

$mVR_E=mV_{〃}R_M$

由能量守恒给出

由上两个方程可得

$v_{〃}=(v_b+v_E)\dfrac{R_F}{R_E}$ $v_{\perp}=\sqrt{(v_b+v_E)^2(1-\dfrac{R_E^2}{R_M^2})-2v_E(1-\dfrac{R_E}{R_M})}$

c)

用 V_s 表示以火星为参照系的飞船的速度(见图 16-8，16-9)，V_M 表示火星速度。飞船在脱离火星的引力场时的速度与进入该场时的速度相同。现 在让我们考虑一个等效问题—以速度 V_s 沿着由θ角给出的方向从火星轨道 发射飞船，从 a)的结果，我们知道这个速度应该是

$V_s \geq V_M(-\cos\theta+$ $)$

因此最小速度是 $V_s=V_M(\quad -1)$

太阳参照系和火星参照系的速度之间的关系是

$V_s=$

利用上述条件

$V_s^2\geq V_M^2(\quad -1)^2=V_M^2(3-2\quad)$

和从 b)得到的速度

$V_{//}=V$

$V_\perp^2+V_{//}^2=V^2+2V_M^2-2V_E^2$

由此得到

$V_s^2-V_M^2-2V_MV_{//}+V^2_{//}+V^2_\perp=V_M^2-2V_MV\quad V^2+2V_M^2-2V_E^2\geq 3V_M^2-2V_M^2$

即 $V^2-2V_M\quad V+2\quad V_M^2-2V_E^2\geq 0$

解这个不等式得

$V\geq V_E($ $\quad +$

最小的发射速度是

$v_b=v_E($ $\quad -1+$

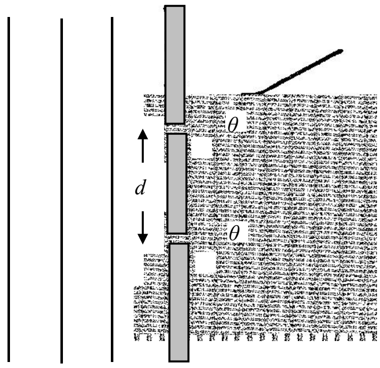
$=v_E\times 0.185=5.5km/s$

d)节省的能量的最大百分比为

$=80\%$

Q1

Figure 1.1



A plane monochromatic light wave, wavelength λ and frequency f, is incident normally on two identical narrow slits, separated by a distance d, as indicated in Figure 1.1. The light wave emerging at each slit is given, at a distance x in a direction θ at time t, by

$$y = a\cos[2\pi(ft - x/\lambda)]$$

where the amplitude a is the same for both waves. (Assume x is much larger than d).

(i) Show that the two waves observed at an angle θ to a normal to the slits, have a resultant amplitude A which can be obtained by adding two vectors, each having magnitude a, and each with an associated direction determined by the phase of the light wave.

Verify geometrically, from the vector diagram, that

$$A = 2a\cos\theta$$

where

$$\beta = \frac{\pi}{\lambda} d\sin\theta$$

(ii) The double slit is replaced by a diffraction grating with N equally spaced slits, adjacent slits being separated by a distance d. Use the vector method of adding amplitudes to show that the vector amplitudes, each of magnitude a, form a part of a regular polygon with vertices on a circle of radius R given by

$$R = \frac{a}{2\sin\beta},$$

Deduce that the resultant amplitude is

$$\frac{a\sin N\beta}{\sin\beta}$$

and obtain the resultant phase difference relative to that of the light from the slit at the edge of the grating.

(iii) Sketch, in the same graph, sin $N\beta$ and ($1/\sin\beta$) as a function of β. On a separate graph show how the intensity of the resultant wave varies as a function of β.

(iv) Determine the intensities of the principal intensity maxima.

(v) Show that the number of principal maxima cannot exceed

$$\left(\frac{2d}{\lambda}+1\right)$$

(vi) Show that two wavelengths λ and $\lambda+\delta\lambda$, where $\delta\lambda \ll \lambda$, produce principal maxima with an angular separation given by

$$\Delta\theta = \frac{n\Delta\lambda}{d\cos\theta} \qquad \text{where } n = 0,\ \pm 1,\ \pm 2....\text{etc}$$

Calculate this angular separation for the sodium D lines for which

$$\lambda = 589.0\text{nm}, \quad \lambda+\Delta\lambda = 589.6\text{nm}, \quad n = 2, \text{ and } d = 1.2\times10^{-6} \text{ m.}$$

$$\left[\text{reminder:} \quad \cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right).\cos\left(\frac{\text{A - B}}{2}\right)\right]$$

Q2

International Physics Olympiad 1956

2. Early this century a model of the earth was proposed in which it was assumed to be a sphere of radius R consisting of a homogeneous isotropic solid mantle down to radius R_c. The core region within radius R_c contained a liquid. Figure 2.1

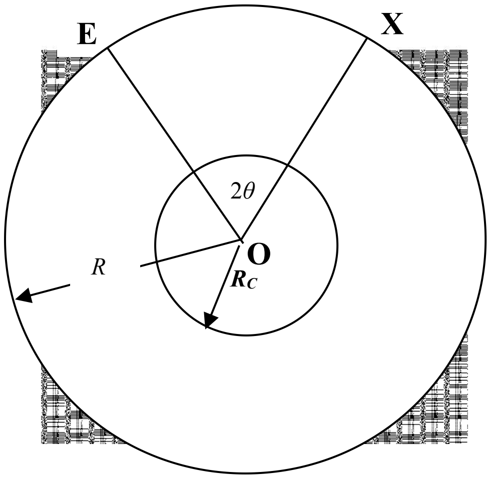


Figure 2.1

The velocities of longitudinal and transverse seismic waves P and S waves respectively, are constant, V_P, and V_S within the mantle. In the core, longitudinal waves have a constant velocity V_{CP}, $< V_P$, and transverse waves are not propagated.

An earthquake at E on the surface of the Earth produces seismic waves that travel through the Earth and are observed by a surface observer who can set up his seismometer at any point X on the Earth's surface. The angular separation between E and X, 2θ given by

$$2\theta = \text{Angle } EOX$$

where O is the centre of the Earth.

(i) Show that the seismic waves that travel through the mantle in a straight line will arrive at X at a time t (the travel time after the earthquake), is given by

$$t = \frac{2R\sin\theta}{v}, \qquad \text{for } \theta > \arccos\left[\frac{R_c}{R}\right],$$

where $v = v_P$ for the P waves and $v = v_S$ for the S waves.

(ii) For some of the positions of X such that the seismic P waves arrive at the observer after two refractions at the mantle-core interface. Draw the path of such a seismic P wave. Obtain a relation between θ and i, the angle of incidence of the seismic P wave at the mantle-core interface, for P waves.

(iii) Using the data

$$
\begin{aligned}
R &= 6370 \text{ km} \\
R_C &= 3470 \text{ km} \\
v_{CP} &= 10.85 \text{ km s}^{-1} \\
v_S &= 6.31 \text{ km s}^{-1} \\
v_{CP} &= 9.02 \text{ km s}^{-1}
\end{aligned}
$$

and the result obtained in (ii),draw a graph of θ against i. Comment on the physical consequences of the form of this graph for observers stationed at different points on the Earth's surface. Sketch the variation of the travel time taken by the P and S waves as a function of θ for $0 \leq \theta \leq 90$ degrees.

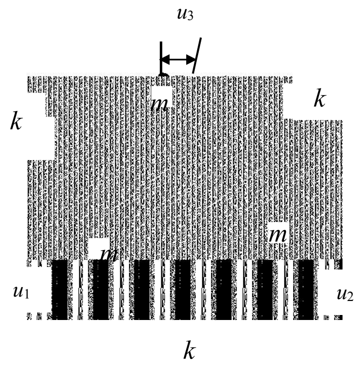
(iv) After an earthquake an observer measures the time delay between the arrival of the S wave, following the P wave, as 2 minutes 11 seconds. Deduce the angular separation of the earthquake from the observer using the data given in Section (iii).

(v) The observer in the previous measurement notices that some time after the arrival of the P and S waves there are two further recordings on the seismometer separated by a time interval of 6 minutes 37 seconds. Explain this result and verify that it is indeed associated with the angular separation determined in the previous section.

Q3

Three particles, each of mass m, are in equilibrium and joined by unstretched massless springs, each with Hooke's Law spring constant k. They are constrained to move in a circular path as indicated in Figure 3.1.

Figure 3.1



(i) If each mass is displaced from equilibrium by small displacements u_1, u_2 and u_3 respectively, write down the equation of motion for each mass.
(ii) Verify that the system has simple harmonic solutions of the form

$$u_n = a_n \cos \omega t \,,$$

with accelerations, $(-\omega^2 u_n)$ where a_n $(n = 1,2,3)$ are constant amplitudes, and ω, the angular frequency, can have 3 possible values,

$$\omega_o \sqrt{3}, \omega_o \sqrt{3} \text{ and } 0. \text{ where } \omega_o^2 = \frac{k}{m}.$$

(iii) The system of alternate springs and masses is extended to N particles, each mass m is joined by springs to its neighbouring masses. Initially the springs are unstretched and in equilibrium. Write down the equation of motion of the nth mass (n = 1,2...N) in terms of its displacement and those of the adjacent masses when the particles are displaced from equilibrium.

$$u_n(t) = a_s \sin\left(\frac{2ns\pi}{N} + \phi\right)\cos \omega_s t,$$

are oscillatory solutions where s = 1, 2,...N , n = 1, 2, ...N and where ϕ is an arbitrary phase, providing the angular frequencies are given by

$$\omega_s = 2\omega_o \sin\left(\frac{s\pi}{N}\right),$$

where a_s $(s = 1,......N)$ are constant amplitudes independent of n.

State the range of possible frequencies for a chain containing an infinite number of masses.

(iv) Determine the ratio

$$u_n / u_{n+1}$$

for large N, in the two cases:

(a) low frequency solutions

(b) $\omega = \omega_{\max}$, where $\omega_{\max}$ is the maximum frequency solution.

Sketch typical graphs indicating the displacements of the particles against particle number along the chain at time t for cases (a) and (b).

(v) If one of the masses is replaced by a mass $m' << m$ estimate any major change one would expect to occur to the angular frequency distribution.

Describe qualitatively the form of the frequency spectrum one would predict for a diatomic chain with alternate masses m and m' on the basis of the previous result.

Reminder

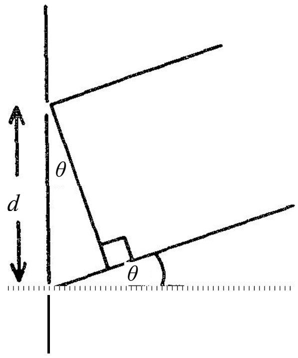
$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$2\sin^2 A = 1 - \cos 2A$$

Answers Question 1

(i) Vector Diagram



If the phase of the light from the first slit is zero, the phase from second slit is

$$\phi = \frac{2\pi}{\lambda} d \sin\theta$$

Adding the two waves with phase difference ϕ where $\xi = 2\pi\left(ft - \frac{x}{\lambda}\right)$,

$$a\cos(\xi+\phi) + a\cos(\xi) = 2a\cos(\phi/2)\left(\xi+\phi/2\right)$$
$$a\cos(\xi+\phi) + a\cos(\xi) = 2a\cos\beta\left\{\cos\left(\xi+\beta\right)\right\}$$

This is a wave of amplitude $A = 2a\cos\beta$ and phase β. From vector diagram, in isosceles triangle OPQ,

$$\beta = \frac{1}{2}\phi = \frac{\pi}{\lambda} d\sin\theta \qquad (NB \quad \phi = 2\beta)$$

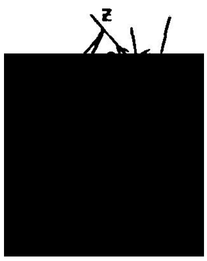
and

$$A = 2a\cos\beta.$$

Thus the sum of the two waves can be obtained by the addition of two vectors of amplitude a and angular directions 0 and ϕ.

(ii) Each slit in diffraction grating produces a wave of amplitude a with phase 2β relative to previous slit wave. The vector diagram consists of a 'regular' polygon with sides of constant length a and with constant angles between adjacent sides.
Let O be the centre of circumscribing circle passing through the vertices of the polygon. Then radial lines such as OS have length R and bisect the internal angles of the polygon. Figure 1.2.

Figure 1.2



$$O\hat{S}T = O\hat{T}S = \frac{1}{2}(180-\phi)$$

$$\text{and} \quad T\hat{O}S = \phi$$

In the triangle *TOS*, for example

$$a = 2R\sin(\phi/2) = 2R\sin\beta \ \text{ as } \ (\phi = 2\beta)$$

$$\therefore R = \frac{a}{2\sin\beta} \qquad (1)$$

As the polygon has *N* faces then:

$$T\hat{O}Z = N(T\hat{O}Z) = N\phi = 2N\beta$$

Therefore in isosceles triangle TOZ, the amplitude of the resultant wave, TZ, is given by

$$2R\sin N\beta \, .$$

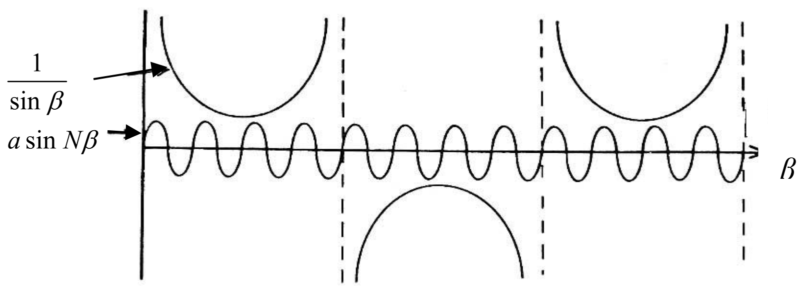
Hence form (1) this amplitude is

$$\frac{a\sin N\beta}{\sin\beta}$$

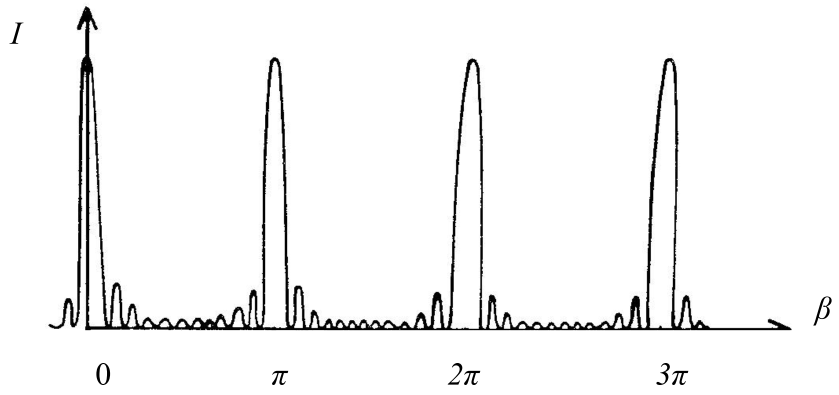
Resultant phase is

$$\begin{aligned}
&= Z\hat{T}S \\
&= O\hat{T}S - O\hat{T}Z \\
&\left(90-\frac{\phi}{2}\right)-\frac{1}{2}(180-N\phi) \\
&-\frac{1}{2}(N-1)\phi \\
&= (N-1)\beta
\end{aligned}$$

(iii)



Intensity $I = \dfrac{a^2\sin^2 N\beta}{\sin^2\beta}$



(iv) For the principle maxima $\beta = \pi p$ where $p = 0 \pm 1 \pm 2........$

$$I_{\max} = a^2\left(\frac{N\beta'}{\beta'}\right) = N^2 a^2 \qquad \beta' = 0 \text{ and } \beta = \pi p + \beta'$$

(v) Adjacent max. estimate I_1 :

$$\sin^2 N\beta = 1, \quad \beta = 2\pi p \mp \frac{3\pi}{2N} \text{ i.e } \beta = \pm\frac{3\pi}{2N}$$

$\left[\beta = \pi p \pm \frac{\pi}{2N}\right]$ does not give a maximum as can be observed from the graph.

$$I_1 = a^2 \frac{1}{\frac{3\pi^2}{2n}} = \frac{a^2 N^2}{23} \text{ for } N >> 1$$

Adjacent zero intensity occurs for $\beta = \pi\rho \pm \frac{\pi}{N}$ i.e. $\delta = \pm\frac{\pi}{N}$

For phase differences much greater than δ, $\mathrm{I} = \mathrm{a}^2\left(\frac{\sin N\beta}{\sin\beta}\right) = a^2$.

(vi)

$$\beta = n\pi \text{ for a principle maximum}$$

$$\text{i.e. } \frac{\pi}{\lambda} d\sin\theta = n\pi \qquad n = 0, \pm1, \pm2..........$$

Differentiating w.r.t, λ

$$d\cos\theta\Delta\theta = n\Delta\lambda$$

$$\Delta\theta = \frac{n\Delta\lambda}{d\cos\theta}$$

Substituting $\lambda = 589.0\text{nm}$, $\lambda + \Delta\lambda = 589.6$ nm. n = 2 and $d = 1.2\times10^{-6}$ m.

$$\Delta\theta = \frac{n\Delta\lambda}{d\sqrt{1-\left(\frac{n\lambda}{d}\right)^2}} \text{ as } \sin\theta = \frac{n\lambda}{d} \text{ and } \cos\theta = \sqrt{1-\left(\frac{n\lambda}{d}\right)^2}$$

$$\Rightarrow \Delta\theta = 5.2\times10^{-3}\, rads \text{ or } 0.30^0$$

Answers Q2

2.(i)

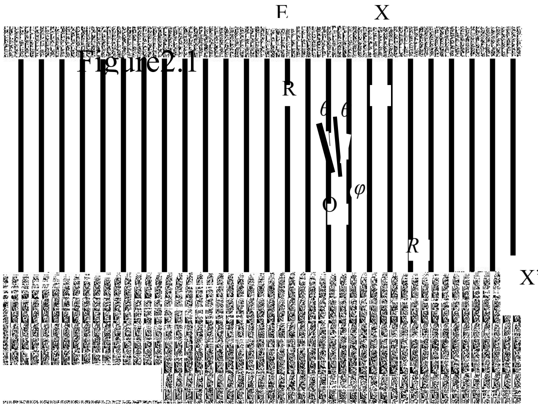


Figure 2.1

$$\mathrm{EX} = 2R\sin\theta \quad \therefore t = \frac{2R\sin\theta}{v}$$

where $v = v_P$ for P waves and $v = v_S$ for S waves.

This is valid providing X is at an angular separation less than or equal to X', the tangential ray to the liquid core. X' has an angular separation given by, from the diagram,

$$2\phi = 2\cos^{-1}\left(\frac{R_C}{R}\right),$$

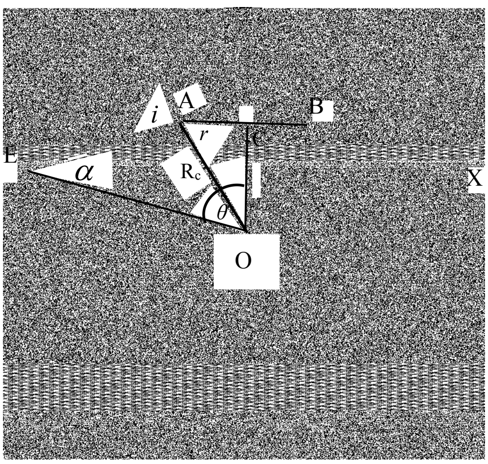
Thus

$$t = \frac{2R\sin\theta}{v} \quad , \qquad \text{for } \theta \le \cos^{-1}\left(\frac{R_C}{R}\right),$$

where $v = v_P$ for P waves and $v = v_S$ for shear waves.

(ii) $\dfrac{R_C}{R} = 0.5447$ and $\dfrac{v_{CP}}{v_P} = 0.831.3$

Figure 2.2



From Figure 2.2

$$\theta = A\hat{O}C + E\hat{O}A \Rightarrow \theta = (90 - r) + (1 - \alpha) \qquad (1)$$

(ii) Continued

Snell's Law gives:

$$\frac{\sin i}{\sin r} = \frac{v_P}{v_{CP}}. \qquad (2)$$

From the triangle EAO, sine rule gives

$$\frac{R_C}{\sin x} = \frac{R}{\sin i}. \qquad (3)$$

Substituting (2) and (3) into (1)

$$\theta = \left[90 - \sin^{-1}\left(\frac{v_{CP}}{v_P} \sin i \right) + i - \sin^{-1}\left(\frac{R_C}{R} \sin i \right) \right] \qquad (4)$$

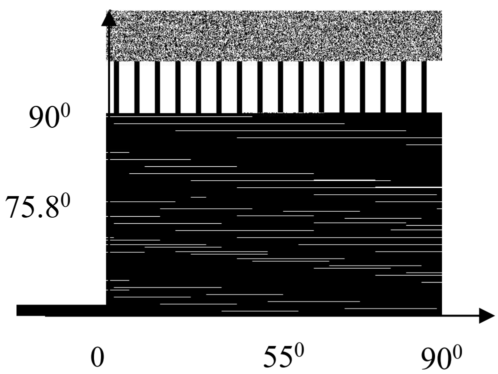
(iii)

<u>For Information Only</u>

For minimum $\theta, \dfrac{d\theta}{di} = 0$. $\quad \Rightarrow 1 - \dfrac{\left(\dfrac{v_{CP}}{v_P}\right)\cos i}{\sqrt{1 - \left(\dfrac{v_{CP}}{v_P}\sin i\right)^2}} - \dfrac{\left(\dfrac{R_C}{R}\right)\cos i}{\sqrt{1 - \left(\dfrac{R_C}{R}\sin i\right)^2}} = 0$

Substituting $i = 55.0^{\circ}$ gives LHS=0, this verifying the minimum occurs at this value of i. Substituting $i = 55.0^{\circ}$ into (4) gives $\theta = 75.8^{0}$.

Plot of θ against i.



Substituting into 4:

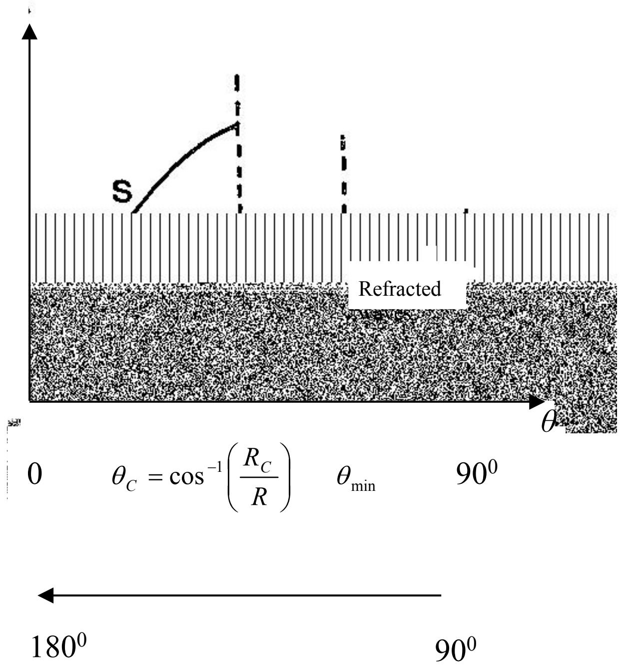
$i = 0$ gives $\theta = 90$

$i = 90°$ gives $\theta = 90.8°$

Substituting numerical values for $i = 0 \rightarrow 90°$ one finds a minimum value at $i = 55^{\circ}$; the minimum values of 0, $\theta_{MIN} = 75•8°$.

Physical Consequence

As θ has a minimum value of 75•8° observers at position for which 2 θ <151•6° will not observe the earthquake as seismic waves are not deviated by angles of less than 151•6°. However for 2 $\theta \leq$ 114° the direct, non-refracted, seismic waves will reach the observer.



(iv) Using the result

$$t = \frac{2r\sin\theta}{v}$$

the time delay Δt is given by

$$\Delta t = 2R\sin\theta\left[\frac{1}{v_S} - \frac{1}{v_P}\right]$$

Substituting the given data

$$131 = 2(6370)\left[\frac{1}{6.31} - \frac{1}{10.85}\right]\sin\theta$$

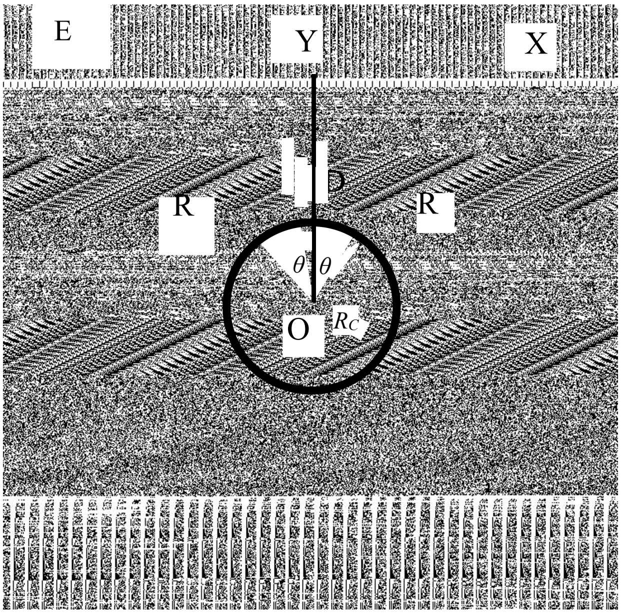
Therefore the angular separation of E and X is

$$2\theta = 17.84^o$$

This result is less than $2\cos^{-1}\left(\frac{R_C}{R}\right) = 2\cos^{-1}\left(\frac{3470}{6370}\right) = 114^o$

And consequently the seismic wave is not refracted through the core.

(v)



The observations are most likely due to reflections from the mantle-core interface. Using the symbols given in the diagram, the time delay is given by

$$\Delta t' = (\mathrm{ED} + \mathrm{DX})\left[\frac{1}{v_S} - \frac{1}{v_P}\right]$$

$$\Delta t' = 2(\mathrm{ED})\left[\frac{1}{v_S} - \frac{1}{v_P}\right] \text{ as ED = EX by symmetry}$$

In the triangle EYD,

$$(\mathrm{ED})^2 = (R\sin\theta)^2 + (R\cos\theta - R_C)^2$$

$$(\mathrm{ED})^2 = R^2 + R_C^2 - 2RR_C\cos\theta \qquad \sin^2\theta + \cos^2\theta = 1$$

Therefore

$$\Delta t' = 2\sqrt{R^2 + R_C^2 - 2RR_C\cos\theta}\left[\frac{1}{v_S} - \frac{1}{v_P}\right]$$

Using (ii)

$$\Delta t' = \frac{\Delta t}{R\sin\theta}\sqrt{R^2 + R_C{}^2 - 2RR_C\cos\theta}$$

$$\Rightarrow 396.7s \text{ or } 6\text{m } 37\text{s}$$

Thus the subsequent time interval, produced by the reflection of seismic waves at the mantle core interface, is consistent with angular separation of 17.84^0.

Answer Q3

Equations of motion:

$$m\frac{d^2u_1}{dt^2} = k(u_2-u_1)+k(u_3-u_1)$$
$$m\frac{d^2u_2}{dt^2} = k(u_3-u_2)+k(u_1-u_2)$$
$$m\frac{d^2u_3}{dt^2} = k(u_1-u_3)+k(u_2-u_3)$$

Substituting $u_n(t) = u_n(0)\cos\omega t$ and $\omega_o^2 = \dfrac{k}{m}$:

$$(2\omega_o^2-\omega^2)u_1(0)-\omega_o^2u_2(0)-\omega_o^2u_3(0)=0 \quad \text{(a)}$$
$$-\omega_o^2u_1(0)+(2\omega_o^2-\omega^2)u_2(0)-\omega_o^2u_3(0)=0 \quad \text{(b)}$$
$$-\omega_o^2u_1(0)-\omega_o^2u_2(0)+(2\omega_o^2-\omega^2)u_3(0)=0 \quad \text{(c)}$$

Solving for $u_1(0)$ and $u_2(0)$ in terms of $u_3(0)$ using (a) and (b) and substituting into (c) gives the equation equivalent to

$$(3\omega_o^2-\omega^2)^2\omega^2=0$$
$$\omega^2 = 3\omega_o^2,\ 3\omega_o^2 \text{ and } 0$$
$$\omega = \sqrt{3}\omega_o,\ \sqrt{3}\omega_o \text{ and } 0$$

(ii) Equation of motion of the n'th particle:

$$m\frac{d^2u_n}{dt^2} = k(u_{1+n}-u_n)+k(u_{n-1}-u_n) \qquad \boxed{n = 1,2\ldots\ldots N}$$
$$\frac{d^2u_n}{dt^2} = k(u_{1+n}-u_n)+\omega_o^2(u_{n-1}-u_n)$$

Substituting $u_n(t) = u_n(0)\sin\left(2ns\dfrac{\pi}{N}\right)\cos\omega_s t$

$$-\omega_s^2\left(\sin\left(2ns\frac{\pi}{N}\right)\right) = \omega_o^2\left[\sin\left(2(n+1)s\frac{\pi}{N}\right)-2\sin\left(2ns\frac{\pi}{N}\right)+\sin\left(2(n-1)s\frac{\pi}{N}\right)\right]$$
$$-\omega_s^2\left(\sin\left(2ns\frac{\pi}{N}\right)\right) = 2\omega_o^2\left[\frac{1}{2}\sin\left(2(n+1)s\frac{\pi}{N}\right)+\sin\left(2ns\frac{\pi}{N}\right)-\frac{1}{2}\sin\left(2(n-1)s\frac{\pi}{N}\right)\right]$$
$$-\omega_s^2\left(\sin\left(2ns\frac{\pi}{N}\right)\right) = 2\omega_o^2\left[\sin\left(2ns\frac{\pi}{N}\right)\cos\left(2s\frac{\pi}{N}\right)-\sin\left(2ns\frac{\pi}{N}\right)\right]$$
$$\therefore \omega_s^2 = 2\omega_o^2\left[1-\cos\left(2s\frac{\pi}{N}\right)\right]: \quad (s=1,2,\ldots..N)$$

As $2\sin^2\theta = 1-\cos 2\theta$

This gives

$$\omega_s = 2\omega_o\sin\left(\frac{s\pi}{N}\right) \quad (s=1,2,\ldots N)$$

ω_s can have values from 0 to $2\omega_o = 2\sqrt{\dfrac{k}{m}}$ when $N\to\infty$; corresponding to range $s=1$ to $\dfrac{N}{2}$.

(iv) For s'th mode

$$\frac{u_n}{u_{n+1}} = \frac{\sin\left(2ns\dfrac{\pi}{N}\right)}{\sin\left(2(n+1)s\dfrac{\pi}{N}\right)}$$

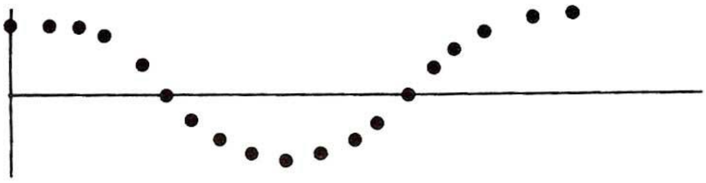
$$\frac{u_n}{u_{n+1}} = \frac{\sin\left(2ns\dfrac{\pi}{N}\right)}{\sin\left(2ns\dfrac{\pi}{N}\right)\cos\left(2s\dfrac{\pi}{N}\right)+\cos\left(2ns\dfrac{\pi}{N}\right)\sin\left(2s\dfrac{\pi}{N}\right)}$$

(a) For small ω, $\left(\dfrac{s}{N}\right)\approx 0$, thus $\cos\left(2ns\dfrac{\pi}{N}\right)\cong 1$ and $=\sin\left(2ns\dfrac{\pi}{N}\right)\approx 0$, and so $\dfrac{u_n}{u_{n+1}}\cong 1$.

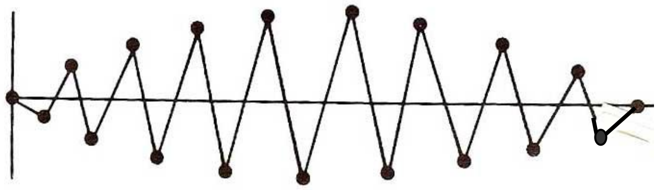
(b) The highest mode, $\omega_{\max} = 2\omega_o$, corresponds to $s = N/2$

$$\therefore \frac{u_n}{u_{n+1}} = -1 \quad \text{as} \quad \frac{\sin(2n\pi)}{\sin(2(n+1)\pi)} = -1$$

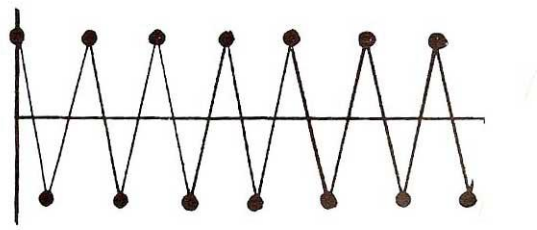
Case (a)



Case (b)
N odd

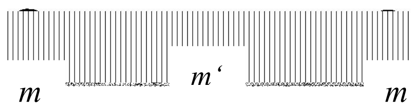


N even



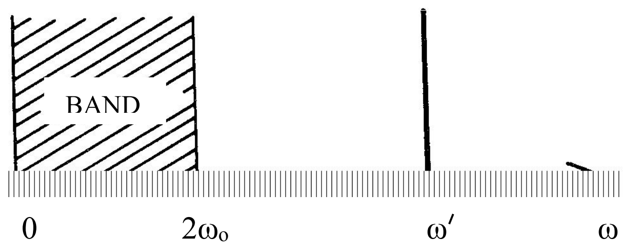
(vi) If $m' << m$, one can consider the frequency associated with m' as due to vibration of m' between two adjacent, much heavier, masses which can be considered stationary relative to m'.

The normal mode frequency of m', in this approximation, is given by



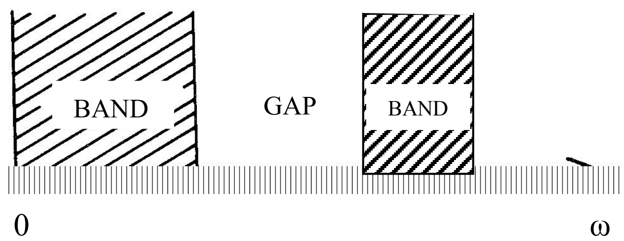
$$m'\ddot{x} = -2kx$$
$$\omega'^2 = \frac{2k}{m}$$
$$\omega' = \sqrt{\frac{2k}{m'}}$$

For small m', ω' will be much greater than ω_{max},



DIATOMIC SYSTEM

More light masses, m', will increase the number of frequencies in region of ω' giving a band-gap-band spectrum.



Problems of the 18th International Physics Olympiad
(Jena, 1987)

Gunnar Friege & Gunter Lind[1]

Leibniz-Institute for Science Education (IPN) at the University of Kiel, Germany

Abstract

The 18th International Physics Olympiad took place in 1987 in the German Democratic Republic (GDR). This article contains the competition problems, their solutions and also a (rough) grading scheme.

Introduction

The 18th international Physics olympics in 1987 was the second International Physics Olympiad hosted by the German Democratic Republic (GDR) . The organisation was lead by the ministry for education and the problems were formulated by a group of professors of different universities. However, the main part of the work was done by the physics department of the university of Jena. The company Carl-Zeiss and a special scientific school in Jena were involved also.

In the competition three theoretical and one experimental problem had to be solved. The theoretical part was quite difficult. Only the first of the three problems ("ascending moist air") had a medium level of difficulty. The points given in the markings were equal distributed. Therefore, there were lots of good but also lots of unsatisfying solutions. The other two theoretical problems were rather difficult. About half of the pupils even did not find an adequate start in solving these problems. The third problem ("infinite LC-grid") revealed quite a few complete solutions. The high level of difficulty can probably be explained with the fact that many pupils nearly had no experience with the subject. Concerning the second problem ("electrons in a magnetic field") only a few pupils worked on the last part 3 (see below).

The experimental problem ("refracting indices") was much more easier than the theoretical problems. There were lots of different possibilities of solution and most of the pupils

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[1] Contact: Leibniz-Institute for Science Education (IPN) at the University of Kiel
Olshausenstrasse 62, 24098 Kiel, Germany
ipho@ipn.uni-kiel.de

managed to come up with partial or complete solutions. Over the half of all teams got more points in the experimental part than in the theoretical part of the competition.

The problems and their solutions are based on the original German and English versions of the competition problems. Only minor changes have been made. Despite the fact that nowadays almost all printed figures are generated with the aid of special computer programmes, the original hand-made figures are published here.

Theoretical Problems

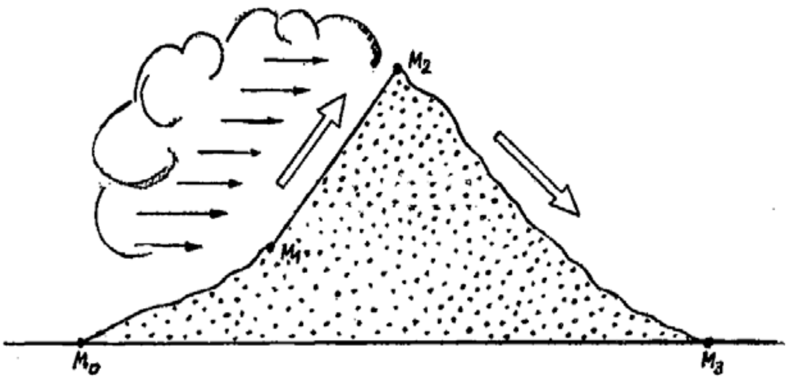
Problem 1: Ascending moist air

Moist air is streaming adiabatically across a mountain range as indicated in the figure.

Equal atmospheric pressures of 100 kPa are measured at meteorological stations M_0 and M_3 and a pressure of 70 kPa at station M_2. The temperature of the air at M_0 is 20° C.

As the air is ascending, cloud formation sets in at 84.5 kPa.

Consider a quantity of moist air ascending the mountain with a mass of 2000 kg over each square meter. This moist air reaches the mountain ridge (station M_2) after 1500 seconds. During that rise an amount of 2.45 g of water per kilogram of air is precipitated as rain.



1. Determine temperature T_1 at M_1 where the cloud ceiling forms.
2. What is the height h_1 (at M_1) above station M_0 of the cloud ceiling assuming a linear decrease of atmospheric density?
3. What temperature T_2 is measured at the ridge of the mountain range?
4. Determine the height of the water column (precipitation level) precipitated by the air stream in 3 hours, assuming a homogeneous rainfall between points M_1 and M_2.

5. What temperature T_3 is measured in the back of the mountain range at station M_3? Discuss the state of the atmosphere at station M_3 in comparison with that at station M_0.

Hints and Data

The atmosphere is to be dealt with as an ideal gas. Influences of the water vapour on the specific heat capacity and the atmospheric density are to be neglected; the same applies to the temperature dependence of the specific latent heat of vaporisation. The temperatures are to be determined to an accuracy of 1 K, the height of the cloud ceiling to an accuracy of 10 m and the precipitation level to an accuracy of 1 mm.

Specific heat capacity of the atmosphere in the pertaining temperature range:

$c_p = 1005\ \mathrm{J \cdot kg^{-1} \cdot K^{-1}}$

Atmospheric density for p_0 and T_0 at station M_0: $\rho_0 = 1.189\ \mathrm{kg \cdot m^{-3}}$

Specific latent heat of vaporisation of the water within the volume of the cloud:

$L_v = 2500\ \mathrm{kJ \cdot kg^{-1}}$

$\frac{c_p}{c_v} = \chi = 1.4$ and $g = 9.81\ \mathrm{m \cdot s^{-2}}$

Solution of problem 1:

1. Temperature T_1 where the cloud ceiling forms

$$T_1 = T_0 \cdot \left(\frac{p_1}{p_0}\right)^{1-\frac{1}{\chi}} = 279\,\mathrm{K} \qquad (1)$$

2. Height h_1 of the cloud ceiling:

$$p_0 - p_1 = \frac{\rho_0 + \rho_1}{2} \cdot g \cdot h_1, \text{ with } \rho_1 = \rho_0 \cdot \frac{p_1}{p_0} \cdot \frac{T_0}{T_1}.$$

$$h_1 = 1410\,\mathrm{m} \qquad (2)$$

3. Temperature T_2 at the ridge of the mountain.

   The temperature difference when the air is ascending from the cloud ceiling to the mountain ridge is caused by two processes:

   – adiabatic cooling to temperature T_x,

– heating by ΔT by condensation.

$$T_2 = T_x + \Delta T \tag{3}$$

$$T_x = T_1 \cdot \left(\frac{p_2}{p_1}\right)^{1-\frac{1}{\chi}} = 265\,\mathrm{K} \tag{4}$$

For each kg of air the heat produced by condensation is $L_v \cdot 2.45$ g = 6.125 kJ.

$$\Delta T = \frac{6.125}{c_p} \cdot \frac{\mathrm{kJ}}{\mathrm{kg}} = 6.1\,\mathrm{K} \tag{5}$$

$$T_2 = 271\ \mathrm{K} \tag{6}$$

4. Height of precipitated water column

$$h = 35\ \mathrm{mm} \tag{7}$$

5. Temperature T_3 behind the mountain

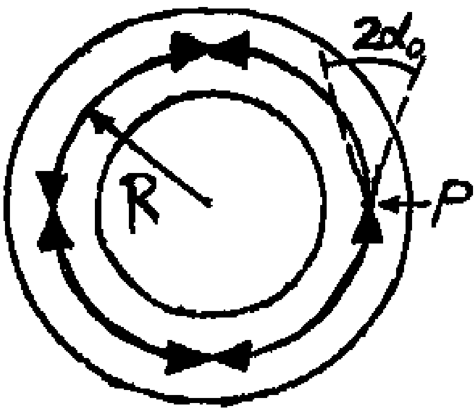
$$T_3 = T_2 \cdot \left(\frac{p_3}{p_2}\right)^{1-\frac{1}{\chi}} = 300\,\mathrm{K} \tag{8}$$

The air has become warmer and dryer. The temperature gain is caused by condensation of vapour.

Problem 2: Electrons in a magnetic field

A beam of electrons emitted by a point source P enters the magnetic field $\vec{B}$ of a toroidal coil (toroid) in the direction of the lines of force. The angle of the aperture of the beam $2 \cdot \alpha_0$ is assumed to be small ($2 \cdot \alpha_0 \ll 1$). The injection of the electrons occurs on the mean radius R of the toroid with acceleration voltage V_0.

Neglect any interaction between the electrons. The magnitude of $\vec{B}$, B, is assumed to be constant.



1. To guide the electron in the toroidal field a homogeneous magnetic deflection field $\vec{B}_1$ is required. Calculate $\vec{B}_1$ for an electron moving on a circular orbit of radius R in the torus.

2. Determine the value of $\vec{B}$ which gives four focussing points separated by $\pi/2$ as indicated in the diagram.

   Note: When considering the electron paths you may disregard the curvature of the magnetic field.

3. The electron beam cannot stay in the toroid without a deflection field $\vec{B}_1$, but will leave it with a systematic motion (drift) perpendicular to the plane of the toroid.

   a) Show that the radial deviation of the electrons from the injection radius is finite.

   b) Determine the direction of the drift velocity.

   Note: The angle of aperture of the electron beam can be neglected. Use the laws of conservation of energy and of angular momentum.

Data:

$\frac{e}{m} = 1.76 \cdot 10^{11}\, C \cdot kg^{-1}$; V_0 = 3 kV; R = 50 mm

Solution of problem 2:

1. Determination of B:

The vector of the velocity of any electron is divided into components parallel with and perpendicular to the magnetic field $\vec{B}$:

$$\vec{v} = \vec{v}_{\parallel} + \vec{v}_{\perp} \tag{1}$$

The Lorentz force $\vec{F} = -e \cdot (\vec{v} \times \vec{B})$ influences only the perpendicular component, it acts as a radial force:

$$m \cdot \frac{v_{\perp}^{2}}{r} = e \cdot v_{\perp} \cdot B \tag{2}$$

Hence the radius of the circular path that has been travelled is

$$r = \frac{m}{e} \cdot \frac{v_{\perp}}{B} \tag{3}$$

and the period of rotation which is independent of $v_\perp$ is

$$T = \frac{2\cdot\pi\cdot r}{v_\perp} = \frac{2\cdot\pi\cdot m}{B\cdot e} \tag{4}$$

The parallel component of the velocity does not vary. Because of $\alpha_0 << 1$ it is approximately equal for all electrons:

$$v_{\parallel 0} = v_0 \cdot \cos\alpha_0 \approx v_0 \tag{5}$$

Hence the distance b between the focusing points, using eq. (5), is

$$b = v_{\parallel 0} \cdot T \approx v_0 \cdot T \tag{6}$$

From the law of conservation of energy follows the relation between the acceleration voltage V_0 and the velocity v_0:

$$\frac{m}{2} \cdot {v_0}^2 = e \cdot V_0 \tag{7}$$

Using eq. (7) and eq. (4) one obtains from eq. (6)

$$b = \frac{2\cdot\pi}{B} \cdot \sqrt{2 \cdot \frac{m}{e} \cdot V_0} \tag{8}$$

and because of $b = \frac{2\cdot\pi\cdot R}{4}$ one obtains

$$B = \frac{4}{R} \cdot \sqrt{2 \cdot \frac{m}{e} \cdot V_0} = 1.48 \cdot 10^{-2} \, \frac{Vs}{m^2} \tag{9}$$

2. Determination of B_1:

   Analogous to eq. (2)

$$m \cdot \frac{v_0^2}{R} = e \cdot v_0 \cdot B_1 \tag{10}$$

   must hold.

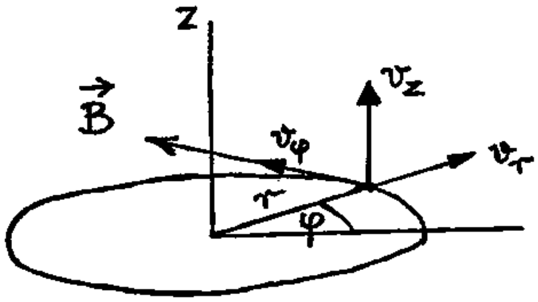
   From eq. (7) follows

$$B_1 = \frac{1}{R} \cdot \sqrt{2 \cdot \frac{m}{e} \cdot V_0} = 0.37 \cdot 10^{-2} \, \frac{Vs}{m^2} \tag{11}$$

3. Finiteness of r_1 and direction of the drift velocity

In the magnetic field the lines of force are circles with their centres on the symmetry axis (z-axis) of the toroid.

In accordance with the symmetry of the problem, polar coordinates r and φ are introduced into the plane perpendicular to the z-axis (see figure below) and the occurring vector quantities (velocity, magnetic field $\vec{B}$, Lorentz force) are divided into the corresponding components.



Since the angle of aperture of the beam can be neglected examine a single electron injected tangentially into the toroid with velocity v_0 on radius R.

In a static magnetic field the kinetic energy is conserved, thus

$$E = \frac{m}{2}\left(v_r^2 + v_\varphi^2 + v_z^2\right) = \frac{m}{2} v_0^2 \tag{12}$$

The radial points of inversion of the electron are defined by the condition

$$v_r = 0$$

Using eq. (12) one obtains

$$v_0^2 = v_\varphi^2 + v_z^2 \tag{13}$$

Such an inversion point is obviously given by

$$r = R \cdot \left(v_\varphi = v_0,\ v_r = 0,\ v_z = 0\right).$$

To find further inversion points and thus the maximum radial deviation of the electron the components of velocity v_φ and v_z in eq. (13) have to be expressed by the radius.

v_φ will be determined by the law of conservation of angular momentum. The Lorentz force obviously has no component in the φ - direction (parallel to the magnetic field). Therefore it cannot produce a torque around the z-axis. From this follows that the z-component of the angular momentum is a constant, i.e. $L_z = m \cdot v_\varphi \cdot r = m \cdot v_0 \cdot R$ and

therefore $v_\varphi = v_0 \cdot \dfrac{R}{r}$ (14)

v_z will be determined from the equation of motion in the z-direction. The z-component of the Lorentz force is $F_z = - e \cdot B \cdot v_r$. Thus the acceleration in the z-direction is

$$a_z = -\frac{e}{m} \cdot B \cdot v_r \,. \tag{15}$$

That means, since B is assumed to be constant, a change of v_z is related to a change of r as follows:

$$\Delta v_z = -\frac{e}{m} \cdot B \cdot \Delta r$$

Because of $\Delta r = r - R$ and $\Delta v_z = v_z$ one finds

$$v_z = -\frac{e}{m} \cdot B \cdot (r - R) \tag{16}$$

Using eq. (14) and eq. (15) one obtains for eq. (13)

$$1 = \left(\frac{R}{r}\right)^2 + A^2 \cdot \left(\frac{r}{R} - 1\right)^2 \tag{17}$$

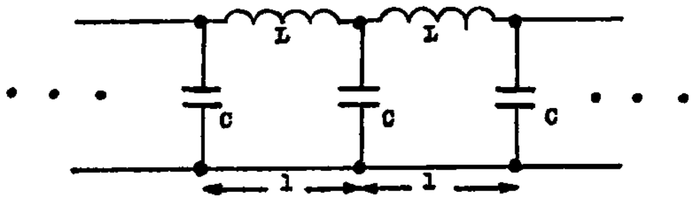
where $A = \dfrac{e}{m} \cdot B \cdot \dfrac{R}{v_0}$

Discussion of the curve of the right side of eq. (17) gives the qualitative result shown in the following diagram:

Hence r_1 is finite. Since $R \le r \le r_1$ eq. (16) yields $v_z < 0$. Hence the drift is in the direction of the negative z-axis.

Problem 3: Infinite LC-grid

When sine waves propagate in an infinite LC-grid (see the figure below) the phase of the ac-voltage across two successive capacitors differs by Φ.



a) Determine how Φ depends on ω, L and C (ω is the angular frequency of the sine wave).

b) Determine the velocity of propagation of the waves if the length of each unit is ℓ .

c) State under what conditions the propagation velocity of the waves is almost independent of ω. Determine the velocity in this case.

d) Suggest a simple mechanical model which is an analogue to the above circuit and derive equations which establish the validity of your model.

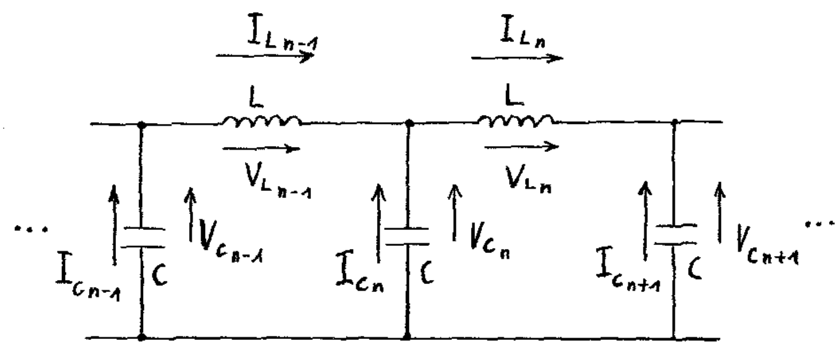
Formulae:

$$\cos\alpha - \cos\beta = -2\cdot\sin\left(\frac{\alpha+\beta}{2}\right)\cdot\sin\left(\frac{\alpha-\beta}{2}\right)$$

$$\sin\alpha - \sin\beta = 2\cdot\cos\left(\frac{\alpha+\beta}{2}\right)\cdot\sin\left(\frac{\alpha-\beta}{2}\right)$$

Solution of problem 3:

a)



Current law: $\quad I_{L_{n-1}} + I_{C_n} - I_{L_n} = 0 \qquad (1)$

Voltage law: $\quad V_{C_{n-1}} + V_{L_{n-1}} - V_{C_n} = 0 \qquad (2)$

Capacitive voltage drop: $V_{C_{n-1}} = \dfrac{1}{\omega \cdot C} \cdot \tilde{I}_{C_{n-1}}$ (3)

Note: In eq. (3) $\tilde{I}_{C_{n-1}}$ is used instead of $I_{C_{n-1}}$ because the current leads the voltage by 90°.

Inductive voltage drop: $V_{L_{n-1}} = \omega \cdot L \cdot \tilde{I}_{L_{n-1}}$ (4)

Note: In eq. (4) $\tilde{I}_{L_{n-1}}$ is used instead of $I_{L_{n-1}}$ because the current lags behind the voltage by 90°.

The voltage V_{C_n} is given by: $V_{C_n} = V_0 \cdot \sin(\omega \cdot t + n \cdot \varphi)$ (5)

Formula (5) follows from the problem.

From eq. (3) and eq. (5): $I_{C_n} = \omega \cdot C \cdot V_0 \cdot \cos(\omega \cdot t + n \cdot \varphi)$ (6)

From eq. (4) and eq. (2) and with eq. (5)

$$I_{L_{n-1}} = \frac{V_0}{\omega \cdot L} \cdot \left[2 \cdot \sin\left(\omega \cdot t + \left(n - \frac{1}{2} \right) \cdot \varphi \right) \cdot \sin \frac{\varphi}{2} \right] \tag{7}$$

$$I_{L_n} = \frac{V_0}{\omega \cdot L} \cdot \left[2 \cdot \sin\left(\omega \cdot t + \left(n + \frac{1}{2} \right) \cdot \varphi \right) \cdot \sin \frac{\varphi}{2} \right] \tag{8}$$

Eqs. (6), (7) and (8) must satisfy the current law. This gives the dependence of φ on ω, L and C.

$$0 = V_0 \cdot \omega \cdot C \cdot \cos(\omega \cdot t + n \cdot \varphi) + 2 \cdot \frac{V_0}{\omega \cdot L} \cdot \sin \frac{\varphi}{2} \cdot \left[2 \cdot \cos(\omega \cdot t + n \cdot \varphi) \cdot \sin\left(-\frac{\varphi}{2} \right) \right]$$

This condition must be true for any instant of time. Therefore it is possible to divide by $V_0 \cdot \cos(\omega \cdot t + n \cdot \varphi)$.

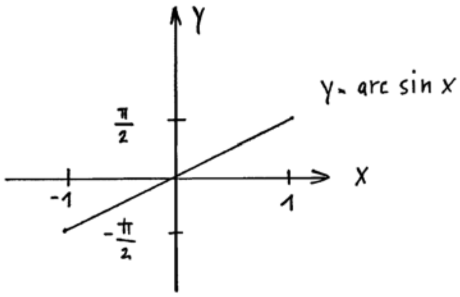
Hence $\omega^2 \cdot L \cdot C = 4 \cdot \sin^2\left(\frac{\varphi}{2}\right)$. The result is

$$\varphi = 2 \cdot \arcsin\left(\frac{\omega \cdot \sqrt{L \cdot C}}{2} \right) \quad \text{with} \quad 0 \le \omega \le \frac{2}{\sqrt{L \cdot C}} \tag{9}$$

b) The distance ℓ is covered in the time Δt thus the propagation velocity is

$$v = \frac{\ell}{\Delta t} = \frac{\omega \cdot \ell}{\varphi} \quad \text{or} \quad v = \frac{\omega \cdot \ell}{2 \cdot \arcsin\left(\dfrac{\omega \cdot \sqrt{L \cdot C}}{2} \right)} \tag{10}$$

c)



Slightly dependent means arc sin $\left(\frac{\omega \cdot \sqrt{L \cdot C}}{2}\right) \sim \omega$, since v is constant in that case.

This is true only for small values of ω. That means $\frac{\omega \cdot \sqrt{L \cdot C}}{2} \ll 1$ and therefore

$$v_0 = \frac{\ell}{\sqrt{L \cdot C}} \tag{11}$$

d) The energy is conserved since only inductances and capacitances are involved. Using the terms of a) one obtains the capacitive energy

$$W_C = \sum_n \frac{1}{2} \cdot C \cdot V_{C_n}{}^2 \tag{12}$$

and the inductive energy

$$W_L = \sum_n \frac{1}{2} \cdot L \cdot I_{L_n}{}^2 \tag{13}$$

From this follows the standard form of the law of conservation of energy

$$W_C = \sum_n \frac{1}{2}\left(C \cdot V_{C_n}{}^2 + L \cdot I_{L_n}{}^2\right) \tag{14}$$

The relation to mechanics is not recognizable in this way since two different physical quantities (V_{C_n} and I_{L_n}) are involved and there is nothing that corresponds to the relation between the locus x and the velocity v = $\dot{x}$.

To produce an analogy to mechanics the energy has to be described in terms of the charge Q, the current I = $\dot{Q}$ and the constants L and C. For this purpose the voltage V_{C_n} has to be expressed in terms of the charges Q_{L_n} passing through the coil.

One obtains:

$$W = \sum_n \left[\underbrace{\frac{L}{2} \cdot \dot{Q}_{L_n}^{\ 2}}_{A} + \underbrace{\frac{1}{2 \cdot C} \left(Q_{L_n} - Q_{L_{n-1}} \right)^2}_{B} \right] \tag{15}$$

Mechanical analogue:

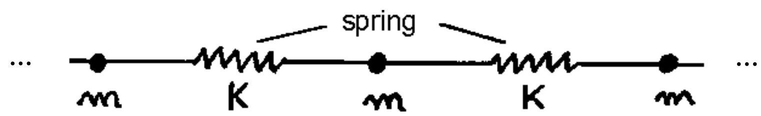
A (kinetic part): $\dot{Q}_{L_n} \longrightarrow v_n$; $L \longrightarrow m$

B (potential part): $Q_{L_n} \longrightarrow x_n$

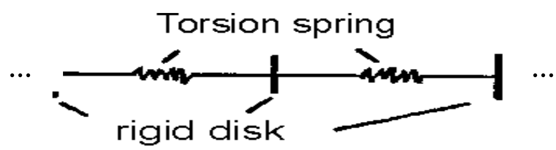
x_n: displacement and v_n: velocity.

However, Q_{L_n} could equally be another quantity (e.g. an angle). L could be e.g. a moment of inertia.

From the structure of the problems follows: Interaction only with the nearest neighbour (the force rises linearly with the distance). A possible model could be:



Another model is:



Experimental Problems

Problem 4: Refractive indices

Find the refractive indices of a prism, n_p, and a liquid, n_l. Ignore dispersion.

a) Determine the refractive index n_p of a single prism by <u>two</u> different experimental methods.
   Illustrate your solution with accurate diagrams and deduce the relations necessary to calculate the refractive index. (One prism only should be used).

b) Use two identical prisms to determine the refractive index n_L of a liquid with $n_L < n_p$. Illustrate your solution with accurate diagrams and deduce the relations necessary to calculate the refractive index.

Apparatus:

Two identical prisms with angles of 30°, 60° and 90°; a set square, a glass dish, a round table, a liquid, sheets of graph paper, other sheets of paper and a pencil.

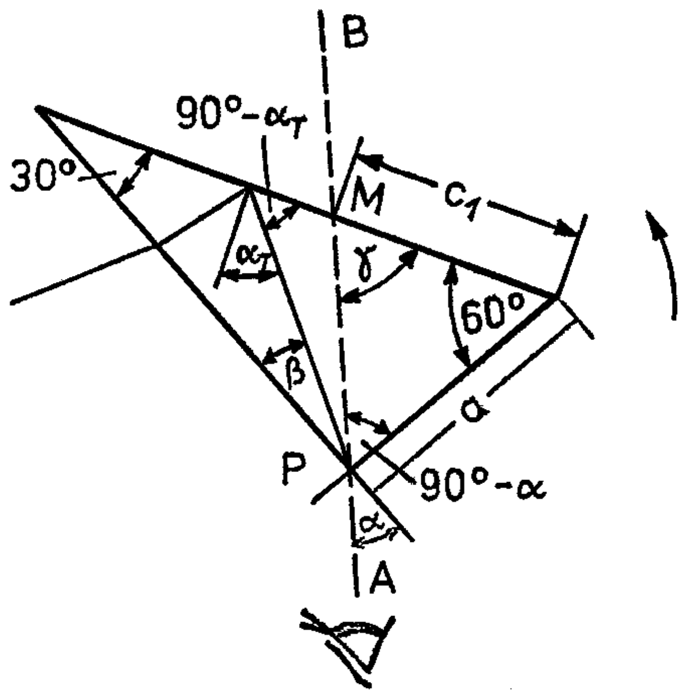
Formulae: $\sin(\alpha \pm \beta) = \sin\alpha \cdot \cos\beta \pm \cos\alpha \cdot \sin\beta$

Additional remarks: You may mark the opaque sides of the prisms with a pencil. The use of the lamp is optional.

Solution of problem 4:

a) Calculation of the refractive index of the prism

First method:



Draw a straight line A – B on a sheet of paper and let this be your line of sight. Place the prism with its rectangular edge facing you onto the line (at point P on the line). Now turn the prism in the direction of the arrow until the dark edge of total reflection which can be seen in the short face of the prism coincides with the 90° edge of the prism. Mark a point M and measure the length c_1. Measure also the length of the short face of the prism.

The following equations apply:

$$\sin\alpha_T = \frac{1}{n_p} \tag{1}$$

$$\frac{\sin\alpha}{\sin\beta} = n_p \tag{2}$$

$$\beta = 60° - \alpha_T \tag{3}$$

$$\gamma = 30° + \alpha \tag{4}$$

$$\frac{\sin\gamma}{\sin(90° - \alpha)} = \frac{a}{c_1} \tag{5}$$

From eq. (5) follows with eq. (4) and the given formulae:

$$\frac{a}{c_1} \cdot \cos\alpha = \sin(30° + \alpha) = \frac{1}{2} \cdot \cos\alpha + \frac{1}{2} \cdot \sqrt{3} \cdot \sin\alpha$$

$$\sin\alpha = \frac{2a - c_1}{2 \cdot \sqrt{a^2 - a \cdot c_1 + c_1^{\,2}}} \tag{6}$$

From eqs. (2), (3) and (1) follows:

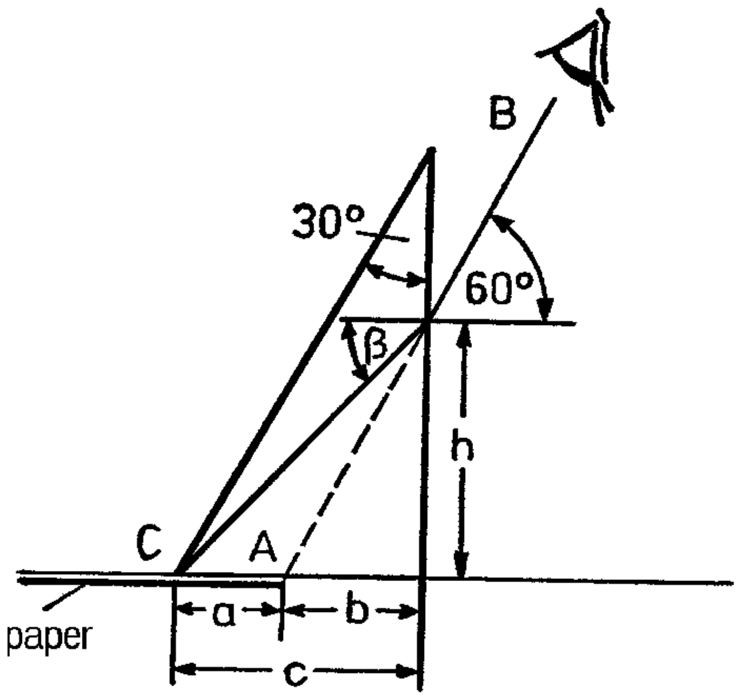
$$\sin\alpha = n_p \cdot \sin(60° - \alpha_T) = \frac{n_p}{2} \cdot \left(\sqrt{3} \cdot \cos\alpha_T - \sin\alpha_T\right)$$

$$n_p = +\left\{\frac{1}{3} \cdot (2 \cdot \sin\alpha + 1)^2 + 1\right\}^{1/2} \tag{7}$$

When measuring c_1 and a one notices that within the error limits of ± 1 mm a equals c_1.

Hence: $\sin\alpha = \frac{1}{2}$ and $n_p = 1.53$. (8)

Second method:



Place edge C of the prism on edge A of a sheet of paper and look along the prism hypotenuse at edge A so that your direction of sight B-A and the table surface form an angle of 60°. Then shift the prism over the edge of the paper into the position shown, such that prism edge C can be seen inside the prism collinear with edge A of the paper outside the prism. The direction of sight must not be changed while the prism is being displaced.

The following equations apply:

$$\left.\begin{array}{l}\tan\beta = \dfrac{h}{c}\\ \tan 60° = \sqrt{3} = \dfrac{h}{b}\end{array}\right\} => h = b\cdot\sqrt{3} = \frac{c\cdot\sin\beta}{\sqrt{1-\sin^2\beta}} \tag{9}$$

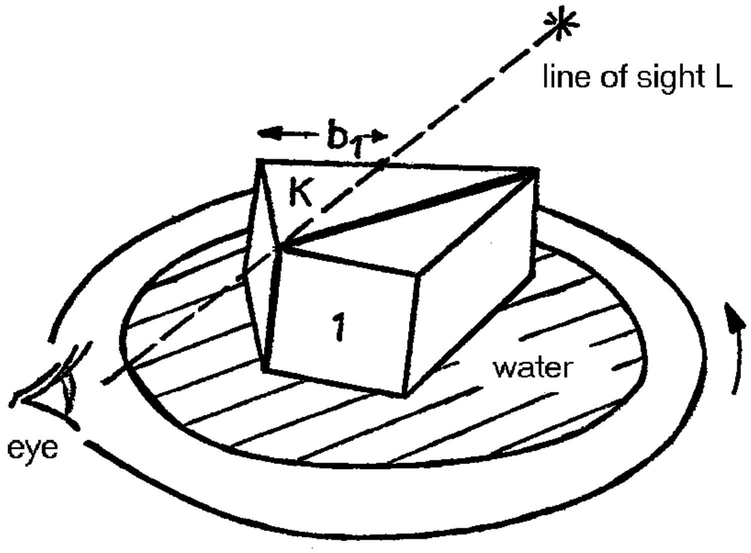
$$\sin\beta = \sin 60°\cdot\frac{1}{n_p} = \frac{\sqrt{3}}{2\cdot n_p} \tag{10}$$

$$n_p = \frac{1}{2}\cdot\sqrt{\left(\frac{c}{b}\right)^2 + 3} \tag{11}$$

With the measured values c = 29 mm and b = 11.5 mm, it follows

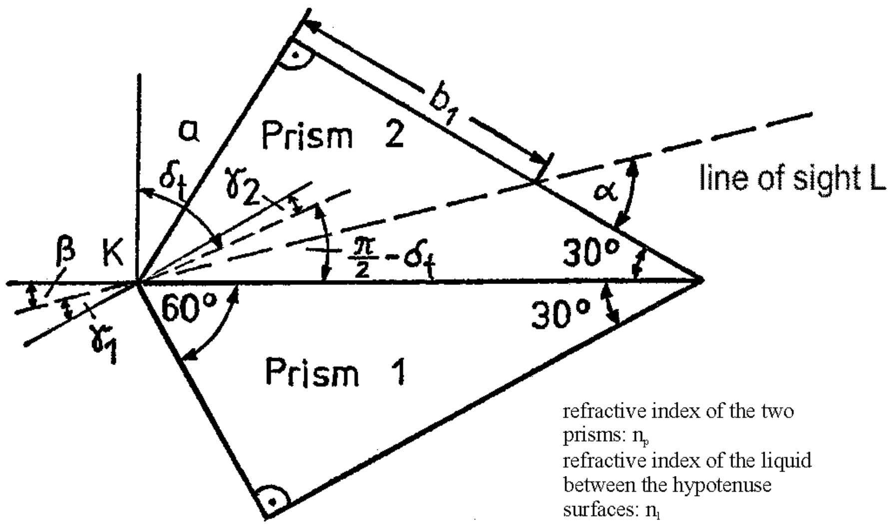
$$n_p = 1.53. \tag{12}$$

b) Determination of the refractive index of the liquid by means of two prisms



Place the two prisms into a glass dish filled with water as shown in the figure above. Some water will rise between the hypotenuse surfaces. By pressing and moving the prisms slightly against each other the water can be made to cover the whole surface. Look over the 60° edges of the prisms along a line of sight L (e.g. in the direction of a fixed point on an illuminated wall). Turn the glass dish together with the two prisms in such a way that the dark shadow of total reflection which can be seen in the short face of prism 1 coincides with the 60° edge of that prism (position shown in the figure below).

While turning the arrangement take care to keep the 60° edge (point K) on the line of sight L. In that position measure the length b_1 with a ruler (marking, reading). The figure below illustrates the position described.



If the refractive index of the prism is known (see part a) the refractive index of the liquid may be calculated as follows:

$$\sin\alpha = \frac{a}{\sqrt{a^2 + b_1^{\,2}}} \tag{13}$$

$$\beta = \alpha - 30°;\ \ \gamma_1 = 30° - \beta = 60° - \alpha \tag{14, 15}$$

$$\frac{\sin\gamma_1}{\sin\gamma_2} = n_p \quad \text{refraction at the short face of prism 1.} \tag{16}$$

The angle of total reflection δ_t at the hypotenuse surface of prism 1 in the position described is:

$$\frac{\pi}{2} - \delta_t = 30° - \gamma_2 \tag{17}$$

$$\delta_t = 60° + \arcsin\left(\frac{\sin\gamma_1}{n_p}\right) \tag{18}$$

From this we can easily obtain n_1:

$$n_1 = n_p \cdot \sin\delta_t = n_p \cdot \sin\left\{60° + \arcsin\frac{\sin\gamma_1}{n_p}\right\} \tag{19}$$

Numerical example for water as liquid:

b_1 = 1.9 cm; α = 55.84°; γ_1 = 4.16°; δ_t = 62.77°; a = 2.8 cm; with n_p = 1.5 follows

n_1= 1.33. (20)

Grading Scheme

Theoretical problems

Problem 1: Ascending moist art	
part 1	2
part 2	2
part 3	2
part 4	2
part 5	2
	10

Problem 2: Electron in a magnetic field	
part 1	3
part 2	1
part 3	6
	10

Problem 3: Infinite LC-grid	
part a	4
part b	1
part c	1
part d	4
	10

Problem 4: Refractive indices	
part a, first method	5
part a, second method	5
part b	10
	20

19th International Physics Olympiad - 1988

Bad Ischl / Austria

THEORY 1
Spectroscopy of Particle Velocities

Basic Data

The absorption and emission of a photon is a reversible process. A good example is to be found in the excitation of an atom from the ground state to a higher energy state and the atoms´ subsequent return to the ground state. In such a case we may detect the absorption of a photon from the phenomenon of spontaneous emission or fluorescence. Some of the more modern instrumentation make use of this principle to identify atoms, and also to measure or calculate the value of the velocity in the velocity spectrum of the electron beam.

In an idealised experiment (see fig. 19.1) a single-charged ion travels in the opposite direction to light from a laser source with velocity v. The wavelength of light from the laser source is adjustable. An ion with velocity Zero can be excited to a higher energy state by the application of laser light having a wavelength of $\lambda = 600$ nm. If we excite a moving ion, our knowledge on Dopplers´ effect tells us that we need to apply laser light of a wavelength other than the value given above.

There is given a velocity spectrum embracing velocity magnitude from $v_1 = 0\,\frac{m}{s}$ to $v_2 = 6{,}000\,\frac{m}{s}$. (see fig. 19.1)

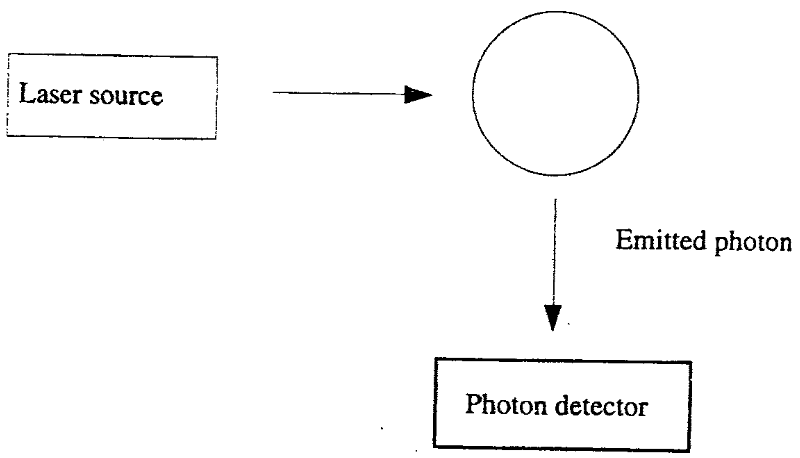


Fig. 19.1

Questions

1.1

1.1.1

What range of wavelength of the laser beam must be used to excite ions of all velocities in the velocity spectrum given above ?

1.1.2

A rigorous analysis of the problem calls for application of the principle from the theory of special relativity

$$\nu' = \nu \cdot \sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}$$

Determine the error when the classical formula for Dopplers´ effect is used to solve the problem.

1.2

Assuming the ions are accelerated by a potential U before excited by the laser beam, determine the relationship between the width of the velocity spectrum of the ion beam and the accelerating potential. Does the accelerating voltage increase or decrease the velocity spectrum width ?

1.3

Each ion has the value $\frac{e}{m} = 4 \cdot 10^6 \frac{A \cdot s}{kg}$, two energy levels corresponding to wavelength $\lambda^{1)} = 600\ \text{nm}$ and wavelength $\lambda^{2)} = \lambda^{1)} + 10^{-3}\ \text{nm}$. Show that lights of the two wavelengths used to excite ions overlap when no accelerating potential is applied. Can accelerating voltage be used to separate the two spectra of laser light used to excite ions so that they no longer overlap ? If the answer is positive, calculate the minimum value of the voltage required.

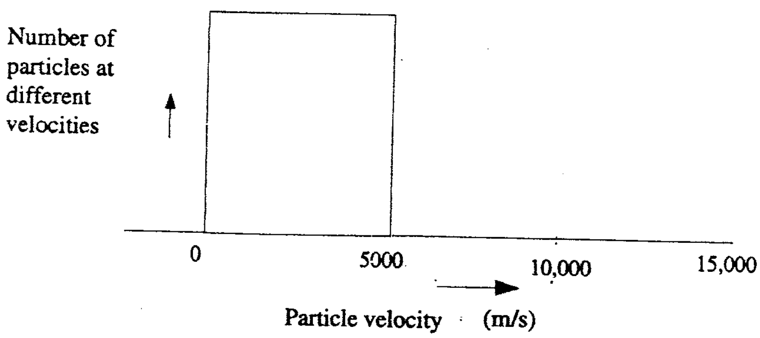


Fig. 19.2

Solution

1.1

1.1.1

Let v be the velocity of the ion towards the laser source relative to the laser source,
ν´ the frequency of the laser light as observed by the observer moving with the ion (e.g. in the frame in which the velocity of the ion is 0) and
ν the frequency of the laser light as observed by the observer at rest with respect to the laser source.

Classical formula for Doppler´s effect is given as

$$\nu' = \nu \cdot \left(1 + \frac{v}{c}\right) \quad \text{......} \quad (1)$$

Let ν^* be the frequency absorbed by an ion (characteristic of individual ions) and
ν_L be the frequency of the laser light used to excite an ion at rest,
hence:

$\nu^* = \nu_L$

For a moving ion, the frequency used to excite ions must be lower than ν^*.

Let ν_H be the frequency used to excite the moving ion.

When no accelerating voltage is applied

frequency of laser light used to excite ions	magnitude of velocity of ions	frequency of laser light absorbed by ions	wavelength of laser light used to excite ions
ν_H	0	ν^*	λ_1
ν_L	v = 6 . 10^3 m/s	ν^*	λ_2

$\nu_L < \nu_H$

$\nu_L = \nu^*$

Calculation of frequency ν_H absorbed by moving ions.

$$\nu^* = \nu_L \cdot \left(1 + \frac{v}{c}\right) \qquad \text{where } \nu^* = \nu_H = 5 \,.\, 10^{14}\ \text{Hz and } v = 6 \,.\, 10^3\ \text{m/s} \ \text{..........} \quad (2)$$

The difference in the values of the frequency absorbed by the stationary ion and the ion moving with the velocity v $\qquad \Delta\nu = \nu_H - \nu_L$
The difference in the values of the wavelengths absorbed by the stationary ion and the ion moving with the velocity v $\qquad \Delta\lambda = \lambda_L - \lambda_H$
(higher frequency implies shorter wavelength)

$$\lambda_L - \lambda_H = \frac{c}{\nu_L} - \frac{c}{\nu_H}$$

from (2)

$$\lambda_L - \lambda_H = \frac{c}{\nu^*} \cdot \left(1 + \frac{\nu}{c}\right) - \frac{c}{\nu^*} = \frac{\nu}{\nu^*}$$

In this case

$$\boxed{\lambda_L - \lambda_H = \frac{6 \cdot 10^3}{5 \cdot 10^{14}}\ \mathrm{m} = 12 \cdot 10^{-3}\ \mathrm{nm}}$$

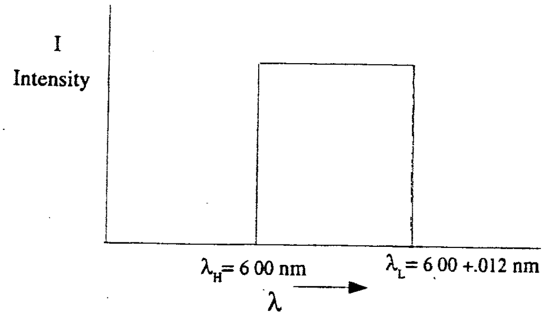


Fig. 19.3 Spectrum of laser light used to excite ions

1.1.2

The formula for calculation of ν´ as observed by the observer moving towards light source based on the principle of the theory of special relativity,

$$\nu' = \nu \cdot \sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}}$$

where v is the magnitude of the velocity of the observer towards the light source,
ν´ is the frequency absorbed by the ion moving with the velocity v towards the light source (also observed by the observer moving with velocity v towards the laser source) and
ν is the frequency of laser light as observed by an observer at rest.

(To put in a metaphoric way, the moving ion "sees" the laser light of frequency ν´ even though the scientist who operates the laser source insists that he is sending a laser beam of frequency ν).

$$\nu' = \nu \cdot \sqrt{\left(1 + \frac{v}{c}\right) \cdot \left(1 + \frac{v}{c} + \frac{v^2}{c^2} + \ldots\ldots\ldots\ldots\right)} = \nu \cdot \sqrt{\left(1 + \frac{v}{c}\right)^2 + \left(1 + \frac{v}{c}\right) \cdot \frac{v^2}{c^2} + \ldots\ldots\ldots\ldots}$$

$$\nu' = \nu \cdot \left(1 + \frac{v}{c}\right) \cdot \left[1 + \frac{v^2}{c^2} \cdot \frac{1}{1 + \frac{v}{c}} + \ldots\ldots\ldots\right]^{\frac{1}{2}} = \nu \cdot \left(1 + \frac{v}{c}\right) \cdot \left[1 + \frac{v^2}{2 \cdot c^2} \cdot \frac{1}{\left(1 + \frac{v}{c}\right)} + \ldots\ldots\ldots\right]$$

The second term in the brackets represents the error if the classical formula for Doppler´s effect is employed.

$$\frac{v}{c} = 2 \cdot 10^{-5}$$

$$\boxed{\frac{v^2}{2 \cdot c^2} \cdot \frac{1}{\left(1 + \frac{v}{c}\right)} = \frac{1}{2} \cdot \frac{4 \cdot 10^{-10}}{1 + 2 \cdot 10^{-5}} \approx 2 \cdot 10^{-10}}$$

The error in the application of classical formula for Doppler´s effect however is of the order of the factor 2.10^{-10}. This means that classical formula for Doppler´s effect can be used to analyze the problem without loosing accuracy.

1.2 When acceleration voltage is used

frequency of laser light used to excite ions	magnitude of velocity of ions	frequency of laser light absorbed by ions	wavelength of laser light used to excite ions
$\nu_H{}'$	$v_H{}'$	$\nu^* = 5 \cdot 10^{14}$ Hz	$\lambda_H{}'$
$\nu_L{}'$	$v_L{}'$	$\nu^* = 5 \cdot 10^{14}$ Hz	$\lambda_L{}'$

Lowest limit of the kinetic energy of ions $\frac{1}{2} \cdot m \cdot (v'_L)^2 = e \cdot U$ and $v'_L = \sqrt{\frac{2 \cdot e \cdot U}{m}}$

Highest limit of the kinetic energy of ions $\frac{1}{2} \cdot m \cdot (v'_H)^2 = \frac{1}{2} \cdot m \cdot v^2 + e \cdot U$

and $v'_H = \sqrt{v^2 + \frac{2 \cdot e \cdot U}{m}}$

Spectrum width of velocity spectrum $\boxed{v'_H - v'_L = \sqrt{v^2 + \frac{2 \cdot e \cdot U}{m}} - \sqrt{\frac{2 \cdot e \cdot U}{m}}}$ .... (3)

(Note that the final velocity of accelerated ions is not the sum of v and $\sqrt{\frac{2 \cdot e \cdot U}{m}}$ as velocity changes with time).

In equation (3) if $\sqrt{\frac{2 \cdot e \cdot U}{m}}$ is negligibly small, the change in the width of the spectrum is negligible, by the same token of argument if $\sqrt{\frac{2 \cdot e \cdot U}{m}}$ is large or approaches ∞, the width of the spectrum of the light used in exciting the ions becomes increasingly narrow and approaches 0.

1.3

Given two energy levels of the ion, corresponding to wavelength $\lambda^{(1)} = 600$ nm and $\lambda^{(2)} = 600+10^{-2}$ nm

For the sake of simplicity, the following sign notations will be adopted:

The superscript in the bracket indicates energy level (1) or (2) as the case may be. The sign ´ above denotes the case when accelerating voltage is applied, and also the subscripts H and L apply to absorbed frequencies (and also wavelengths) correspond to the high velocity and low velocity ends of the velocity spectrum of the ion beam respectively.

The subscript following λ (or ν) can be either 1 or 2, with number 1 corresponding to lowest velocity of the ion and number 2 the highest velocity of the ion. When no accelerating voltage is applied, the subscript 1 implies that minimum velocity of the ion is 0, and the highest velocity of the ion is 6000 m/s. If accelerating voltage U is applied, number 1 indicates that the wavelength of laser light pertains to the ion of lowest velocity and number 2 indicates the ion of the highest velocity.

Finally the sign * indicates the value of the wavelength (λ^*) or frequency (ν^*) absorbed by the ion (characteristic absorbed frequency).

When no accelerating voltage is applied:

For the first energy level:

frequency of laser light used to excite ions	magnitude of velocity of ions	frequency of laser light absorbed by ions	wavelength of laser light used to excite ions
$\nu_H^{(1)}$	0	$\nu^{(1)*} = 5 \,.\, 10^{14}$ Hz	$\lambda_1^{(1)}$
$\nu_L^{(1)}$	$v=6.10^3$ m/s	$\nu^{(1)*} = 5 \,.\, 10^{14}$ Hz	$\lambda_2^{(1)}$

$\nu_H^{(1)*} = \nu_L^{(1)*} = \nu^{(1)*} = 5 \,.\, 10^{14}$ Hz

Differences in frequencies of laser light used to excite ions $= \nu_H^{(1)} - \nu_L^{(1)}$
Differences of wavelengths of laser light used to excite ions $= \lambda_L^{(1)} - \lambda_H^{(1)}$

$$\boxed{\frac{v}{\nu_L^{(1)*}} = \frac{6000}{5 \cdot 10^{14}} = 0{,}012 \text{ nm}}$$

For the second energy level:

frequency of laser light used to excite ions	magnitude of velocity of ions	frequency of laser light absorbed by ions	wavelength of laser light used to excite ions
$\nu_H^{(2)}$	0	$\nu^{(2)*} = 5 \,.\, 10^{14}$ Hz	$\lambda_H^{(2)}$
$\nu_L^{(2)}$	$v = 6000$ m/s	$\nu^{(2)*} = 5 \,.\, 10^{14}$ Hz	$\lambda_L^{(2)}$

$\nu_H^{(2)*} = \nu_L^{(2)*} = \nu^{(2)*} = 5 \,.\, 10^{14}$ Hz

Differences in frequencies of laser light used to excite ions $\nu_H^{(2)} - \nu_L^{(2)}$
Differences in wavelengths of laser light used to excite ions $\lambda_L^{(2)} - \lambda_H^{(2)}$

This gives $\boxed{\frac{6000}{5 \cdot 10^{14}} = 0{,}012 \text{ nm}}$

Hence the spectra of laser light (absorption spectrum) used to excite an ion at two energy levels overlap as shown in fig. 19.4.

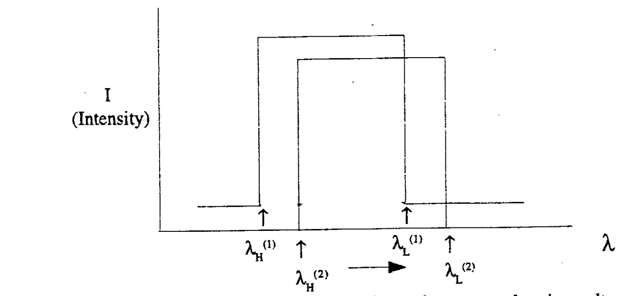


Fig. 19.4 Spectrum of laser light used to excite ions when no accelerating voltage is applied(Absorption Spectrum)

When accelerating voltage is applied:

Let $\lambda_H^{(1)\prime}$ and $\lambda_L^{(1)\prime}$ be the range of the wavelengths used to excite ions in the first energy level, when accelerating voltage is applied. (Note the prime sign to denote the situation in which the accelerating voltage is used), and let $\lambda_H^{(2)\prime}$ and $\lambda_L^{(2)\prime}$ represent the range of the wavelengths used to excite ions in the second energy level also when an accelerating voltage is applied.

Condition for the two spectra not to overlap:

$$\lambda_H^{(2)\prime} \geq \lambda_L^{(1)\prime} \qquad \text{(see fig. 19.4)} \quad \ldots \quad (4)$$

(Keep in mind that lower energy means longer wavelengths and vice versa).

From condition (3):
$$\lambda_L - \lambda_H = \frac{v}{\nu^*} \quad \ldots \quad (5)$$

The meanings of this equation is if the velocity of the ion is v, the wavelength which the ion “sees” is λ_L, when λ_H is the wavelength which the ion of zero-velocity “sees”.

Equation (5) may be rewritten in the context of the applications of accelerating voltage in order for the two spectra of laser light will not overlap as follows:

$$\lambda_L^{(N)\prime} - \lambda_H^{(N)\prime} = \frac{v'}{\nu^*} \qquad \text{where N is the order of the energy level} \quad \ldots \quad (6)$$

The subscript L relates λ to lowest velocity of the ion which “sees” frequency ν^*. The lowest velocity in this case is $\sqrt{\frac{2 \cdot e \cdot U}{m}}$ and the subscript H relates λ to the highest velocity of the ion, in this case $\sqrt{v^2 + \frac{2 \cdot e \cdot U}{m}}$.

Equation (6) will be used to calculate

- width of velocity spectrum of the ion accelerated by voltage U
- potential U which results in condition given by (4)

Let us take up the second energy level (lower energy level of the two ones) of the ion first:

$$\lambda_L^{(2)\prime} - \lambda_H^{(2)} = \frac{v'}{\nu^*} \quad \ldots \quad (7)$$

substitute

$$v' = \sqrt{\frac{2 \cdot e \cdot U}{m}}$$

$\lambda_H^{(1)} = 600 + 10^{-3}$ nm
$v^* = 5 \, . \, 10^{14}$ Hz
$v = 0$ m/s

$$\lambda_H^{(2)\prime} = (600 + 0{,}001) \cdot 10^{-9} + \frac{\sqrt{\frac{2 \cdot e \cdot U}{m}}}{5 \cdot 10^{14}} \text{ m} \quad \ldots \quad (8)$$

Considering the first energy level of the ion

$$\lambda_L^{(1)'} - \lambda_H^{(1)} = \frac{v'}{\nu^*} \qquad (9)$$

In this case

$$v' = \sqrt{v^2 + \frac{2 \cdot e \cdot U}{m}}$$

$\nu^* = 5 \,.\, 10^{14}$ Hz
$v = 6000$ m/s
$\lambda_H^{(1)} = 600 \,.\, 10^{-9}$ m

$$\lambda_L^{(1)'} = 600 \cdot 10^{-9} + \frac{\sqrt{v^2 + \frac{2 \cdot e \cdot U}{m}}}{5 \cdot 10^{14}} \text{ m} \qquad (10)$$

Substitute $\lambda_H^{(2)'}$ from (8) and $\lambda_L^{(1)'}$ from (10) in (4) one gets

$$(600 + 0{,}001) \cdot 10^{-9} + \frac{\sqrt{\frac{2 \cdot e \cdot U}{m}}}{5 \cdot 10^{14}} \geq 600 \cdot 10^{-9} + \frac{\sqrt{v^2 + \frac{2 \cdot e \cdot U}{m}}}{5 \cdot 10^{14}}$$

$$500 \geq \frac{\sqrt{v^2 + \frac{2 \cdot e \cdot U}{m}}}{5 \cdot 10^{14}} - \frac{\sqrt{\frac{2 \cdot e \cdot U}{m}}}{5 \cdot 10^{14}}$$

$$500 \geq \sqrt{36 \cdot 10^6 + 2 \cdot 4 \cdot 10^6 \cdot U} - \sqrt{2 \cdot 4 \cdot 10^6 \cdot U}$$

assume that U is of the order of 100 and over,

then $$\sqrt{8 \cdot 10^6 \cdot U} \cdot \left(1 + \frac{9}{4 \cdot U}\right) - \sqrt{8 \cdot 10^6 \cdot U} \leq 500$$

$$\frac{1}{\sqrt{2 \cdot U}} \cdot 9 \cdot 10^3 \leq 500$$

$$\sqrt{2 \cdot U} \geq 324$$

$$\boxed{U \geq 162 \text{ V}}$$

The minimum value of accelerating voltage to avoid overlapping of absorption spectra is approximately 162 V

THEORY 2
Maxwell´s Wheel

Introduction

A cylindrical wheel of uniform density, having the mass M = 0,40 kg, the radius R = 0,060 m and the thickness d = 0,010 m is suspended by means of two light strings of the same length from the ceiling. Each string is wound around the axle of the wheel. Like the strings, the mass of the axle is negligible. When the wheel is turned manually, the strings are wound up until the centre of mass is raised 1,0 m above the floor. If the wheel is allowed to move downward vertically under the pulling force of the gravity, the strings are unwound to the full length of the strings and the wheel reaches the lowest point. The strings then begin to wound in the opposite sense resulting in the wheel being raised upwards.

Analyze and answer the following questions, assuming that the strings are in vertical position and the points where the strings touch the axle are directly below their respective suspending points (see fig. 19.5).

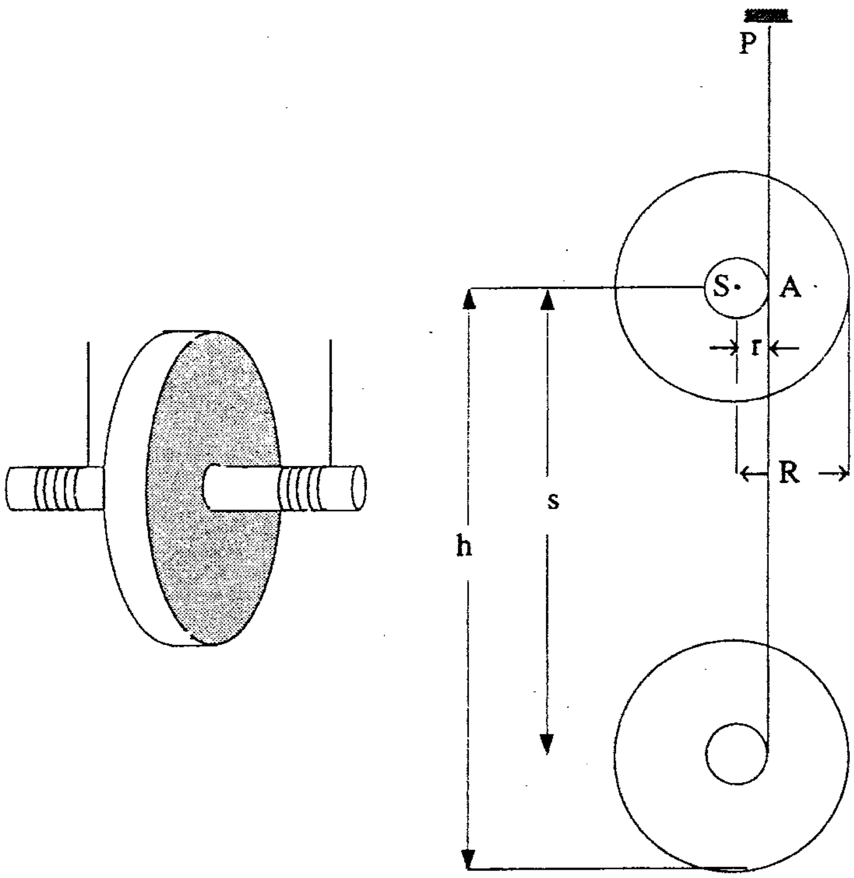


Fig. 19.5

Questions

2.1

Determine the angular speed of the wheel when the centre of mass of the wheel covers the vertical distance s.

2.2

Determine the kinetic energy of the linear motion of the centre of mass E_r after the wheel travels a distance s = 0,50 m, and calculate the ratio between E_r and the energy in any other form in this problem up to this point.

Radius of the axle = 0,0030 m

2.3

Determine the tension in the string while the wheel is moving downward.

2.4

Calculate the angular speed ω' as a function of the angle Φ when the strings begin to unwind themselves in opposite sense as depicted in fig. 19.6.
Sketch a graph of variables which describe the motion (in cartesian system which suits the problem) and also the speed of the centre of mass as a function of Φ.

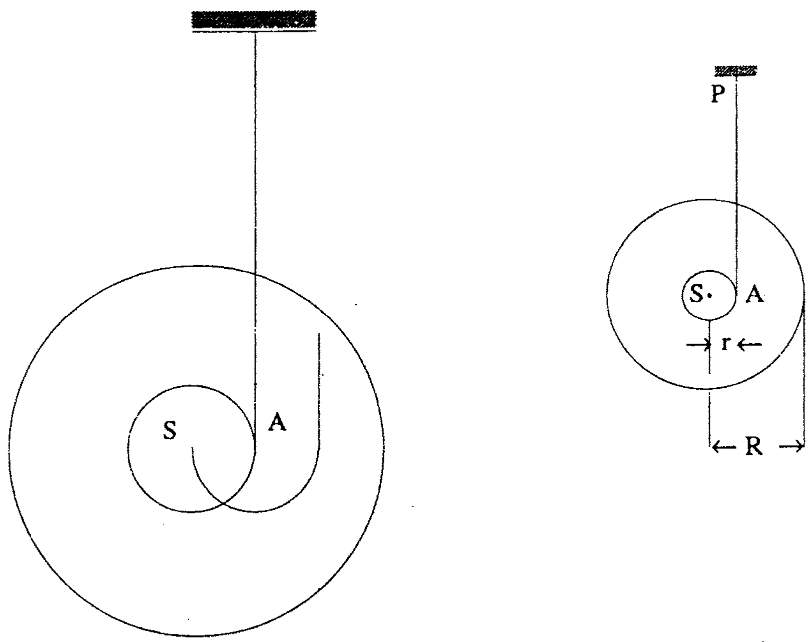


Fig. 19.6

2.5

If the string can withstand a maximum tension T_m = 10 N, find the maximum length of the string which may be unwound without breaking by the wheel.

Solution

2.1

conservation of energy: $$M \cdot g \cdot s = \frac{1}{2} \cdot I_A \cdot \omega^2 \quad \text{(1)}$$

where ω is the angular speed of the wheel and I_A is the moment of inertia about the axis through A.

Note: If we would take the moment of inertia about S instead of A we would have

$$M \cdot g \cdot s = \frac{1}{2} \cdot I_S \cdot \omega^2 + \frac{1}{2} \cdot m \cdot v^2$$

where v is the speed of the centre of mass along the vertical.
This equation is the same as the above one in meanings since

$I_A = I_S + M \cdot r^2$ and $I_S = M \cdot R^2$

From (1) we get $$\omega = \sqrt{\frac{2 \cdot M \cdot g \cdot s}{I_A}}$$

substitute $$I_A = \frac{1}{2} \cdot M \cdot r^2 + M \cdot R^2$$

$$\omega = \sqrt{\frac{2 \cdot g \cdot s}{r^2 + \frac{R^2}{2}}}$$

Putting in numbers we get $$\boxed{\omega = \sqrt{\frac{2 \cdot 9{,}81 \cdot 0{,}50}{9 \cdot 10^{-6} + \frac{1}{2} \cdot 36 \cdot 10^{-4}}} \approx 72{,}4 \, \frac{\text{rad}}{\text{s}}}$$

2.2

Kinetic energy of linear motion of the centre of mass of the wheel is

$$E_T = \frac{1}{2} \cdot M \cdot v^2 = \frac{1}{2} \cdot M \cdot \omega^2 \cdot r^2 = \frac{1}{2} \cdot 0{,}40 \cdot 72{,}4^2 \cdot 9 \cdot 10^{-6} = 9{,}76 \cdot 10^{-3} \text{ J}$$

Potential energy of the wheel

$$E_P = M \cdot g \cdot s = 0{,}40 \cdot 9{,}81 \cdot 0{,}50 = 1{,}962 \text{ J}$$

Rotational kinetic energy of the wheel

$$E_R = \frac{1}{2} \cdot I_S \cdot \omega^2 = \frac{1}{2} \cdot 0{,}40 \cdot 1{,}81 \cdot 10^{-3} \cdot 72{,}4^2 = 1{,}899 \text{ J}$$

$$\boxed{\frac{E_T}{E_R} = \frac{9{,}76 \cdot 10^{-3}}{1{,}899} = 5{,}13 \cdot 10^{-3}}$$

2.3

Let $\frac{T}{2}$ be the tension in each string.

Torque τ which causes the rotation is given by $\tau = M \cdot g \cdot r = I_A \cdot \alpha$

where α is the angular acceleration $\alpha = \frac{M \cdot g \cdot r}{I_A}$

The equation of the motion of the wheel is M.g – T = M.a

Substituting $a = \alpha . r$ and $I_A = \frac{1}{2} \cdot M \cdot r^2 + M \cdot R^2$ we get

$$T = M \cdot g + \frac{M \cdot g \cdot r^2}{\frac{1}{2} \cdot M \cdot R^2 + M \cdot r^2} = M \cdot g \cdot \left(1 + \frac{2 \cdot r^2}{R^2 + 2 \cdot r^2}\right)$$

Thus for the tension $\frac{T}{2}$ in each string we get

$$\frac{T}{2} = \frac{M \cdot g}{2} \cdot \left(1 + \frac{2 \cdot r^2}{R^2 + 2 \cdot r^2}\right) = \frac{0{,}40 \cdot 9{,}81}{2} \cdot \left(1 + \frac{2 \cdot 9 \cdot 10^{-6}}{3{,}6 \cdot 10^{-3} + 2 \cdot 9 \cdot 10^{-6}}\right) = 1{,}96\ \text{N}$$

$$\boxed{\frac{T}{2} = 1{,}96\ \text{N}}$$

2.4

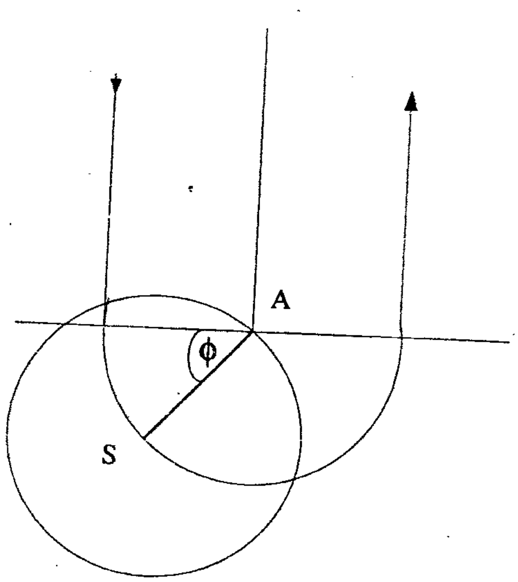


Fig 19.7

After the whole length of the strings is completely unwound, the wheel continues to rotate about A (which is at rest for some interval to be discussed). Let $\dot{\Phi}$ be the angular speed of the centre of mass about the axis through A. The equation of the rotational motion of the wheel about A may be written as $|\tau| = I_A \cdot \ddot{\Phi}$,

where τ is the torque about A, I_A is the moment of inertia about the axis A and $\ddot{\Phi}$ is the angular acceleration about the axis through A.

Hence $M \cdot g \cdot r \cdot \cos\Phi = I_A \cdot \ddot{\phi}$

and $\ddot{\phi} = \frac{M \cdot g \cdot r \cdot \cos\Phi}{I_A}$

Multiplied with $\dot{\Phi}$ gives:

$$\dot{\Phi} \cdot \ddot{\Phi} = \frac{M \cdot g \cdot r \cdot \cos\Phi \cdot \dot{\Phi}}{I_A} \quad \text{or} \quad \frac{1}{2} \cdot \frac{d(\dot{\Phi})^2}{dt} = \frac{M \cdot g \cdot r \cdot \cos\Phi}{I_A} \cdot \frac{d\Phi}{dt}$$

this gives

$$\left(\dot{\Phi}\right)^2 = \frac{2 \cdot M \cdot g \cdot r \cdot \sin\Phi}{I_A} + C \qquad \text{[C = arbitrary constant]}$$

If $\Phi = 0$ [s = H] than is $\dot{\Phi} = \omega$

That gives $\omega = \frac{2 \cdot M \cdot g \cdot H}{I_A}$ and therefore $C = \frac{2 \cdot M \cdot g \cdot H}{I_A}$

Putting these results into the equation above one gets

$$\dot{\Phi} = \omega = \sqrt{\frac{2 \cdot M \cdot g \cdot H \cdot \sin\Phi}{I_A} \cdot \left(1 + \frac{r}{H}\right)}$$

For $\frac{r}{H}$ <<< 1 we get:

$$\omega = \omega'_{MAX} = \sqrt{\frac{2 \cdot M \cdot g \cdot H}{I_A}}$$

and

$$v = r \cdot \omega'_{MAX} = r \cdot \sqrt{\frac{2 \cdot M \cdot g \cdot H}{I_A}}$$

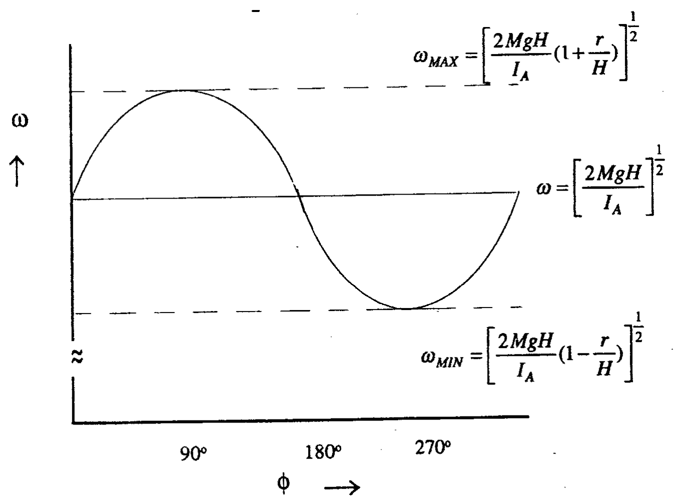


Fig.19.8

Component of the displacement
along x-axis is $x = r.\sin\Phi\text{-}r$
along y-axis is $y = r.\cos\Phi\text{-}r$

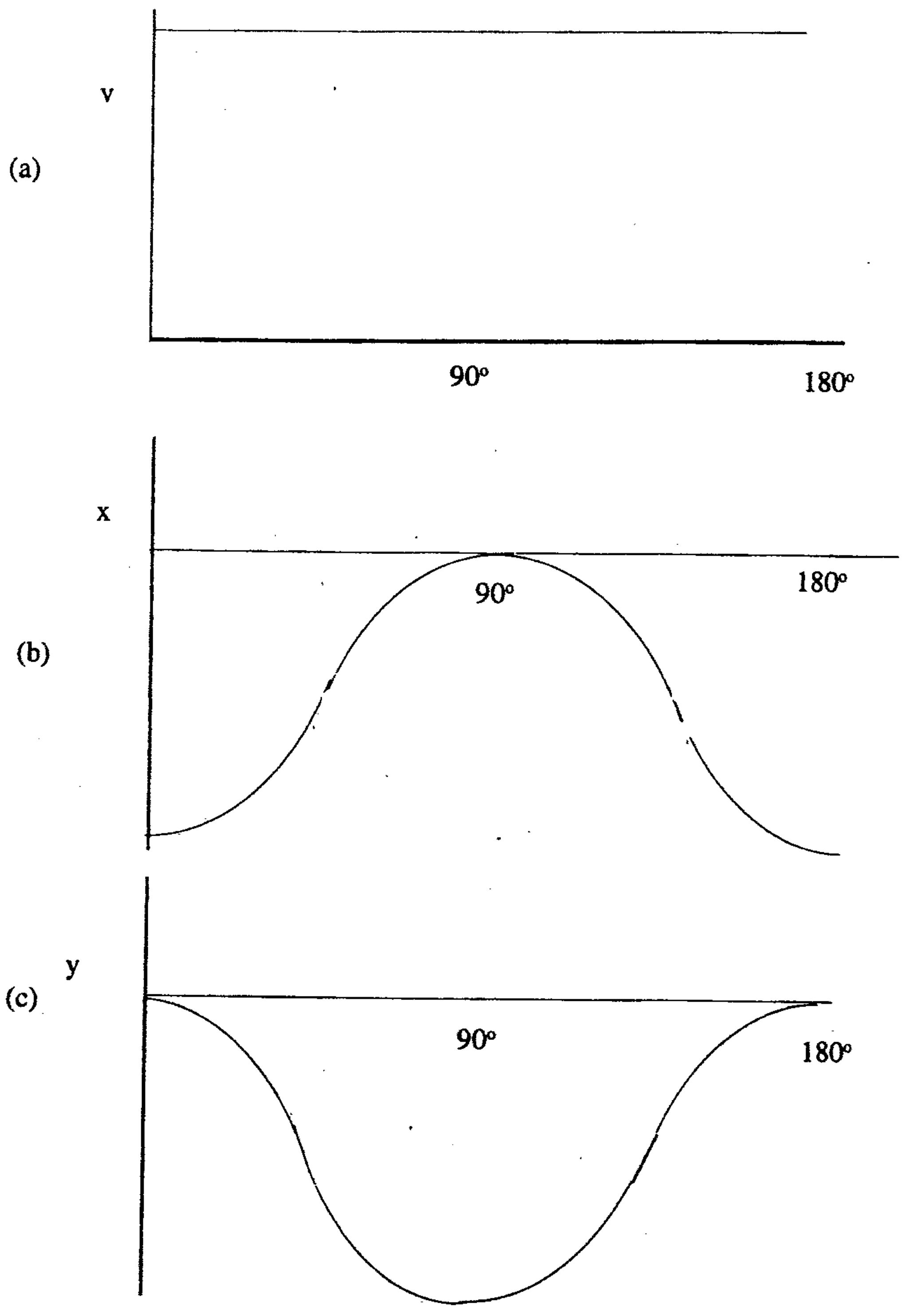


Fig.19.9

2.5

Maximum tension in each string occurs $\dot{\Phi} = \omega'_{MAX}$

The equation of the motion is $T_{MAX} - M \cdot g = M \cdot (\omega'_{MAX})^2 \cdot r$

Putting in T = 20 N and $\omega'_{MAX} = \sqrt{\frac{2 \cdot M \cdot g \cdot s}{I_A}}$ (where s is the maximum length of the strings supporting the wheel without breaking) and $I_A = M \cdot \left(\frac{R^2}{2} + r^2 \right)$ the numbers one gets:

$20 = 0{,}40 \cdot 9{,}81 \cdot \left(1 + \frac{4 \cdot 3 \cdot 10^{-3} \cdot s}{36 \cdot 10^{-4} + 2 \cdot 9 \cdot 10^{-6}} \right)$ This gives: s = 1,24 m

The maximum length of the strings which support maximum tension without breaking is

$$\boxed{1{,}24\ \text{m}}\ .$$

THEORY 3
Recombination of Positive and Negative Ions in Ionized Gas

Introduction

A gas consists of positive ions of some element (at high temperature) and electrons. The positive ion belongs to an atom of unknown mass number Z. It is known that this ion has only one electron in the shell (orbit).

Let this ion be represented by the symbol $A^{(Z-1)+}$

Constants:

electric field constant	$\varepsilon_O = 8{,}85 \cdot 10^{-12}\ \frac{A \cdot s}{V \cdot m}$
elementary charge	$e = \pm 1{,}602 \cdot 10^{-19}\ A \cdot s$
	$q^2 = \frac{e^2}{4 \cdot \pi \cdot \varepsilon_O} = 2{,}037 \cdot 10^{-28}\ J \cdot m$
Planck´s constant	$\hbar = 1{,}054 \cdot 10^{-34}\ J \cdot s$
(rest) mass of an electron	$m_e = 9{,}108 \cdot 10^{-31}\ kg$
Bohr´s atomic radius	$r_B = \frac{\hbar}{m \cdot q^2} = 5{,}92 \cdot 10^{-11}\ m$
Rydberg´s energy	$E_R = \frac{q^2}{2 \cdot r_B} = 2{,}180 \cdot 10^{-18}\ J$
(rest) mass of a proton	$m_P \cdot c^2 = 1{,}503 \cdot 10^{-10}\ J$

Questions:

3.1

Assume that the ion which has just one electron left the shell.
$A^{(Z-1)+}$ is in the ground state.

In the lowest energy state, the square of the average distance of the electron from the nucleus or r^2 with components along x-, y- and z-axis being $(\Delta x)^2$, $(\Delta y)^2$ and $(\Delta z)^2$ respectively and $r_O^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$ and also the square of the average momentum by $p_O^2 = (\Delta p_x)^2 + (\Delta p_y)^2 + (\Delta p_z)^2$, whereas $\Delta p_x \geq \frac{\hbar}{2 \cdot \Delta x}$, $\Delta p_y \geq \frac{\hbar}{2 \cdot \Delta y}$ and $\Delta p_z \geq \frac{\hbar}{2 \cdot \Delta z}$.

Write inequality involving $(p_O)^2 \cdot (r_O)^2$ in a complete form.

3.2

The ion represented by $A^{(Z-1)+}$ may capture an additional electron and consequently emits a photon.

Write down an equation which is to be used for calculation the frequency of an emitted photon.

3.3

Calculate the energy of the ion $A^{(Z-1)+}$ using the value of the lowest energy. The calculation should be approximated based on the following principles:

3.3.A

The potential energy of the ion should be expressed in terms of the average value of $\frac{1}{r}$.

(ie. $\frac{1}{r_o}$; r_0 is given in the problem).

3.3.B

In calculating the kinetic energy of the ion, use the average value of the square of the momentum given in 3.1 after being simplified by $(p_o)^2 \cdot (r_o)^2 \approx (\hbar)^2$

3.4

Calculate the energy of the ion $A^{(Z-2)+}$ taken to be in the ground state, using the same principle as the calculation of the energy of $A^{(Z-1)+}$. Given the average distance of each of the two electrons in the outermost shell (same as r_0 given in 3.3) denoted by r_1 and r_2, assume the average distance between the two electrons is given by r_1+r_2 and the average value of the square of the momentum of each electron obeys the principle of uncertainty ie.

$p_1^2 \cdot r_1^2 \approx \hbar^2$ and $p_2^2 \cdot r_2^2 \approx \hbar^2$

hint: Make use of the information that in the ground state $r_1 = r_2$

3.5

Consider in particular the ion $A^{(Z-2)+}$ is at rest in the ground state when capturing an additional electron and the captured electron is also at rest prior to the capturing. Determine the numerical value of Z, if the frequency of the emitted photon accompanying electron capturing is $2{,}057 \cdot 10^{17}$ rad/s. Identify the element which gives rise to the ion.

Solution

3.1

$r_0^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$

$p_0^2 = (\Delta p_x)^2 + (\Delta p_y)^2 + (\Delta p_z)^2$

since

$$\Delta p_x \geq \frac{\hbar}{2 \cdot \Delta x} \qquad \Delta p_y \geq \frac{\hbar}{2 \cdot \Delta y} \qquad \Delta p_z \geq \frac{\hbar}{2 \cdot \Delta z}$$

gives

$$p_0^2 \geq \frac{\hbar^2}{4} \cdot \left[\frac{1}{(\Delta x)^2} + \frac{1}{(\Delta y)^2} + \frac{1}{(\Delta z)^2} \right]$$

and

$$(\Delta x)^2 = (\Delta y)^2 = (\Delta z)^2 = \frac{r_0^2}{3}$$

thus $\boxed{p_0^2 \cdot r_0^2 \geq \frac{9}{4} \cdot \hbar^2}$

3.2

$|\vec{v}_e|$ ...... speed of the external electron before the capture

$|\vec{V}_i|$ .... speed of $A^{(Z-1)+}$ before capturing

$|\vec{V}_f|$ ...... speed of $A^{(Z-1)+}$ after capturing

$E_n = h.\nu$ ...... energy of the emitted photon

conservation of energy:

$$\frac{1}{2} \cdot m_e \cdot v_e^2 + \frac{1}{2} \cdot (M + m_e) \cdot V_i^2 + E\left[A^{(Z-1)+}\right] = \frac{1}{2} \cdot (M + 2 \cdot m_e) \cdot V_f^2 + E\left[A^{(Z-2)+}\right]$$

where $E[A^{(Z-1)+})$ and $E[A^{(Z-2)+}]$ denotes the energy of the electron in the outermost shell of ions $A^{(Z-1)+}$ and $A^{(Z-2)+}$ respectively.

conservation of momentum:

$$m_e \cdot \vec{v}_e + (M + m) \cdot \vec{V}_i = (M + 2 \cdot m_e) \cdot \vec{V}_f + \frac{h \cdot \nu}{c} \cdot \vec{1}$$

where $\vec{1}$ is the unit vector pointing in the direction of the motion of the emitted photon.

3.3

Determination of the energy of $A^{(Z-1)+}$:

potential energy = $-\dfrac{Z\cdot e^2}{4\cdot\pi\cdot\varepsilon_0\cdot r_0}=-\dfrac{Z\cdot q^2}{r_0}$

kinetic energy = $\dfrac{p^2}{2\cdot m}$

If the motion of the electrons is confined within the x-y-plane, principles of uncertainty in 3.1 can be written as

$$r_0^2=(\Delta x)^2+(\Delta y)^2$$

$$p_0^2=(\Delta p_x)^2+(\Delta p_y)^2$$

$$p_0^2=\frac{\hbar^2}{4}\cdot\left[\frac{1}{(\Delta x)^2}+\frac{1}{(\Delta y)^2}\right]=\frac{\hbar^2}{4}\cdot\left[\frac{2}{r_0^2}+\frac{2}{r_0^2}\right]=\frac{\hbar^2}{4}\cdot\frac{4}{r_0^2}$$

thus

$$p_0^2\cdot r_0^2=\hbar^2$$

$$E\left[A^{(Z-1)+}\right]=\frac{p_0^2}{2\cdot m_e}-\frac{Z\cdot q^2}{r_0}=\frac{\hbar^2}{2\cdot m_e\cdot r_e}-\frac{Z\cdot q^2}{r_0}$$

Energy minimum exists, when $\dfrac{dE}{dr_0}=0$.

Hence

$$\frac{dE}{dr_0}=-\frac{\hbar^2}{m_e\cdot r_e^3}+\frac{Z\cdot q^2}{r_0^2}=0$$

this gives $\dfrac{1}{r_0}=\dfrac{Z\cdot q^2\cdot m_e}{\hbar^2}$

hence

$$E\left[A^{(Z-1)+}\right]=\frac{\hbar^2}{2\cdot m_e}\cdot\left(\frac{Z\cdot q^2\cdot m_e}{\hbar}\right)^2-Z\cdot q^2\cdot\frac{Z\cdot q^2\cdot m_e}{\hbar^2}=-\frac{m_e}{2}\cdot\left(\frac{Z\cdot q^2}{\hbar}\right)^2=-\frac{q^2\cdot Z^2}{2\cdot r_B}=-E_R\cdot Z^2$$

$$\boxed{E\left[A^{(Z-1)+}\right]\ =\ -E_R\cdot Z^2}$$

3.4

In the case of $A^{(Z-1)+}$ ion captures a second electron

potential energy of both electrons = $-2 \cdot \frac{Z \cdot q^2}{r_0}$

kinetic energy of the two electrons = $2 \cdot \frac{p^2}{2 \cdot m} = \frac{\hbar^2}{m_e \cdot r_0^2}$

potential energy due to interaction between the two electrons = $\frac{q^2}{\left|\vec{r}_1 - \vec{r}_2\right|} = \frac{q^2}{2 \cdot r_0}$

$$E\left[A^{(Z-2)+}\right] = \frac{\hbar^2}{m_e \cdot r_0^2} - \frac{2 \cdot Z \cdot q^2}{r_0^2} + \frac{q^2}{2 \cdot r_0}$$

total energy is lowest when $\frac{dE}{dr_0} = 0$

hence

$$0 = -\frac{2 \cdot \hbar^2}{m_e \cdot r_0^3} + \frac{2 \cdot Z \cdot q^2}{r_0^3} - \frac{q^2}{2 \cdot r_0^2}$$

hence

$$\frac{1}{r_0} = \frac{q^2 \cdot m_e}{2 \cdot \hbar^2} \cdot \left(2 \cdot Z - \frac{1}{2}\right) = \frac{1}{r_B} \cdot \left(Z - \frac{1}{4}\right)$$

$$E\left[A^{(Z-2)+}\right] = \frac{\hbar^2}{m_e} \cdot \left(\frac{q^2 \cdot m_e}{2 \cdot \hbar^2}\right)^2 - \frac{q^2 \cdot \left(2 \cdot Z - \frac{1}{2}\right)}{\hbar} \cdot \frac{q^2 \cdot m_e \cdot \left(2 \cdot Z - \frac{1}{2}\right)}{2 \cdot \hbar}$$

$$E\left[A^{(Z-2)+}\right] = -\frac{m_e}{4} \cdot \left[\frac{q^2 \cdot \left(2 \cdot Z - \frac{1}{2}\right)}{\hbar}\right]^2 = -\frac{m_e \cdot \left[q^2 \cdot \left(Z - \frac{1}{4}\right)\right]^2}{\hbar^2} = -\frac{q^2 \cdot \left(Z - \frac{1}{4}\right)^2}{\hbar^2}$$

this gives

$$\boxed{E\left[A^{(Z-2)+}\right] = -2 \cdot E_R \cdot \left(Z - \frac{1}{4}\right)^2}$$

3.5

The ion $A^{(Z-1)+}$ is at rest when it captures the second electron also at rest before capturing.

From the information provided in the problem, the frequency of the photon emitted is given by

$$\nu = \frac{\omega}{2 \cdot \pi} = \frac{2{,}057 \cdot 10^{17}}{2 \cdot \pi} \text{ Hz}$$

The energy equation can be simplified to $E\left[A^{(Z-1)+}\right] - E\left[A^{(Z-2)+}\right] = \hbar \cdot \omega = h \cdot \nu$

that is

$$-E_R \cdot Z^2 - \left[-2 \cdot E_R \cdot \left(Z - \frac{1}{4} \right)^2 \right] = \hbar \cdot \omega$$

putting in known numbers follows

$$2{,}180 \cdot 10^{-18} \cdot \left[-Z^2 + 2 \cdot \left(Z - \frac{1}{4} \right)^2 \right] = 1{,}05 \cdot 10^{-34} \cdot 2{,}607 \cdot 10^{17}$$

this gives

$$Z^2 - Z - 12{,}7 = 0$$

with the physical sensuous result $Z = \dfrac{1 + \sqrt{1 + 51}}{2} = 4{,}1$

$\boxed{\text{This implies Z = 4, and that means Beryllium}}$

EXPERIMENTS

EXPERIMENT 1: Polarized Light

General Information

Equipment:

- one electric tungsten bulb made of frosedt-surface glass complete with mounting stand, 1 set
- 3 wooden clamps, each of which contains a slit for light experiment
- 2 glass plates; one of which is rectangular and the other one is square-shaped
- 1 polaroid sheet (circular-shaped)
- 1 red film or filter
- 1 roll self adhesive tape
- 6 pieces of self-adhesive labelling tape
- 1 cellophane sheet
- 1 sheet of black paper
- 1 drawing triangle with a handle
- 1 unerasable luminocolour pen 312, extra fine and black colour
- 1 lead pencil type F
- 1 lead pencil type H
- 1 pencil sharpener
- 1 eraser
- 1 pair of scissors

Important Instructions to be Followed

1. There are 4 pieces of labelling tape coded for each contestant. Stick the tape one each on the instrument marked with the sign #. Having done this, the contestant may proceed to perform the experiment to answer the questions.
2. Cutting, etching, scraping or folding the polaroid is strictly forbidden.
3. If marking is to be made on the polaroid, use the lumino-colour pen provided and put the cap back in place after finishing.
4. When marking is to be made on white paper sheet, use the white tape.
5. Use lead pencils to draw or sketch a graph.
6. Black paper may be cut into pieces for use in the experiment, but the best way of using the black paper is to roll it into a cylinder as to form a shield around the electric bulb. An aperture of proper size may be cut into the side of the cylinder to form an outlet for light used in the experiment.
7. Red piece of paper is to be folded to form a double layer.

The following four questions will be answered by performing the experiment:

Questions

1.1

1.1.a

Locate the axis of the light transmission of the polaroid film. This may be done by observing light reflected from the surface of the rectangular glass plate provided. (Light transmitting axis is the direction of vibration of the electric field vector of light wave transmitted through the polaroid). Draw a straight line along the light transmission axis as exactly as possible on the polaroid film. (#)

1.1.b

Set up the apparatus on the graph paper for the experiment to determine the refractive index of the glass plate for white light.

When unpolarized light is reflected at the glass plate, reflected light is partially polarized. Polarization of the reflected light is a maximum if the tangens of incident angle is equal to the refractive index of the glass plate, or: $\tan\alpha = n$.
Draw lines or dots that are related to the determination of the refractive index on the graph paper. (#)

1.2

Assemble a polariscope to observe birefringence in birefringent glass plate when light is normally incident on the plastic sheet and the glass plates.

A birefringent object is the object which splits light into two components, with the electric field vectors of the two components perpendicular to each other. The two directions of the electric field vectors are known as birefringent axes characteristic of birefringent material. These two components of light travel with different velocity.

Draw a simple sketch depicting design and functions of the polariscope assembled.

Insert a sheet of clear cellophane in the path of light in the polariscope. Draw lines to indicate birefringent axes (#). Comment briefly but concisely on what is observed, and describe how berefringent axes are located.

1.3

1.3.a

Stick 10 layers of self-adhesive tape provided on the glass plate as shown below. Make sure that each layer recedes in equal steps.

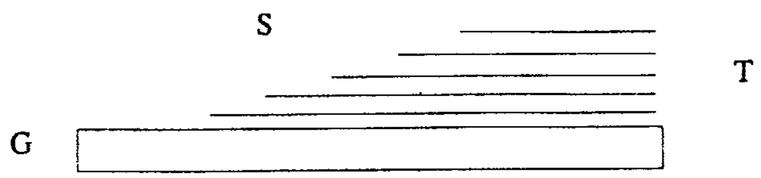


Fig. 19.10

G square glass plate as a substrate for the cellophane layers
T 10 layers of cellophane sheet
S steps about 3 mm up to 4 mm wide

Insert the assembled square plate into the path of light in the polariscope. Describe conditions for observing colours. How can these colours be changed ? Comment on the observations from this experiment.

1.3.b

Prepare monochromatic red light by placing doubly-folded red plastic sheet in the path of white light. Mark on the assembled square plate to show the steps which allow the determination of the difference of the optical paths of the two components of light from berefringent phenomenon, described under 1.2 (#).

Estimate the difference of the optical paths from two consecutive steps.

1.4

1.4.a

With the polariscope assembled, examine the central part of the drawing triangle provided. Describe relevant optical properties of the drawing triangle pertaining to birefringence.

1.4.b

Comment on the results observed. Draw conclusions about the physical properties of the material of which the triangle is made.

Additional Cautions

Be sure that the following items affixed with the coded labels provided accompany the report.

1. (#) Polarized film with the position of the transmission axis clearly marked.
2. (#) Graph paper with lines and dots denoting experimental setup for determining refractive index.
3. (#) Sheet of cellophane paper with marking indicating the positions of birefringent axis.
4. (#) Square glass plate affixed with self-adhesive tape with markings to indicate the positions of birefringent axis.

Solution

In this experiment the results from one experimental stage are used to solve problems in the following experimental stages. Without actually performing all parts of the experiment, solution cannot be meaningfully discussed.

It suffices that some transparent crystals are anisotropic, meaning their optical properties vary with the direction. Crystals which have this property are said to be doubly refracting or exhibit birefringece.

This phenomenon can be understood on the basis of wave theory. When a wavefront enters a birefringent material, two sets of Huygens wavelets propagate from every point of the entering wavefront causing the incident light to split into two components of two different velocities. In some crystals there is a particular direction (or rather a set of parallel directions) in which the velocities of the two components are the same. This direction is known as optic axes. the former is said to be uniaxial, and the latter biaxial.

If a plane polarized light (which may be white light or monochromatic light) is allowed to enter a uniaxial birefringed material, with its plane of polarization making some angle, say 45° with the optic axis, the incident light is splitted into two components (ordinary and extraordinary) travelling with two different velocities. Because of different velocities their phases different.

Upon emerging from the crystal, the two components recombine to from a resultant wave. The phase difference between the two components causes the resultant wave to be either linearly or circularly or elliptical polarized depending on the phase difference between the two components. The type of polarization can be determined by means of an analyser which is a second polaroid sheet provided for this experiment.

EXPERIMENT 2: Electron Tube

Introduction

Free electrons in a metal may be thought of as being “electron gas” confined in potential or energy walls. Under normal conditions or even when a voltage is applied near the surface of the metal, these electrons cannot leave the potential walls (see fig. 19.11)

If however the metal or the electron gas is heated, the electrons have enough thermal energy (kinetic energy) to overcome the energy barrier W (W is known as “work function”). If a voltage is applied across the metal and the anode, these thermally activated electrons may reach the anode.

The number of electrons arriving at the anode per unit time depends on the nature of the cathode and the temperature, i.e. all electrons freed from the potential wall will reach the anode no longer increase with applied voltage (see fig. 19.11)

The saturated current corresponding to the number of thermally activated electrons freed from the metal surface per unit time obeys what is generally known as Richardson´s equation i.e.

$$I_B = C \cdot T^2 \cdot e^{-\frac{W}{k \cdot T}}$$

where

C is a constant

T temperature of the cathode in Kelvin

k Boltzmann´s constant = 1,38 . 10^{-23} J/K

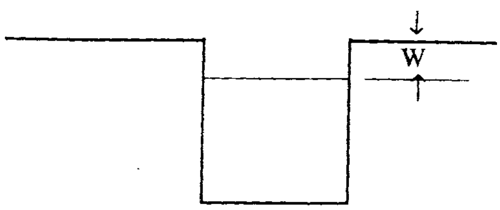


Fig 19.11

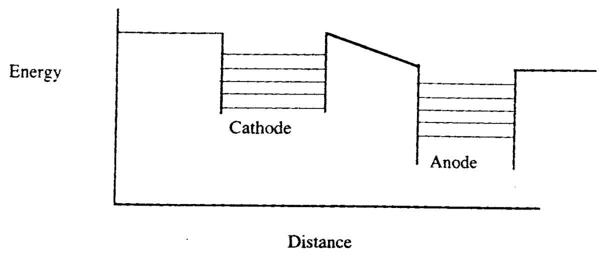


Fig.19.12

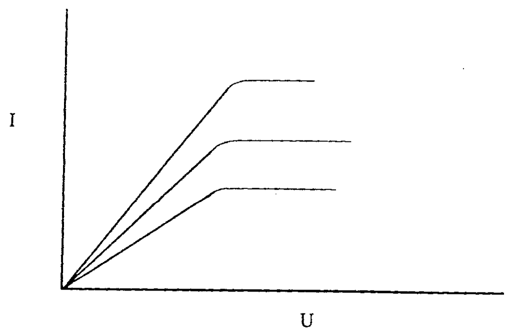


Fig 19.13 Graph of current as a function of voltage across anode-cathode

Determine the value of the work function W of tungsten metal in the form of heating filament of the vacuum tube provided.

The following items of equipment are placed at the disposal of the contestants:

- Electron tube AZ 41 which is a high-vacuum, full-wave rectifying diode. The cathode is made from a coated tungsten filament the work function of which is to be ascertained. According to the manual prepared by its manufacturer, no more than 4 V should be used when applying heating current to the cathode. Since the tube has two anodes, it is most desirable to have them connected for all measurements. The diagram in fig. 19.14 is a guide to identifying the anodes and the cathode.

- multimeter 1 unit, internal resistance for voltage measurement: 10 MΩ
- battery 1,5 V (together with a spare)
- battery 9 V; four units can be connected in series as shown in fig. 19.15
- connectors
- resistors; each of which has specifications as follows:
  1000 Ω ± 2% (brown, black, black, brown, brown, red)
  100 Ω ± 2% (brown, black, black, black, brown, red)
  47,5 Ω ± 1% (yellow, violet, green, gold, brown)
- resistors; 4 units, each of which has the resistance of about 1 Ω and coded
- connecting wires
- screw driver
- graph paper (1 sheet)
- graph of specific resistance of tungsten as a function of temperature; 1 sheet

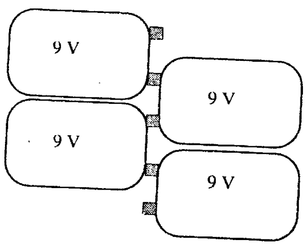


Fig 19.15

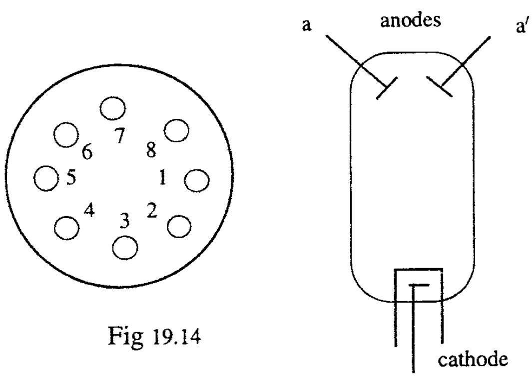


Fig 19.14

Solve the following problems:

2.1

Determine the resistance of 4 numerically-coded resistors. Under no circumstances must the multimeter be used as an ohmmeter.

2.2

Determine the saturated current for 4 different values of cathode temperatures, using 1,5 V battery to heat the cathode filament. A constant value of voltage between 35 V – 40 V between the anode and the cathode is sufficient to produce a saturated current. Obtain this value of voltage by connecting the four 9 V batteries in series. Describe how the different values of temperature are determined.

2.3

Determine the value of W. Explain the procedures used.

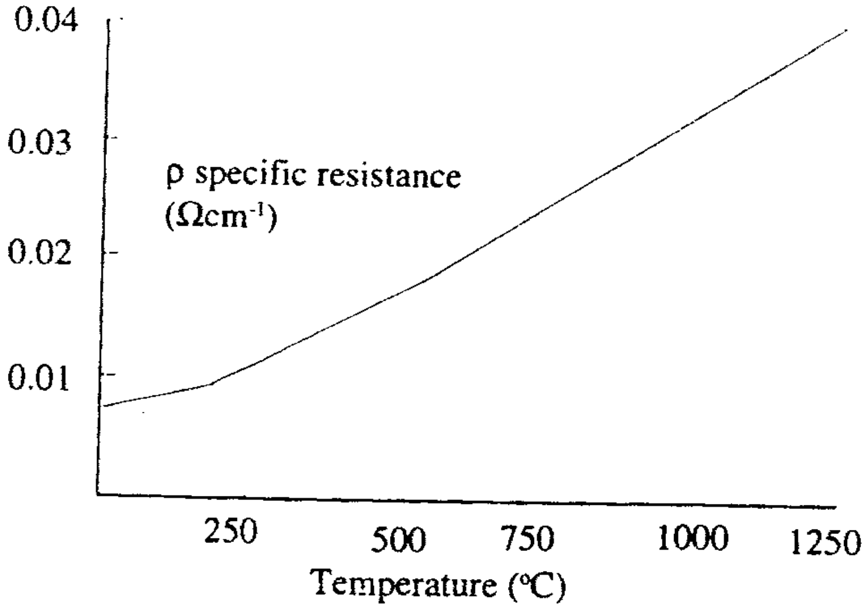


Fig 19.16

Solution

2.1

Connect the circuit as shown in fig. 19.17

R_X .... resistance to be determined
R ...... known value of resistance

Measure potential difference across R_X and R. Chose the value of R which gives comparable value of potential difference across R_X.

In this particular case R = 47,5 Ω

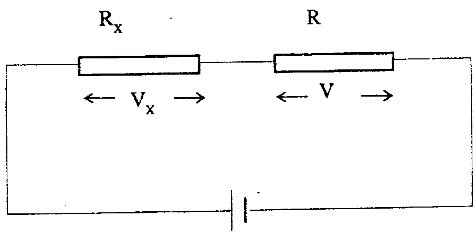


Fig. 19.17

$$\frac{R_X}{R} = \frac{V_X}{V}$$

where V_X and V are values of potential differences across R_X and R respectively.
R_X can be calculated from the above equation.
(The error in R_X depends on the errors of V_X and V_R).

2.2

Connect the circuit as shown in fig. 19.18

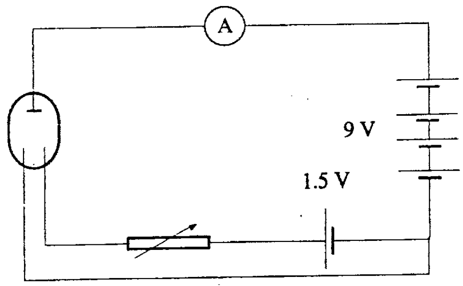


Fig 19.18

- Begin the experiment by measuring the resistance R_0 of the tungsten cathode when there is no heating current
- Add resistor R = 1000 Ω into the cathode circuit, determine resistance R_1 of the tungsten cathode, calculate the resistance of the current-carrying cathode.
- Repeat the experiment, using the resistor R = 100 Ω in the cathode circuit, determine resistance R_2 of tungsten cathode with heating current in the circuit.
- Repeat the experiment, using the resistor R = 47,5 Ω in the cathode circuit, determine resistance R_3 of tungsten cathode with heating current in the circuit.
- Plot a graph of $\frac{R_1}{R_0}, \frac{R_2}{R_0}$ and $\frac{R_3}{R_0}$ as a function of temperature, put the value of $\frac{R_0}{R_0} = 1$ to coincide with room temperature i.e. 18°C approximately and draw the remaining part of the graph parallel to the graph of specific resistance as a function of temperature provided in the problem. From the graph, read values of the temperature of the cathode T_1, T_2 and T_3 in Kelvin.

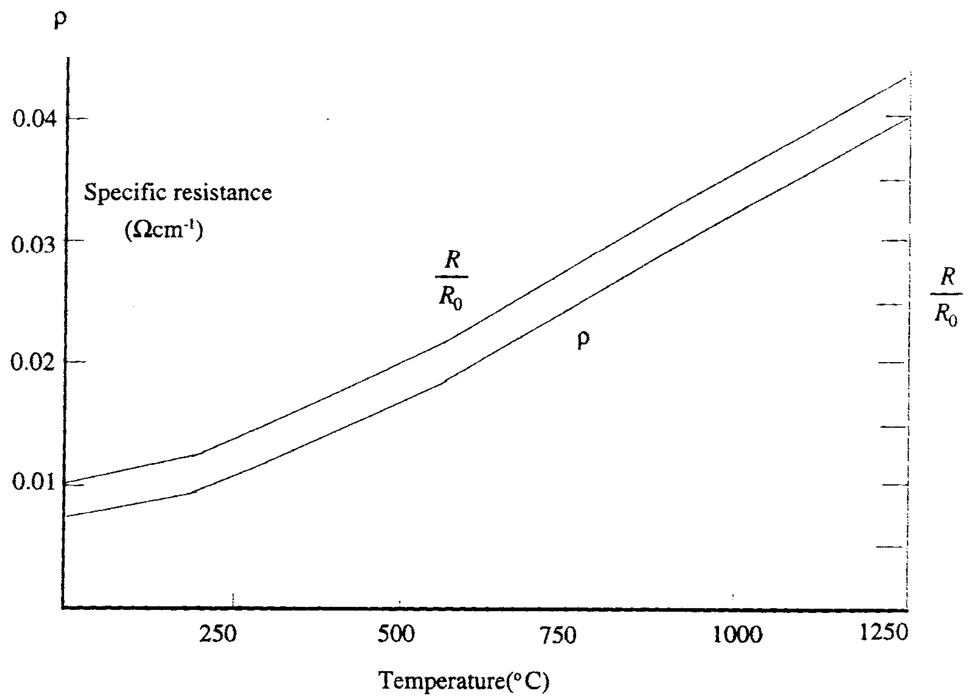


Fig 19.19

From the equation $I = C \cdot T^2 \cdot e^{-\frac{W}{k \cdot T}}$

we get $\ln \frac{I}{T^2} = -\frac{W}{k \cdot T} + \ln C$

Plot a graph of $\ln \frac{I}{T^2}$ against $\frac{1}{T}$.

The curve is linear. Determine the slope m from this graph. $-m = -\frac{W}{k}$

Work function W can be calculated using known values of m and k (given in the problem).

Error in W depends on the error of T which in turn depends on the error of measured R.

Problems of the 20th International Physics Olympiad [1]

(Warsaw, 1989)

Waldemar Gorzkowski

Institute of Physics, Polish Academy of Sciences, Warsaw, Poland [2]

Abstract

The article contains problems given at the 20th International Physics Olympiad (1989) and their solutions. The 20th IPhO was the third IPhO organized in Warsaw, Poland.

Logo

The emblem of the XX International Physics Olympiad contains a picture that is a historical record of the first hypernuclear event observed and interpreted in Warsaw by M. Danysz and J. Pniewski[3]. The collision of a high-energy particle with a heavy nucleus was registered in nuclear emulsion. Tracks of the secondary particles emitted in the event, seen in the picture (upper star), consist of tracks due to fast pions ("thin tracks") and to much slower fragments of the target nucleus ("black tracks"). The "black track" connecting the upper star (greater) with the lower star (smaller) in the figure is due to a hypernuclear fragment, in this case due to a part of the primary nucleus containing an unstable hyperon Λ instead of a nucleon. Hyperfragments (hypernuclei) are a new kind of matter in which the nuclei contain not only protons and neutrons but also some other heavy particles.

In the event observed above the hyperon Λ, bound with nucleon, decays like a free particle through a week (slow) process only. This fact strongly suggested the existence of a new quantum number that could explain suppression of the decay, even in presence of nucleons. Indeed, this was one of the observations that, 30 months later, led to the concept of strangeness.

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[1] This article has been sent for publication in *Physics Competitions* in October 2003

[2] e-mail: gorzk@ifpan.edu.pl

[3] M. Danysz and J. Pniewski, *Bull. Acad. Polon. Sci.*, **3(1)** 42 (1952) and *Phil. Mag.*, **44**, 348 (1953). Later the same physicists, Danysz and Pniewski, discovered the first case of a nucleus with two hyperons (double hyperfragment).

Introduction

Theoretical problems (including solutions and marking schemes) were prepared especially for the 20th IPhO by Waldemar Gorzkowski. The experimental problem (including the solution and marking scheme) was prepared especially for this Olympiad by Andrzej Kotlicki. The problems were refereed independently (and many times) by at least two persons after any change was made in the text to avoid unexpected difficulties at the competition. This work was done by:

First Problem:
Andrzej Szadkowski, Andrzej Szymacha, Włodzimierz Ungier

Second Problem:
Andrzej Szadkowski, Andrzej Szymacha, Włodzimierz Ungier, Stanisław Woronowicz

Third Problem:
Andrzej Rajca, Andrzej Szymacha, Włodzimierz Ungier

Experimental Problem:
Krzysztof Korona, Anna Lipniacka, Jerzy Łusakowski, Bruno Sikora

Several English versions of the texts of the problems were given to the English-speaking students. As far as I know it happened for the first time (at present it is typical). The original English version was accepted (as a version for the students) by the leaders of the Australian delegation only. The other English-speaking delegations translated the English originals into English used in their countries. The net result was that there were at least four English versions. Of course, physics contained in them was exactly the same, while wording and spelling were somewhat different (the difference, however, were not too great).

This article is based on the materials quoted at the end of the article and on personal notes of the author.

THEORETICAL PROBLEMS

Problem 1

Consider two liquids A and B insoluble in each other. The pressures p_i (i = A or B) of their saturated vapors obey, to a good approximation, the formula:

$$\ln(p_i / p_o) = \frac{\alpha_i}{T} + \beta_i; \qquad i = \text{A or B},$$

where p_o denotes the normal atmospheric pressure, T – the absolute temperature of the vapor, and α_i and β_i (i = A or B) – certain constants depending on the liquid. (The symbol ln denotes the natural logarithm, i.e. logarithm with base e = 2.7182818…)

The values of the ratio p_i/p_0 for the liquids A and B at the temperature 40°C and 90°C are given in Tab. 1.1.

Table 1.1

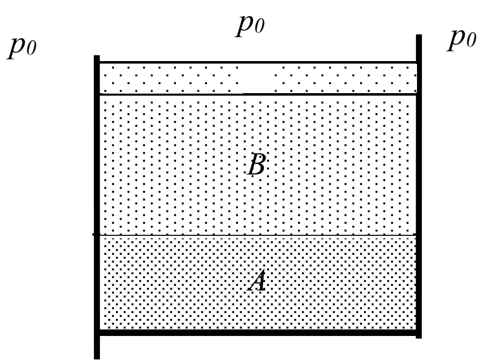
t [°C]	p_i/p_0	
	i = A	i = B
40	0.284	0.07278
90	1.476	0.6918

The errors of these values are negligible.

A. Determine the boiling temperatures of the liquids A and B under the pressure p_0.

B. The liquids A and B were poured into a vessel in which the layers shown in Fig. 1.1 were formed. The surface of the liquid B has been covered with a thin layer of a non-volatile liquid C, which is insoluble in the liquids A and B and vice versa, thereby preventing any free evaporation from the upper surface of the liquid B, The ratio of molecular masses of the liquids A and B (in the gaseous phase) is:

$$\gamma = \mu_A / \mu_B = 8.$$



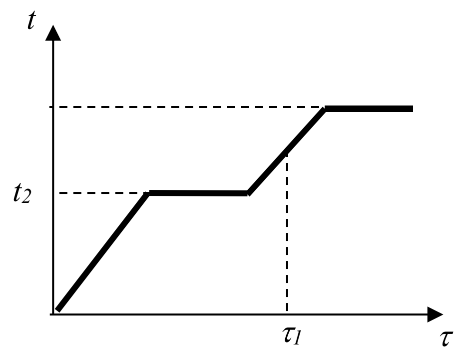


Fig. 1.1
Fig. 1.2

The masses of the liquids A and B were initially the same, each equal to m = 100g. The heights of the layers of the liquids in the vessel and the densities of the liquids are small enough to make the assumption that the pressure in any point in the vessel is practically equal to the normal atmospheric pressure p_0.

The system of liquids in the vessel is slowly, but continuously and uniformly, heated. It was established that the temperature t of the liquids changed with time τ as shown schematically in the Fig. 1.2.

Determine the temperatures t_1 and t_2 corresponding to the horizontal parts of the diagram and the masses of the liquids A and B at the time τ_1. The temperatures should be rounded to the nearest degree (in °C) and the masses of the liquids should be determined to one-tenth of gram.

REMARK: Assume that the vapors of the liquids, to a good approximation,

(1) obey the Dalton law stating that the pressure of a mixture of gases is equal to the sum of the partial pressures of the gases forming the mixture and

(2) can be treated as perfect gases up to the pressures corresponding to the saturated vapors.

Solution

PART A

The liquid boils when the pressure of its saturated vapor is equal to the external pressure. Thus, in order to find the boiling temperature of the liquid i (i - A or B), one should determine such a temperature T_{bi} (or t_{bi}) for which $p_i/p_0 = 1$.

Then $\ln(p_i / p_0) = 0$, and we have:

$$T_{bi} = -\frac{\alpha_i}{\beta_i}.$$

The coefficients α_i and β_i are not given explicitly. However, they can be calculated from the formula given in the text of the problem. For this purpose one should make use of the numerical data given in the Tab. 1.1.

For the liquid A, we have:

$$\ln 0.284 = \frac{\alpha_A}{(40+273.15)\mathrm{K}} + \beta_A,$$

$$\ln 1.476 = \frac{\alpha_A}{(90+273.15)\mathrm{K}} + \beta_A.$$

After subtraction of these equations, we get:

$$\ln 0.284 - \ln 1.476 = \alpha_A \left(\frac{1}{40+273.15} - \frac{1}{90+273.15} \right) \mathrm{K}^{-1}.$$

$$\alpha_A = \frac{\ln \frac{0.284}{1.476}}{\frac{1}{40+273.15} - \frac{1}{90+273.15}} \mathrm{K} \approx \text{-}3748.49\mathrm{K}.$$

Hence,

$$\beta_A = \ln 0.284 - \frac{\alpha_A}{(40+273.15)\mathrm{K}} \approx 10.711.$$

Thus, the boiling temperature of the liquid A is equal to

$$T_{bA} = 3748.49\mathrm{K}/10.711 \approx 349.95 \mathrm{~K}.$$

In the Celsius scale the boiling temperature of the liquid A is

$$t_{bA} = (349.95 - 273.15)°\mathrm{C} = 76.80°\mathrm{C} \approx 77°\mathrm{C}.$$

For the liquid B, in the same way, we obtain:

$$\alpha_B \approx \text{-}5121.64 \mathrm{~K},$$

$$\beta_B \approx 13.735,$$

$$T_{bB} \approx 372\text{-}89 \mathrm{~K},$$

$$t_{bB} \approx 99.74°\mathrm{C} \approx 100°\mathrm{C}.$$

PART B

As the liquids are in thermal contact with each other, their temperatures increase in time in the same way.

At the beginning of the heating, what corresponds to the left sloped part of the diagram, no evaporation can occur. The free evaporation from the upper surface of the liquid B cannot occur - it is impossible due to the layer of the non-volatile liquid C. The evaporation from the inside of the system is considered below.

Let us consider a bubble formed in the liquid A or in the liquid B or on the surface that separates these liquids. Such a bubble can be formed due to fluctuations or for many other reasons, which will not be analyzed here.

The bubble can get out of the system only when the pressure inside it equals to the external pressure p_0 (or when it is a little bit higher than p_0). Otherwise, the bubble will collapse.

The pressure inside the bubble formed in the volume of the liquid A or in the volume of the liquid B equals to the pressure of the saturated vapor of the liquid A or B, respectively. However, the pressure inside the bubble formed on the surface separating the liquids A and B is equal to the sum of the pressures of the saturated vapors of both these liquids, as then the bubble is in a contact with the liquids A and B at the same time. In the case considered the pressure inside the bubble is greater than the pressures of the saturated vapors of each of the liquids A and B (at the same temperature).

Therefore, when the system is heated, the pressure p_0 is reached first in the bubbles that were formed on the surface separating the liquids. Thus, the temperature t_1 corresponds to a kind of common boiling of both liquids that occurs in the region of their direct contact. The temperature t_1 is for sure lower than the boiling temperatures of the liquids A and B as then the pressures of the saturated vapors of the liquids A and B are less then p_0 (their sum equals to p_0 and each of them is greater than zero).

In order to determine the value of t_1 with required accuracy, we can calculate the values of the sum of the saturated vapors of the liquids A and B for several values of the temperature *t* and look when one gets the value p_0.

From the formula given in the text of the problem, we have:

$$\frac{p_A}{p_0} = e^{\frac{\alpha_A}{T} + \beta_A}, \tag{1}$$

$$\frac{p_B}{p_0} = e^{\frac{\alpha_B}{T} + \beta_B}. \tag{2}$$

$p_A + p_B$ equals to p_0 if

$$\frac{p_A}{p_0} + \frac{p_B}{p_0} = 1.$$

Thus, we have to calculate the values of the following function:

$$y(x) = e^{\frac{\alpha_A}{t+t_0}+\beta_A} + e^{\frac{\alpha_B}{t+t_0}+\beta_B},$$

(where $t_0 = 273.15$ °C) and to determine the temperature $t = t_1$, at which $y(t)$ equals to 1. When calculating the values of the function $y(t)$ we can divide the intervals of the temperatures t by 2 (approximately) and look whether the results are greater or less than 1.

We have:

Table 1.2

t	$y(t)$
40°C	< 1 (see Tab. 1.1)
77°C	> 1 (as t_1 is less than t_{bA})
59°C	0.749 < 1
70°C	1.113 > 1
66°C	0.966 < 1
67°C	1.001 > 1
66.5°C	0.983 < 1

Therefore, $t_1 \approx$ 67° C (with required accuracy).

Now we calculate the pressures of the saturated vapors of the liquids A and B at the temperature $t_1 \approx$ 67°C, i.e. the pressures of the saturated vapors of the liquids A and B in each bubble formed on the surface separating the liquids. From the equations (1) and (2), we get:

$$p_A \approx 0.734\, p_0 ,$$

$$p_B \approx 0.267\, p_0 ,$$

$$(p_A + p_B = 1.001 p_0 \approx p_0) .$$

These pressures depend only on the temperature and, therefore, they remain constant during the motion of the bubbles through the liquid B. The volume of the bubbles during this motion also cannot be changed without violation of the relation $p_A + p_B = p_0$. It follows from the above remarks that the mass ratio of the saturated vapors of the liquids A and B in each bubble is the same. This conclusion remains valid as long as both liquids are in the system. After total evaporation of one of the liquids the temperature of the system will increase again (second sloped part of the diagram). Then, however, the mass of the system remains constant until the temperature reaches the value t_2 at which the boiling of the liquid (remained in the vessel) starts. Therefore, the temperature t_2 (the higher horizontal part of the diagram) corresponds to the boiling of the liquid remained in the vessel.

The mass ratio m_A / m_B of the saturated vapors of the liquids A and B in each bubble leaving the system at the temperature t_1 is equal to the ratio of the densities of these vapors ρ_A / ρ_B. According to the assumption 2, stating that the vapors can be treated as ideal gases, the last ratio equals to the ratio of the products of the pressures of the saturated vapors by the molecular masses:

$$\frac{m_A}{m_B} = \frac{\rho_A}{\rho_B} = \frac{p_A \mu_A}{p_B \mu_B} = \frac{p_A}{p_B} \mu .$$

Thus,

$$\frac{m_A}{m_B} \approx 22.0 .$$

We see that the liquid A evaporates 22 times faster than the liquid B. The evaporation of 100 g of the liquid A during the “surface boiling” at the temperature t_1 is associated with the evaporation of 100 g / 22 $\approx$ 4.5 g of the liquid B. Thus, at the time τ_1 the vessel contains 95.5 g of the liquid B (and no liquid A). The temperature t_2 is equal to the boiling temperature of the liquid B: $t_2 = 100°$C.

Marking Scheme

1 . physical condition for boiling 1 point
2. boiling temperature of the liquid A (numerical value) 1 point
3 . boiling temperature of the liquid B (numerical value) 1 point
4 . analysis of the phenomena at the temperature t_1 3 points
5 . numerical value of t_1 1 point
6 . numerical value of the mass ratio of the saturated vapors in the bubble 1 point
7 . masses of the liquids at the time τ_1 1 point
8 . determination of the temperature t_2 1 point

REMARK: As the sum of the logarithms is not equal to the logarithm of the sum, the formula given in the text of the problem should not be applied to the mixture of the saturated vapors in the bubbles formed on the surface separating the liquids. However, the numerical data have been chosen in such a way that even such incorrect solution of the problem gives the correct value of the temperature t_1 (within required accuracy). The purpose of that was to allow the pupils to solve the part B of the problem even if they determined the temperature t_1 in a wrong way. Of course, one cannot receive any points for an incorrect determination of the temperature t_1 even if its numerical value is correct.

Typical mistakes in the pupils' solutions

Nobody has received the maximum possible number of points for this problem, although several solutions came close. Only two participants tried to analyze proportion of pressures of the vapors during the upward movement of the bubble trough the liquid B. Part of the students confused Celsius degrees with Kelvins. Many participants did not take into account the boiling on the surface separating the liquids A and B, although this effect was the essence of the problem. Part of the students, who did notice this effect, assumed a priori that the liquid with lower boiling temperature "must" be the first to evaporate. In general, this need not be true: if γ were, for example, 1/8 instead 8, then liquid A rather than B would remain in the vessel. As regards the boiling temperatures, practically nobody had any essential difficulties.

Problem 2

Three non-collinear points P_1, P_2 and P_3, with known masses m_1, m_2 and m_3, interact with one another through their mutual gravitational forces only; they are isolated in free space and do not interact with any other bodies. Let σ denote the axis going through the center-of-mass of the three masses, and perpendicular to the triangle $P_1P_2P_3$. What conditions should the angular velocities ω of the system (around the axis σ) and the distances:

$$P_1P_2 = a_{12}, \qquad P_2P_3 = a_{23}, \qquad P_1P_3 = a_{13},$$

fulfill to allow the shape and size of the triangle $P_1P_2P_3$ unchanged during the motion of the system, i.e. under what conditions does the system rotate around the axis σ as a rigid body?

Solution

As the system is isolated, its total energy, i.e. the sum of the kinetic and potential energies, is conserved. The total potential energy of the points P_1, P_2 and P_3 with the masses m_1, m_2 and m_3 in the inertial system (i.e. when there are no inertial forces) is equal to the sum of the gravitational potential energies of all the pairs of points (P_1,P_2), (P_2,P_3) and (P_1,P_3). It depends only on the distances a_{12}, a_{23} and a_{23} which are constant in time. Thus, the total potential energy of the system is constant. As a consequence the kinetic energy of the system is constant too. The moment of inertia of the system with respect to the axis σ depends only on the distances from the points P_1, P_2 and P_3 to the axis σ which, for fixed a_{12}, a_{23} and a_{23} do not depend on time. This means that the moment of inertia I is constant. Therefore, the angular velocity of the system must also be constant:

$$\omega = \text{const.} \qquad (1)$$

This is the first condition we had to find. The other conditions will be determined by using three methods described below. However, prior to performing calculations, it is desirable to specify a convenient coordinates system in which the calculations are expected to be simple.

Let the positions of the points P_1, P_2 and P_3 with the masses m_1, m_2 and m_3 be given by the vectors $\mathbf{r}_1$, $\mathbf{r_2}$ and $\mathbf{r_3}$. For simplicity we assume that the origin of the coordinate system is localized at the center of mass of the points P_1, P_2 and P_3 with the masses m_1, m_2 and m_3 and that all the vectors $\mathbf{r}_1$, $\mathbf{r_2}$ and $\mathbf{r_3}$ are in the same coordinate plane, e.g. in the plane (x,y). Then the axis σ is the axis z.

In this coordinate system, according to the definition of the center of mass, we have:

$$m_1\mathbf{r_1} + m_2\mathbf{r_2} + m_3\mathbf{r_2} = 0 \qquad (2)$$

Now we will find the second condition by using several methods.

FIRST METHOD

Consider the point P_1 with the mass m_1. The points P_2 and P_3 act on it with the forces:

$$\mathbf{F}_{21} = G\frac{m_1 m_2}{a_{12}^3}(\mathbf{r}_2 - \mathbf{r}_1), \qquad (3)$$

$$\mathbf{F}_{31} = G\frac{m_1 m_3}{a_{13}^3}(\mathbf{r}_3 - \mathbf{r}_1). \qquad (4)$$

where G denotes the gravitational constant.

In the inertial frame the sum of these forces is the centripetal force

$$\mathbf{F}_{r1} = -m_1\omega^2\mathbf{r}_1,$$

which causes the movement of the point P_1 along a circle with the angular velocity ω. (The moment of this force with respect to the axis σ is equal to zero.) Thus, we have:

$$\mathbf{F}_{21} + \mathbf{F}_{31} = \mathbf{F}_{r1}. \qquad (5)$$

In the non-inertial frame, rotating around the axis σ with the angular velocity ω, the sum of the forces (3), (4) and the centrifugal force

$$\mathbf{F'}_{r1} = m_1\omega^2\mathbf{r}_1$$

should be equal to zero:

$$\mathbf{F}_{21} + \mathbf{F}_{31} + \mathbf{F}'_{r1} = 0. \qquad (6)$$

(The moment of this sum with respect to any axis equals to zero.)

The conditions (5) and (6) are equivalent. They give the same vector equality:

$$G\frac{m_1 m_2}{a_{12}^3}(\mathbf{r}_2 - \mathbf{r}_1) + G\frac{m_1 m_3}{a_{13}^3}(\mathbf{r}_3 - \mathbf{r}_1) + m_1\omega^2\mathbf{r}_1 = 0, \qquad (7')$$

$$G\frac{m_1}{a_{12}^3}m_2\mathbf{r}_2 + G\frac{m_1}{a_{13}^3}m_3\mathbf{r}_3 + m_1\mathbf{r}_1\left(\omega^2 - \frac{Gm_2}{a_{12}^3} - \frac{Gm_3}{a_{13}^3}\right) = 0 \qquad (7'')$$

From the formula (2), we get:

$$m_2\mathbf{r}_2 = -m_1\mathbf{r}_1 - m_3\mathbf{r}_3 \qquad (8)$$

Using this relation, we write the formula (7) in the following form:

$$G\frac{m_1}{a_{12}^3}(-m_1\mathbf{r}_1 - m_3\mathbf{r}_3) + G\frac{m_1}{a_{13}^3}m_3\mathbf{r}_3 + m_1\mathbf{r}_1\left(\omega^2 - \frac{Gm_2}{a_{12}^3} - \frac{Gm_3}{a_{13}^3}\right) = 0,$$

i.e.

$$\mathbf{r}_1 m_1\left(\omega^2 - \frac{Gm_2}{a_{12}^3} - \frac{Gm_3}{a_{13}^3} - \frac{Gm_1}{a_{12}^3}\right) + \mathbf{r}_3\left(\frac{1}{a_{13}^3} - \frac{1}{a_{12}^3}\right)Gm_1m_3 = 0.$$

The vectors $\mathbf{r}_1$ and $\mathbf{r}_3$ are non-collinear. Therefore, the coefficients in the last formula must be equal to zero:

$$\left(\frac{1}{a_{13}^3} - \frac{1}{a_{12}^3}\right)Gm_1m_3 = 0,$$

$$m_1\left(\omega^2 - \frac{Gm_2}{a_{12}^3} - \frac{Gm_3}{a_{13}^3} - \frac{Gm_1}{a_{12}^3}\right) = 0.$$

The first equality leads to:

$$\frac{1}{a_{13}^3} = \frac{1}{a_{12}^3}$$

and hence,

$$a_{13} = a_{12}.$$

Let $a_{13} = a_{12} = a$. Then the second equality gives:

$$\omega^2 a^3 = GM \qquad (9)$$

where

$$M = m_1 + m_2 + m_3 \qquad (10)$$

denotes the total mass of the system.

In the same way, for the points P_2 and P_3, one gets the relations:

a) the point P_2:

$$a_{23} = a_{12}; \qquad \omega^2 a^3 = GM$$

b) the point P_3:

$$a_{13} = a_{23}; \qquad \omega^2 a^3 = GM$$

Summarizing, the system can rotate as a rigid body if all the distances between the masses are equal:

$$a_{12} = a_{23} = a_{13} = a, \qquad (11)$$

the angular velocity ω is constant and the relation (9) holds.

SECOND METHOD

At the beginning we find the moment of inertia I of the system with respect to the axis σ. Using the relation (2), we can write:

$$0 = (m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + m_3\mathbf{r}_3)^2 = m_1^2\mathbf{r}_1^2 + m_2^2\mathbf{r}_2^2 + m_3^2\mathbf{r}_3^2 + 2m_1m_2\mathbf{r}_1\mathbf{r}_2 + 2m_1m_3\mathbf{r}_1\mathbf{r}_3 + 2m_3m_2\mathbf{r}_3\mathbf{r}_2.$$

Of course,

$$\mathbf{r}_i^2 = r_i^2 \qquad i = 1, 2, 3$$

The quantities $2\mathbf{r}_i\mathbf{r}_j$ (i, j = 1, 2, 3) can be determined from the following obvious relation:

$$a_{ij}^2 = \left|\mathbf{r}_i - \mathbf{r}_j\right|^2 = (\mathbf{r}_i - \mathbf{r}_j)^2 = \mathbf{r}_i^2 + \mathbf{r}_j^2 - 2\mathbf{r}_i\mathbf{r}_j = r_i^2 + r_j^2 - 2\mathbf{r}_i\mathbf{r}_j.$$

We get:

$$2\mathbf{r}_i\mathbf{r}_j = r_i^2 + r_j^2 - a_{ij}^2.$$

With help of this relation, after simple transformations, we obtain:

$$0=(m_1\mathbf{r}_1+m_2\mathbf{r}_2+m_3\mathbf{r}_3)^2=(m_1+m_2+m_3)(m_1r_1^2+m_2r_2^2+m_3r_3^2)-\sum_{i<j}m_im_ja_{ij}^2.$$

The moment of inertia I of the system with respect to the axis σ, *a*ccording to the definition of this quantity, is equal to

$$I=m_1r_1^2+m_2r_2^2+m_3r_3^2.$$

The last two formulae lead to the following expression:

$$I=\frac{1}{M}\sum_{i<j}m_im_ja_{ij}^2$$

where M (the total mass of the system) is defined by the formula (10).

In the non-inertial frame, rotating around the axis σ with the angular velocity ω, the total potential energy V_{tot} is the sum of the gravitational potential energies

$$V_{ij}=-G\frac{m_im_j}{a_{ij}}; \qquad i,j=1,2,3;\ i<j$$

of all the masses and the potential energies

$$V_i=-\tfrac{1}{2}\omega^2m_ir_i^2; \qquad i=1,2,3$$

of the masses m_i (i = 1, 2, 3) in the field of the centrifugal force:

$$V_{tot}=G\sum_{i<j}\frac{m_im_j}{a_{ij}}-\tfrac{1}{2}\omega^2\sum_{i=1}^{3}m_ir_i^2=G\sum_{i<j}\frac{m_im_j}{a_{ij}}-\tfrac{1}{2}\omega^2I=G\sum_{i<j}\frac{m_im_j}{a_{ij}}-\tfrac{1}{2}\omega^2\frac{1}{M}\sum_{i<j}m_im_ja_{ij}^2=$$
$$=-\sum_{i<j}m_im_j\left(\frac{\omega^2}{2M}a_{ij}^2+\frac{G}{a_{ij}}\right)$$

i.e.

$$V_{tot}=-\sum_{i<j}m_im_j\left(\frac{\omega^2}{2M}a_{ij}^2+\frac{G}{a_{ij}}\right).$$

A mechanical system is in equilibrium if its total potential energy has an extremum. In our case the total potential energy V_{tot} is a sum of three terms. Each of them is proportional to:

$$f(a)=\frac{\omega^2}{2M}a^2+\frac{G}{a}.$$

The extrema of this function can be found by taking its derivative with respect to a and requiring this derivative to be zero. We get:

$$\frac{\omega^2}{M}a - \frac{G}{a^2} = 0.$$

It leads to:

$$\omega^2 a^3 = GM \qquad \text{or} \qquad \omega^2 a^3 = G(m_1 + m_2 + m_3).$$

We see that all the terms in V_{tot} have extrema at the same values of $a_{ij} = a$. (In addition, the values of a and ω should obey the relation written above.) It is easy to show that it is a maximum. Thus, the quantity V_{tot} has a maximum at $a_{ij} = a$.

This means that our three masses can remain in fixed distances only if these distances are equal to each other:

$$a_{12} = a_{23} = a_{13} = a$$

and if the relation

$$\omega^2 a^3 = GM \ ,$$

where M the total mass of the system, holds. We have obtained the conditions (9) and (11) again.

THIRD METHOD

Let us consider again the point P_1 with the mass m_1 and the forces $\mathbf{F}_{21}$ and $\mathbf{F}_{31}$ given by the formulae (3) and (4). It follows from the text of the problem that the total moment (with respect to any fixed point or with respect to the mass center) of the forces acting on the point P_1 must be equal to zero. Thus, we have:

$$\mathbf{F}_{21} \times \mathbf{r}_1 + \mathbf{F}_{31} \times \mathbf{r}_1 = 0$$

where the symbol $\times$ denotes the vector product. Therefore

$$G\frac{m_1 m_2}{a_{12}^3}(\mathbf{r}_2 - \mathbf{r}_1) \times \mathbf{r}_1 + G\frac{m_1 m_3}{a_{13}^3}(\mathbf{r}_3 - \mathbf{r}_1) \times \mathbf{r}_1 = 0.$$

But

$$\mathbf{r}_1 \times \mathbf{r}_1 = 0.$$

Thus:

$$\frac{m_2}{a_{12}^3}\mathbf{r}_2 \times \mathbf{r}_1 + \frac{m_3}{a_{13}^3}\mathbf{r}_3 \times \mathbf{r}_1 = 0.$$

Using the formula (8), the last relation can be written as follows:

$$\frac{1}{a_{12}^3}(-m_1\mathbf{r}_1 - m_3\mathbf{r}_3) \times \mathbf{r}_1 + \frac{m_3}{a_{13}^3}\mathbf{r}_3 \times \mathbf{r}_1 = 0,$$

$$-\frac{m_3}{a_{12}^3}\mathbf{r}_3 \times \mathbf{r}_1 + \frac{m_3}{a_{13}^3}\mathbf{r}_3 \times \mathbf{r}_1 = 0,$$

$$\left(\frac{1}{a_{13}^3} - \frac{1}{a_{12}^3}\right)\mathbf{r}_3 \times \mathbf{r}_1 = 0.$$

The vectors $\mathbf{r}_1$ and $\mathbf{r}_3$ are non-col1inear (and different from 0). Therefore

$$\mathbf{r}_3 \times \mathbf{r}_1 \neq 0$$

and

$$\frac{1}{a_{13}^3} - \frac{1}{a_{12}^3} = 0,$$

hence,

$$a_{12} = a_{13}.$$

Similarly, one gets:

$$a_{12} = a_{23} \ \ (= a).$$

We have re-derived the condition (11).

Taking into account that all the distances a_{ij} have the same value *a*, from the equation (7) concerning the point P_1, using the relation (2) we obtain:

$$G\frac{m_1 m_2}{a^3}(\mathbf{r}_2 - \mathbf{r}_1) + G\frac{m_1 m_3}{a^3}(\mathbf{r}_3 - \mathbf{r}_1) + m_1\omega^2\mathbf{r}_1 = 0,$$

$$-\left(G\frac{m_1}{a^3} + G\frac{m_2}{a^3}G\frac{m_3}{a^3}\right)m_1\mathbf{r}_1 + m_1\omega^2\mathbf{r}_1 = 0,$$

$$\frac{GM}{a^3} = \omega^2 .$$

This is the condition (9). The same condition is got in result of similar calculations for the points P_2 and P_3.

The method described here does not differ essentially from the first method. In fact they are slight modifications of each other. However, it is interesting to notice how application of a proper mathematical language, e.g. the vector product, simplifies the calculations.

The relation (9) can be called a "generalized Kepler's law" as, in fact, it is very similar to the Kepler's law but with respect to the many-body system. As far as I know this generalized Kepler's law was presented for the first time right at the 20th IPhO.

Marking scheme

1. the proof that $\omega = \text{const}$ 1 point
2. the conditions at the equilibrium (conditions for the forces and their moments or extremum of the total potential energy) 3 points
3. the proof of the relation $a_{ij} = a$ 4 points
4. the proof of the relation $\omega^2 a^3 = GM$ 2 points

Remarks and typical mistakes in the pupils' solutions

No type of error was observed as predominant in the pupils' solutions. Practically all the mistakes can be put down to the students' scant experience in calculations and general lack of skill. Several students misunderstood the text of the problem and attempted to prove that the three masses should be equal. Of course, this was impossible. Moreover, it was pointless, since the masses were given. Almost all the participants tried to solve the problem by analyzing equilibrium of forces and/or their moments. Only one student tried to solve the problem by looking for a minimum of the total potential energy (unfortunately, his solution was not fully correct). Several participants solved the problem using a convenient reference system: one mass in the origin and one mass on the *x*-axis. One of them received a special prize.

Problem 3

The problem concerns investigation of transforming the electron microscope with magnetic guiding of the electron beam (which is accelerated with the potential difference $U = 511$ kV) into a proton microscope (in which the proton beam is accelerated with the potential difference $-U$). For this purpose, solve the following two problems:

A. An electron after leaving a device, which accelerated it with the potential difference U, falls into a region with an inhomogeneous field $\boldsymbol{B}$ generated with a system of stationary coils $L_1, L_2, \dots, L_n$. The known currents in the coils are $i_1, i_2, \dots, i_n$, respectively.

What should the currents $i_1', i_2', \dots, i_n'$ in the coils $L_1, L_2, \dots, L_n$ be, in order to guide the proton (initially accelerated with the potential difference $-U$) along the same trajectory (and in the same direction) as that of the electron?

HINT: The problem can be solved by finding a condition under which the equation describing the trajectory is the same in both cases. It may be helpful to use the relation:

$$\boldsymbol{p}\frac{d}{dt}\boldsymbol{p} = \frac{1}{2}\frac{d}{dt}\boldsymbol{p}^2 = \frac{1}{2}\frac{d}{dt}p^2.$$

B. How many times would the resolving power of the above microscope increase or decrease if the electron beam were replaced with the proton beam? Assume that the resolving power of the microscope (i.e. the smallest distance between two point objects whose circular images can be just separated) depends only on the wave properties of the particles.

Assume that the velocities of the electrons and protons before their acceleration are zero, and that there is no interaction between own magnetic moment of either electrons or protons and the magnetic field. Assume also that the electromagnetic radiation emitted by the moving particles can be neglected.

NOTE: Very often physicists use 1 electron-volt (1 eV), and its derivatives such as 1 keV or 1 MeV, as a unit of energy. 1 electron-volt is the energy gained by the electron that passed the potential difference equal to 1 V.

Perform the calculations assuming the following data:

Rest energy of electron: $E_e = m_e c^2 = 511$ keV

Rest energy of proton: $E_p = m_p c^2 = 938$ MeV

Solution

PART A

At the beginning one should notice that the kinetic energy of the electron accelerated with the potential difference U = 511 kV equals to its rest energy E_0. Therefore, at least in the case of the electron, the laws of the classical physics cannot be applied. It is necessary to use relativistic laws.

The relativistic equation of motion of a particle with the charge e in the magnetic field $\mathbf{B}$ has the following form:

$$\frac{d}{dt}\mathbf{p} = \mathbf{F}_L$$

where $\mathbf{p} = m_0\gamma\mathbf{v}$ denotes the momentum of the particle (vector) and

$$\mathbf{F}_L = e\mathbf{v} \times \mathbf{B}$$

is the Lorentz force (its value is evB and its direction is determined with the right hand rule). m_0 denotes the (rest) mass of the particle and v denotes the velocity of the particle. The quantity γ is given by the formula:

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

The Lorentz force $\mathbf{F}_L$ is perpendicular to the velocity $\mathbf{v}$ of the particle and to its momentum $\mathbf{p} = m_0\gamma\mathbf{v}$. Hence,

$$\mathbf{F}_L \cdot \mathbf{v} = \mathbf{F}_L \cdot \mathbf{p} = 0\,.$$

Multiplying the equation of motion by $\mathbf{p}$ and making use of the hint given in the text of the problem, we get:

$$\frac{1}{2}\frac{d}{dt}p^2 = 0\,.$$

It means that the value of the particle momentum (and the value of the velocity) is constant during the motion:

$$p = m_0 v\gamma = \text{ const}; \qquad\qquad v = \text{const}.$$

The same result can be obtained without any formulae in the following way:

The Lorentz force $\mathbf{F}_L$ is perpendicular to the velocity $\mathbf{v}$ (and to the momentum p as $\mathbf{p} = m_0\gamma\mathbf{v}$) and, as a consequence, to the trajectory of the particle. Therefore, there is no force that could change the component of the momentum tangent to the trajectory. Thus, this component, whose value is equal to the length of $\mathbf{p}$, should be constant: $\mathbf{p} = \text{const}$. (The same refers to the component of the velocity tangent to the trajectory as $\mathbf{p} = m_0\gamma\mathbf{v}$).

Let s denotes the path passed by the particle along the trajectory. From the definition of the velocity, we have:

$$\frac{ds}{dt} = v.$$

Using this formula, we can rewrite the equation of motion as follows:

$$v\frac{d}{ds}\mathbf{p} = \frac{ds}{dt}\frac{d}{ds}\mathbf{p} = \frac{d}{dt}\mathbf{p} = \mathbf{F}_L,$$

$$\frac{d}{ds}\mathbf{p} = \frac{\mathbf{F}_L}{v}.$$

Dividing this equation by p and making use of the fact that $p = \text{const}$, we obtain:

$$v\frac{d}{ds}\frac{\mathbf{p}}{p} = \frac{\mathbf{F}_L}{vp}$$

and hence

$$\frac{d}{ds}\mathbf{t} = \frac{\mathbf{F}_L}{vp}$$

where $\mathbf{t} = \mathbf{p}/p = \mathbf{v}/v$ is the versor (unit vector) tangent to the trajectory. The above equation is exactly the same for both electrons and protons if and only if the vector quantity:

$$\frac{\mathbf{F}_L}{vp}$$

is the same in both cases.

Denoting corresponding quantities for protons with the same symbols as for the electrons, but with primes, one gets that the condition, under which both electrons and protons can move along the same trajectory, is equivalent to the equality:

$$\frac{\mathbf{F}_L}{vp} = \frac{\mathbf{F}'_L}{v'p'}.$$

However, the Lorentz force is proportional to the value of the velocity of the particle, and the directions of any two vectors of the following three: **t** (or **v**), $\mathbf{F}_L$, **B** determine the direction of the third of them (right hand rule). Therefore, the above condition can be written in the following form:

$$\frac{e\mathbf{B}}{p} = \frac{e'\mathbf{B}'}{p'} .$$

Hence,

$$\mathbf{B}' = \frac{e}{e'}\frac{p'}{p}\mathbf{B} = \frac{p'}{p}\mathbf{B} .$$

This means that at any point the direction of the field **B** should be conserved, its orientation should be changed into the opposite one, and the value of the field should be multiplied by the same factor p'/p. The magnetic field **B** is a vector sum of the magnetic fields of the coils that are arbitrarily distributed in the space. Therefore, each of this fields should be scaled with the same factor $-p'/p$. However, the magnetic field of any coil is proportional to the current flowing in it. This means that the required scaling of the fields can only be achieved by the scaling of all the currents with the same factor $-p'/p$:

$$i'_n = -\frac{p'}{p} i_n .$$

Now we shall determine the ratio p'/p. The kinetic energies of the particles in both cases are the same; they are equal to $E_k = e|U| =$ 511 keV. The general relativistic relation between the total energy E of the particle with the rest energy E_0 and its momentum p has the following form:

$$E^2 = E_0^2 + p^2c^2$$

where c denotes the velocity of light.

The total energy of considered particles is equal to the sum of their rest and kinetic energies:

$$E = E_0 + E_k .$$

Using these formulae and knowing that in our case $E_k = e|U| = E_e$, we determine the momenta of the electrons (p) and the protons (p’). We get:

a) electrons:

$$(E_e + E_e)^2 = E_e^2 + p^2c^2 ,$$

$$p = \frac{E_e}{c}\sqrt{3}.$$

b) protons

$$(E_p + E_e)^2 = E_p^2 + p'^2 c^2,$$

$$p' = \frac{E_e}{c}\sqrt{\left(\frac{E_p}{E_e}+1\right)^2 - \left(\frac{E_p}{E_e}\right)^2}.$$

Hence,

$$\frac{p'}{p} = \frac{1}{\sqrt{3}}\sqrt{\left(\frac{E_p}{E}+1\right)^2 - \left(\frac{E_p}{E_e}\right)^2} \approx 35.0$$

and

$$i'_n = -35.0\, i_n.$$

It is worthwhile to notice that our protons are 'almost classical', because their kinetic energy $E_k (= E_e)$ is small compared to the proton rest energy E_p. Thus, one can expect that the momentum of the proton can be determined, with a good accuracy, from the classical considerations. We have:

$$E_e = E_k = \frac{p'^2}{2m_p} = \frac{p'^2 c^2}{2m_p c^2} = \frac{p'^2 c^2}{2E_p},$$

$$p' = \frac{1}{c}\sqrt{2E_e E_p}.$$

On the other hand, the momentum of the proton determined from the relativistic formulae can be written in a simpler form since E_p/E_e » 1. We get:

$$p' = \frac{E_e}{c}\sqrt{\left(\frac{E_p}{E_e}+1\right)^2 - \left(\frac{E_p}{E_e}\right)^2} = \frac{E_e}{c}\sqrt{2\frac{E_p}{E_e}+1} \approx \frac{E_e}{c}\sqrt{2\frac{E_p}{E_e}} = \frac{1}{c}\sqrt{2E_e E_p}.$$

In accordance with our expectations, we have obtained the same result as above.

PART B

The resolving power of the microscope (in the meaning mentioned in the text of the problem) is proportional to the wavelength, in our case to the length of the de Broglie wave:

$$\lambda = \frac{h}{p}$$

where h denotes the Planck constant and p is the momentum of the particle. We see that λ is inversely proportional to the momentum of the particle. Therefore, after replacing the electron beam with the proton beam the resolving power will be changed by the factor $p/p' \approx 1/35$. It means that our proton microscope would allow observation of the objects about 35 times smaller than the electron microscope.

Marking scheme

1. the relativistic equation of motion	1 point
2. independence of p and v of the time	1 point
3. identity of $e\mathbf{B}/p$ in both cases	2 points
4. scaling of the fields and the currents with the same factor	1 point
5. determination of the momenta (relativistically)	1 point
6. the ratio of the momenta (numerically)	1 point
7. proportionality of the resolving power to λ	1 point
8. inverse proportionality of λ to p	1 point
9. scaling of the resolving power	1 point

Remarks and typical mistakes in the pupils' solutions

Some of the participants tried to solve the problem by using laws of classical mechanics only. Of course, this approach was entirely wrong. Some students tried to find the required condition by equating "accelerations" of particles in both cases. They understood the "acceleration" of the particle as a ratio of the force acting on the particle to the "relativistic" mass of the particle. This approach is incorrect. First, in relativistic physics the relationship between force and acceleration is more complicated. It deals with not one "relativistic" mass, but with two "relativistic" masses: transverse and longitudinal. Secondly, identity of trajectories need not require equality of accelerations.

The actual condition, i.e. the identity of $e\mathbf{B}/p$ in both cases, can be obtained from the following two requirements:

1° in any given point of the trajectory the curvature should be the same in both cases;

2° in the vicinity of any given point the plane containing a small arc of the trajectory should be oriented in space in both cases in the same way.

Most of the students followed the approach described just above. Unfortunately, many forgot about the second requirement (they neglected the vector character of the quantity $e\mathbf{B}/p$).

EXPERIMENTAL PROBLEM[4]

The following equipment is provided:

1. Two piezoelectric discs of thickness 10 mm with evaporated electrodes (Fig. 4.1) fixed in holders on the jaws of the calipers;

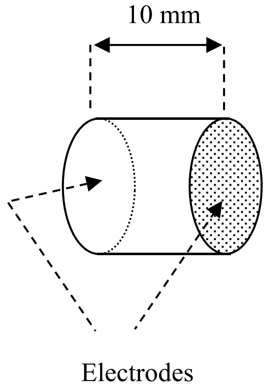


Fig. 4.1

2. The calibrated sine wave oscillator with a photograph of the control panel, explaining the functions of the switches and regulators;
3. A double channel oscilloscope with a photograph of the control panel, explaining the functions of the switches and regulators;
4. Two closed plastic bags containing liquids;
5. A beaker with glycerin (for wetting the discs surfaces to allow better mechanical coupling);
6. Cables and a three way connector;
7. A stand for support the bags with the liquids;
8. Support and calipers.

A piezoelectric material changes its linear dimensions under the influence of an electric field and vice-versa, the distortion of a piezoelectric material induces an electrical field. Therefore, it is possible to excite the mechanical vibrations in a piezoelectric material by applying an alternating electric field, and also to induce an alternating electric field by mechanical vibrations.

A. Knowing that the velocity of longitudinal ultrasonic waves in the material of the disc is about $4 \cdot 10^3$ m/s, estimate roughly the resonant frequency of the mechanical vibrations parallel to the disc axis. Assume that the disc holders do no

[4] The Organizing Committee planned to give another experimental problem: a problem on high T_c superconductivity. Unfortunately, the samples of superconductors, prepared that time by a factory, were of very poor quality. Moreover, they were provided after a long delay. Because of that the organizers decided to use this problem, which was also prepared, but considered as a second choice.

restrict the vibrations. (Note that other types of resonant vibrations with lower or higher frequencies may occur in the discs.)

Using your estimation, determine experimentally the frequency for which the piezoelectric discs work best as a transmitter-receiver set for ultrasound in the liquid. Wetting surfaces of the discs before putting them against the bags improves penetration of the liquid in the bag by ultrasound.

B. Determine the velocity of ultrasound for both liquids without opening the bags and estimate the error.

C. Determine the ratio of the ultrasound velocities for both liquids and its error.

Complete carefully the synopsis sheet. Your report should, apart from the synopsis sheet, contain the descriptions of:

- method of resonant frequency estimation;
- methods of measurements;
- methods of estimating errors of the measured quantities and of final results.

Remember to define all the used quantities and to explain the symbols.

	Synopsis Sheet[5]			
	Formula for estimating the resonant frequency:		Results (with units):	
	Measured best transmitter frequency (with units):			Error:
	Definition of measured quantity:	Symbol:	Results:	Error:
	Final formula for ultrasound velocity in liquid:			
		Velocity of ultrasound (with units):		Error:
	Liquid A Liquid B			

5 In the real Synopsis Sheet the students had more space for filling.

	Ratio of velocities: Error:

Solution (draft)[6]

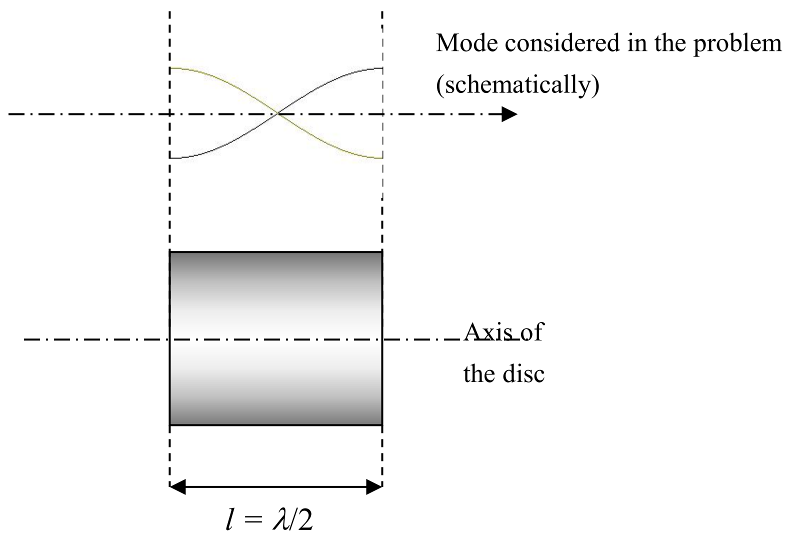
A. As the holders do not affect vibrations of the disc we may expect antinodes on the flat surfaces of the discs (Fig. 4.2; geometric proportions not conserved). One of the frequencies is expected for

$$l = \tfrac{1}{2}\lambda = \frac{v}{2f},$$

where v denotes the velocity of longitudinal ultrasonic wave (its value is given in the text of the problem), f - the frequency and l - the thickness of the disc. Thus:

$$f = \frac{v}{2l}.$$

Numerically $f = 2 \cdot 10^5$ Hz = 200 kHz.



[6] This draft solution is based on the camera-ready text of the more detailed solution prepared by Dr. Andrzej Kotlicki and published in the proceedings [3]

Fig. 4.2

One should stress out that different modes of vibrations can be excited in the disc with height comparable to its diameter. We confine our considerations to the modes related to longitudinal waves moving along the axis of the disc as the sound waves in liquids are longitudinal. We neglect coupling between different modes and require antinodes exactly at the flat parts of the disc. We assume also that the piezoelectric effect does not affect velocity of ultrasound. For these reasons the frequency just determined should be treated as only a rough approximation. However, it indicates that one should look for the resonance in vicinity of 200 kHz.

The experimental set-up is shown in Fig. 4.3. The oscillator (generator) is connected to one of the discs that works as a transmitter and to one channel of the oscilloscope. The second disc is connected to the second channel of the oscilloscope and works as a receiver. Both discs are placed against one of the bags with liquid (Fig. 4.4). The distance d can be varied.

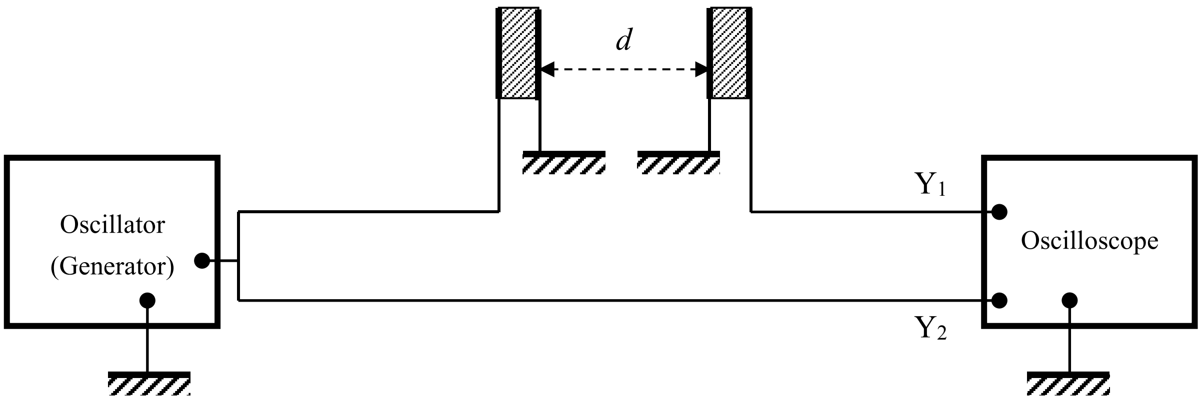


Fig. 4.3

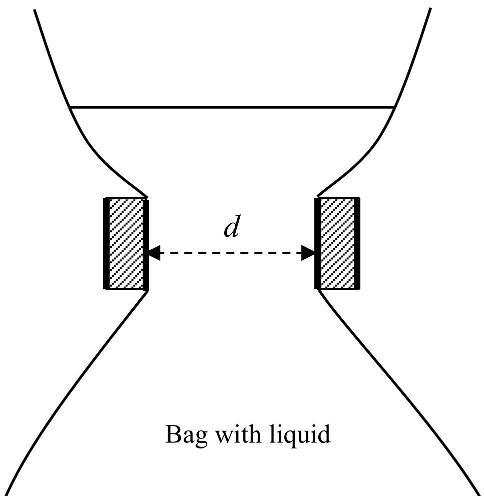


Fig. 4.4

One searches for the resonance by slowly changing the frequency of the oscillator in the range 100 – 1000 kHz and watching the signal on the oscilloscope. In this way the students could find a strong resonance at frequency $f \approx 220$ kHz. Other resonance peaks could be found at about 110 kHz and 670 kHz. They should have been neglected as they are substantially weaker. (They correspond to some other modes of vibrations.) Accuracy of these measurements was 10 kHz (due to the width of the resonance and the accuracy of the scale on the generator).

B. The ultrasonic waves pass through the liquid and generate an electric signal in the receiver. Using the same set-up (Fig. 4.3 and 4.4) we can measure dependence of the phase shift between the signals at Y_1 and Y_2 vs. distance between the piezoelectric discs d at the constant frequency found in point **A**. This phase shift is $\Delta\varphi = 2\pi df / v_l + \varphi_0$, where v_l denotes velocity of ultrasound in the liquid. φ_0 denotes a constant phase shift occurring when ultrasound passes trough the bag walls (possibly zero). The graph representing dependence $d(\Delta\varphi)$ should be a straight line. Its slope allows to determine v_l and its error. In general, the measurements of $\Delta\varphi$ are difficult for many reflections in the bag, which perturb the signal. One of the best ways is to measure *d* only for $\Delta\varphi = n\pi$ (*n* - integer) as such points can be found rather easy. Many technical details concerning measurements can be found in [3] (pp. 37 and 38).

The liquids given to the students were water and glycerin. In the standard solution the author of the problem received the following values:

$$v_{\text{water}} = (1.50 \pm 0.10) \cdot 10^3 \text{ m/s}; \qquad v_{\text{glycerin}} = (1.96 \pm 0.10) \cdot 10^3 \text{ m/s}.$$

The ratio of these values was 1.31 ± 0.15.

The ultrasonic waves are partly reflected or scattered by the walls of the bag. This effect somewhat affects measurements of the phase shift. To minimize its role one can measure the phase shift (for a given distance) or distance (at the same phase shift) several times, each time changing the shape of the bag. As regards errors in determination of velocities it is worth to mention that the most important factor affecting them was the error in determination of the frequency. This error, however, practically does not affect the ratio of velocities.

Marking Scheme

Frequency estimation

1. Formula 1 point
2. Result (with units) 1 point
3. Method of experimental determining the resonance frequency 1 point
4. Result (if within 5% of standard value) 2 points
5. Error 1 point

Measurements of velocities

1. Explanation of the method 2 points
2. Proper number of measurements in each series 3 points
3. Result for velocity in the first liquid (if within 5% of standard value) 2 points
4. Error of the above 1 point
5. Result for velocity in the second liquid (if within 5% of standard value) 2 points
6. Error of the above 1 point

Ratio of velocities

1. Result (if within 3% of standard value) 2 points
2. Error of the above 1 point

Typical mistakes

The results of this problem were very good (more than a half of competitors obtained more than 15 points). Nevertheless, many students encountered some difficulties in estimation of the frequency. Some of them assumed presence of nodes at the flat surfaces of the discs (this assumption is not adequate to the situation, but accidentally gives proper formula). In part B some students tried to find distances between nodes and antinodes for ultrasonic standing wave in the liquid. This approach gave false results as the pattern of standing waves in the bag is very complicated and changes when the shape of the bag is changed.

Acknowledgement

I would like to thank very warmly to Prof. Jan Mostowski and Dr. Andrzej Wysmołek for reading the text of this article and for valuable critical remarks. I express special thanks to Dr. Andrzej Kotlicki for critical reviewing the experimental part of the article and for a number of very important improvements.

Literature

[1] **Waldemar Gorzkowski** and **Andrzej Kotlicki**, *XX Międzynarodowa Olimpiada Fizyczna - cz. I*, Fizyka w Szkole *nr* **1/90**, pp. 34 - 39

[2] **Waldemar Gorzkowski**, *XX Międzynarodowa Olimpiada Fizyczna - cz. II*, Fizyka w Szkole *nr* **2/3-90**, pp. 23 - 32

[3] ***XX International Physics Olympiad*** *- Proceedings of the XX International Physics Olympiad, Warsaw (Poland), July 16 - 24, 1989*, ed. by W. Gorzkowski, World Scientific Publishing Company, Singapore 1990 [ISBN 981-02-0084-6]

THE 21st INTERNATIONAL PHYSICS OLYMPIAD - 1990
GRONINGEN, THE NETHERLANDS

Hans Jordens
Department of Science and Technology, University of Groningen, The Netherlands.

Question 1. X-ray Diffraction from a crystal.

We wish to study X-ray diffraction by a cubic crystal lattice. To do this we start with the diffraction of a plane, monochromatic wave that falls perpendicularly on a 2-dimensional grid that consists of N_1 x N_2 slits with separations d_1 and d_2. The diffraction pattern is observed on a screen at a distance L from the grid. The screen is parallel to the grid and L is much larger than d_1 and d_2.

a - Determine the positions and widths of the principal maximum on the screen. The width is defined as the distance between the minima on either side of the maxima.

We consider now a cubic crystal, with lattice spacing a and size N_0.a x N_0.a x N_1.a. N_1 is much smaller than N_0. The crystal is placed in a parallel X-ray beam along the z-axis at an angle Θ (see Fig. 1). The diffraction pattern is again observed on a screen at a great distance from the crystal.

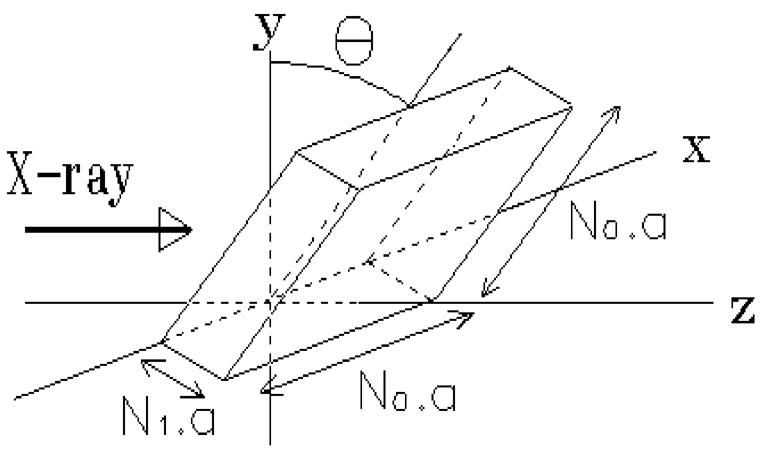


Figure 1 Diffraction of a parallel X-ray beam along the z-axis.
The angle between the crystal and the y-axis is Θ.

b - Calculate the position and width of the maxima as a function of the angle Θ (for small Θ).
- What in particular are the consequences of the fact that $N_1 << N_0$.

The diffraction pattern can also be derived by means of Bragg's theory, in which it is assumed that the X-rays are reflected from atomic planes in the lattice. The diffraction pattern then arises from interference of these reflected rays with each other.

c - Show that this so-called Bragg reflection yields the same conditions for the maxima as those that you found in b.

In some measurements the so-called powder method is employed. A beam of X-rays is scattered by a powder of very many, small crystals. (Of course the sizes of the crystals are much larger than the lattice spacing, a).
Scattering of X-rays of wavelength 0.15 nm by Potassium Chloride [KCl] (which has a cubic lattice, see Fig.2) results in the production of concentric dark circles on a photographic plate. The distance between the crystals and the plate is 0.10 m, and the radius of the smallest circle is 0.053 m (see Fig. 3). K^+ and Cl^- ions have almost the same size, and they may be treated as identical scattering centres.

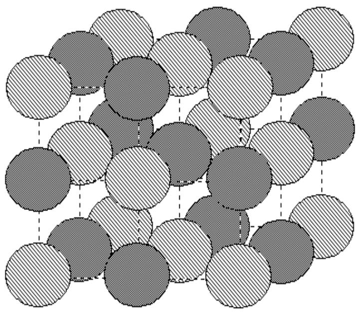


Figure 2. The cubic latice of Potassium Chloride in which the K^+ and Cl^-wons have almost the same size.

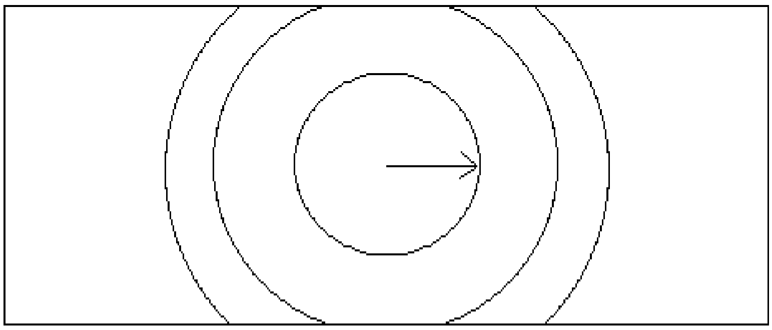


Figure 3. Scattering of X-rays by a powder of KCl crystals results in the production of concentric dark circles on a photographic plate.

d - Calculate the distance between two neighbouring K ions in the crystal.

Question 2. Electric experiments in the magnetosphere of the earth.

In May 1991 the spaceship Atlantis will be placed in orbit around the earth. We shall assume that this orbit will be circular and that it lies in the earth's equatorial plane.
At some predetermined moment the spaceship will release a satellite S, which is attached to a conducting rod of length L. We suppose that the rod is rigid, has negligible mass, and is covered by an electrical insulator. We also neglect all friction. Let α be the angle that the rod makes to the line between the Atlantis and the centre of the earth. (see Fig. 1).
S also lies in the equatorial plane. Assume that the mass of the satellite is much smaller than that of the Atlantis, and that L is much smaller than the radius of the orbit.

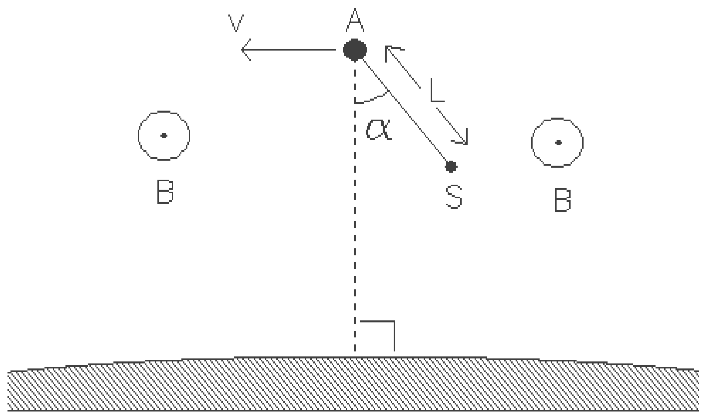


Figure 1 The spaceship Atlantis (A) with a satellite (S) in an orbit around the earth. The orbit lies in the earth's equatorial plane.
The magnetic field (B) is perpendicular to the diagram and is directed towards the reader.

a_1 - Deduce for which value(s) of α the configuration of the spaceship and satellite remain unchanged (with respect to the earth)? In other words, for which value(s) of α is α constant?

a_2 - Discuss the stability of the equilibrium for each case.

Suppose that, at a given moment,
the rod deviates from the stable configuration by a small angle. The system will begin to swing like a pendulum.

b - Express the period of the swinging in terms of the period of revolution of the system around the earth.

In Fig. 1 the magnetic field of the earth is perpendicular to the diagram and is directed towards the reader. Due to the orbital velocity of the rod, a potential difference arises between its ends. The environment (the magnetosphere) is a rarefied, ionised gas with a very good electrical conductivity. Contact with the ionised gas is made by means of electrodes in A (the Atlantis) and S (the satellite). As a consequence of the motion, a current, I, flows through the rod.

c_1 - In which direction does the current flow through the rod? (Take $\alpha = 0$)

Data:	- the period of the orbit	$T = 5{,}4 \cdot 10^3$ s
	- length of the rod	$L = 2{,}0 \cdot 10^4$ m
	- magnetic field strength of the eart at the height of the satellite	$B = 5{,}0 \cdot 10^{-5}$ Wb.m^{-2}
	- the mass of the shuttle Atlantis	$m = 1{,}0 \cdot 10^5$ kg

Next, a current source inside the shuttle is included in the circuit, which maintaines a net direct current of 0.1 A in the opposite direction.

c_2 - How long must this current be maintained to change the altitude of the orbit by 10 m.
Assume that α remains zero. Ignore all contributions from currents in the magnetosphere.
- Does the altitude decrease or increase?

Question 3. The rotating neutron star.

A 'millisecond pulsar' is a source of radiation in the universe that emits very short pulses with a period of one to several milliseconds. This radiation is in the radio range of wavelengths; and a suitable radio receiver can be used to detect the separate pulses and thereby to measure the period with great accuracy.

These radio pulses originate from the surface of a particular sort of star, the so-called neutron star. These stars are very compact: they have a mass of the same order of magnitude as that of the sun, but their radius is only a few tens of kilometers. They spin very quickly. Because of the fast rotation, a neutron star is slightly flattened (oblate). Assume the axial cross-section of the surface to be an ellipse with almost equal axes. Let r_p be the polar and r_e the equatorial radii; and let us define the flattening factor by:

$$\epsilon = \frac{(r_e - r_p)}{r_p}$$

Consider a neutron star with	a mass of	$2.0 . 10^{30}$ kg,
	an average radius of	$1.0 . 10^{4}$ m,
	and a rotation period of	$2.0 . 10^{-2}$ s.

a - Calculate the flattening factor, given that the gravitational constant is $6.67 . 10^{-11}$ $N.m^2.kg^{-2}$.

In the long run (over many years) the rotation of the star slows down, due to energy loss, and this leads to a decrease in the flattening. The star has however a solid crust that floats on a liquid interior. The solid crust resists a continuous adjustment to equilibrium shape. Instead, starquakes occur with sudden changes in the shape of the crust towards equilibrium. During and after such a star-quake the angular velocity is observed to change according to figure 1.

angular velocity of the crust (s^{-1})

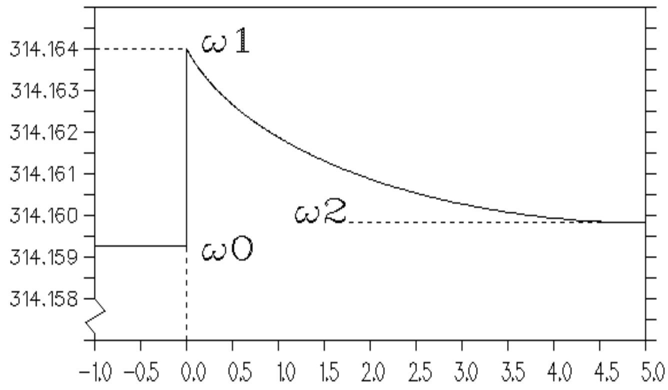


Figure 1 time (days) -->

A sudden change in the shape of the crust of a neutron star results in a sudden change of the angular velocity.

b - Calculate the average radius of the liquid interior, using the data of Fig. 1. Make the approximation that the densities of the crust and the interior are the same. (Ignore the change in shape of the interior).

Question 4. Determination of the efficiency of a LED.

Introduction

In this experiment we shall use two modern semiconductors: the light-emitting diode (LED) and the photo-diode (PD). In a LED, part of the electrical energy is used to excite electrons to higher energy levels. When such an excited electron falls back to a lower energy level, a photon with energy E_{photon} is emitted, where

$$E_{photon} = \frac{h.c}{\lambda}$$

Here h is Planck's constant, c is the speed of light, and λ is the wavelength of the emitted light. The efficiency of the LED is defined to be the ratio between the radiated power, ϕ , and the electrical power used, P_{LED}:

$$\eta = \frac{\phi}{P_{LED}}$$

In a photo-diode, radiant energy is transformed into electrical energy. When light falls on the sensitive surface of a photo-diode, some (but not all) of the photons free some (but not all) of the electrons from the crystal structure. The ratio between the number of incoming photons per second, N_p, and the number of freed electrons per second, N_e, is called the quantum efficiency, q_p

$$q_p = \frac{N_e}{N_p}$$

The experiment

The purpose of this experiment is to determine the efficiency of a LED as a function of the current that flows through the LED. To do this, we will measure the intensity of the emitted light with a photo-diode. The LED and the PD have been mounted in two boxes, and they are connected to a circuit panel (Fig. 1). By measuring the potential difference across the LED, and across the resistors R_1 and R_3, one can determine both the potential differences across, and the currents flowing through the LED and the PD.

We use the multimeter to measure VOLTAGES only!! This is done by turning the knob to position 'V'. The meter selects the appropriate sensitivity range automatically. If the display is not on "AUTO" switch "off" and push on "V" again. Connection: "COM" and "V-Ω".

The box containing the photo-diode and the box containing the LED can be moved freely over the board. If both boxes are positioned opposite to each other, then the LED, the PD and the hole in the box containing the PD remain in a straight line.

Data:- The quantum efficiency of the photo-diode	$q_p = 0.88$
- The detection surface of the PD is	2.75 x 2.75 mm^2
- The wave-length of the light emitted from the LED is	635 nm.
- The internal resistance of the voltmeter is:	100 MΩ in the range up to 200 mV 10 MΩ in the other ranges.
The range is indicated by small numbers on the display.	
- Planck's constant	$h = 6.63 \cdot 10^{-34}$ J.s
- The elementary quantum of charge	$e = 1.6 \cdot 10^{-19}$ C
- The speed of light in vacuo	$c = 3.00 \cdot 10^{8}$ $m.s^{-1}$

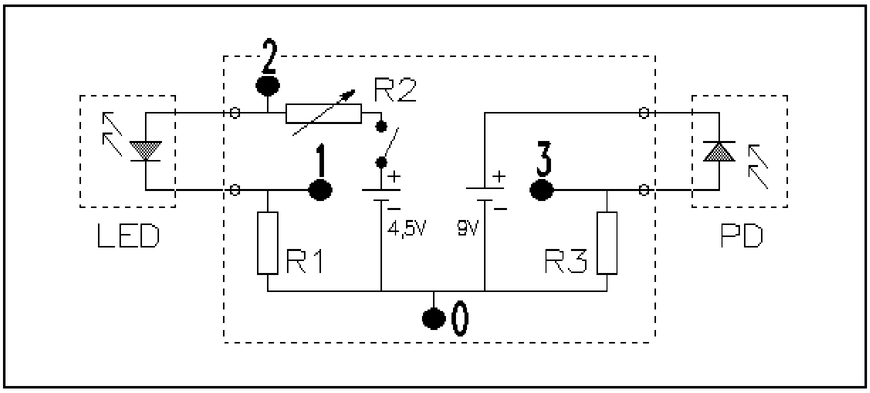


Figure 1.

R_1 = 100 Ω
R_2 = variable resistor
R_3 = 1 MΩ

The points labelled 0, 1, 2 and 3 are measuring points.

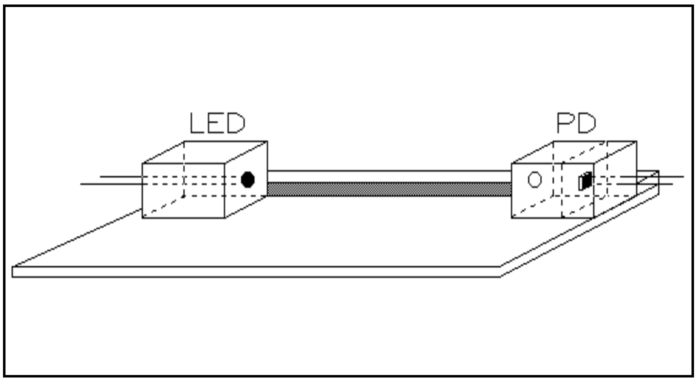


Figure 2 The experimental setup: a board and the two boxes containing the LED and the photo-diode.

Instructions

1. Before we can determine the efficiency of the LED, we must first calibrate the photo-diode. The problem is that we know nothing about the LED.

   Show experimentally that the relation between the current flowing through the photo-diode and the intensity of light falling on it, I [$J.s^{-1}.m^{-2}$], is linear.

2. Determine the current for which the LED has maximal efficiency.

3. Carry out an experiment to measure the maximal (absolute) efficiency of the LED.

No marks (points) will be allocated for an error analysis (in THIS experiment only). Please summarize data in tables and graphs with clear indications of quantities (and units).

Question 5. Determination of the ratio of the magnetic field strengths of two different magnets.

Introduction

When a conductor moves in a magnetic field, currents are induced: these are the so-called eddy currents. As a consequence of the interaction between the magnetic field and the induced currents, the moving conductor suffers a reactive force. Thus an aluminium disk that rotates in the neighbourhood of a stationary magnet experiences a braking force.

Material available

1. A stand.
2. A clamp.
3. An homogenius aluminium disk on an axle, in a holder, that can rotate.
4. Two magnets. The geometry of each is the same (up to 1%); each consists of a clip containing two small magnets of identical magnetization and area, the whole producing a homogenius field, B_1 or B_2.
5. Two weights. One weight has twice the mass (up to 1%) of the other.
6. A stop-watch.
7. A ruler.

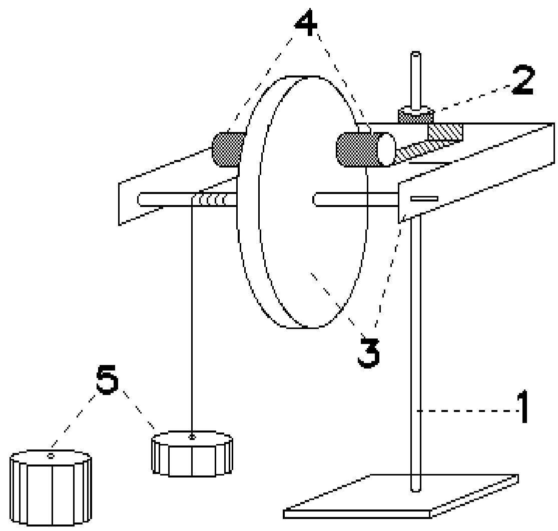


Figure 1.

The experiment

The aluminium disk is fixed to an axle, around which a cord is wrapped. A weight hangs from the cord; and when the weight is released, the disk accelerates until a constant angular velocity is reached. The terminal speed depends, among other things, on the magnitude of the magnetic field strength of the magnet.
Two magnets of different field strengths B_1 or B_2, are available. Either can be fitted on to the holder that carries the aluminium disk: they may be interchanged.

Instructions

1. Think of an experiment in which the ratio of the magnetic field strengths B_1 and B_2, of the two magnets can be measured as accurately as possible.

2. Give a - short - theoretical treatment, indicating how one can obtain the ratio from the measurements.

3. Carry out the experiment and determine the ratio.

4. GIVE AN ERROR ESTIMATION.

Use of the stopwatch

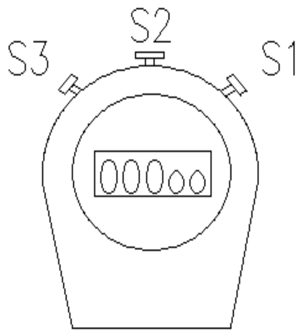


Figure 2.

The stop-watch has three buttons: S_1, S_2 and S_3 (see Fig. 2).

Button S_2 toggles between the date-time and the stop-watch modes. Switch to the stop-watch mode. One should see this:

$\boxed{OOOoo}$

On pressing S_1 once, the stop-watch begins timing. To stop it, press S_1 a second time.

The stop-watch can be reset to zero by pressing S_3 once.

Solution of question 1.

a - Consider first the x-direction. If waves coming from neighbouring slits (with separation d_1) traverse paths of lengths that differ by:

$$\Delta_1 = n_1.\lambda$$

where n_1 is an integer, then a principal maximum occurs. The position on the screen (in the x-direction) is:

$$x_{n_1} = \frac{n_1.\lambda.L}{d_1}$$

since $d_1 << d_2$.
The path difference between the middle slit and one of the slits at the edge is then:

$$\Delta_{(\frac{N_1}{2})} = \frac{N_1}{2}.n_1.\lambda$$

If on the other hand this path difference is:

$$\Delta_{(\frac{N_1}{2})} = \frac{N_1}{2}.n_1.\lambda + \frac{\lambda}{2}$$

then the first minimum, next to the principal maximum, occurs. The position of this minimum on the screen is given by:

$$x_{n_1} + \Delta x = \frac{\left(\frac{N_1}{2}.n_1.\lambda + \frac{\lambda}{2}\right).L}{\frac{N_1}{2}.d_1} = \frac{n_1.\lambda.L}{d_1} + \frac{\lambda.L}{N_1.d_1}$$

$$\rightarrow \quad \Delta x = \frac{\lambda.L}{N_1.d_1}$$

The width of the principal maximum is accordingly:

$$2.\Delta x = 2.\frac{\lambda.L}{N_1.d_1}$$

A similar treatment can be made for the y-direction, in which there are N_2 slits with separation d_2. The positions and widths of the principal maximal are:

$$(x_{n_1}, y_{n_2}) = \left(\frac{n_1.\lambda.L}{d_1}, \frac{n_2.\lambda.L}{d_2}\right)$$

$$2.\Delta x = 2.\frac{\lambda.L}{N_1.d_1} \quad ; \qquad 2.\Delta y = 2.\frac{\lambda.L}{N_2.d_2}$$

An alternative method of solution is to calculate the intensity for the 2-dimensional grid as a function of the angle that the beam makes with the screen.

b - In the x-direction the beam 'sees' a grid with spacing a, so that in this direction we have:

$$x_{n_1} = \frac{n_1.\lambda.L}{a} \qquad \Delta x = 2.\frac{\lambda.L}{N_0.a}$$

In the y-direction, the beam 'sees' a grid with effective spacing a.cos(Θ). Analogously, we obtain:

$$y_{n_2} = \frac{n_2.\lambda.L}{a.\cos(\theta)} \qquad \Delta y = 2.\frac{\lambda.L}{N_0.a.\cos(\theta)}$$

In the z-direction, the beam 'sees' a grid with effective spacing a.sin(Θ). This gives rise to principal maxima with position and width:

$$y'_{n_3} = \frac{n_3.\lambda.L}{a.\sin(\theta)} \qquad \Delta y' = 2.\frac{\lambda.L}{N_1.a.\sin(\theta)}$$

This pattern is superimposed on the previous one. Since sin(Θ) is very small, only the zeroth-order pattern will be seen, and it is very broad, since $N_1.\sin(\Theta) << N_0$. The diffraction pattern from a plane wave falling on a thin plate of a cubic crystal, at a small angle of incidence to the normal, will be almost identical to that from a two-dimensional grid.

c - In Bragg reflection, the path difference for constructive interference between neighbouring planes:

$$\Delta = 2.a.\sin(\phi) \approx 2.a.\phi = n.\lambda \quad \rightarrow \quad \frac{x}{L} \approx 2.\phi \approx \frac{n.\lambda}{a} \quad \rightarrow \quad x \approx \frac{n.\lambda.L}{a}$$

Here ϕ is the angle of diffraction.
This is the same condition for a maximum as in section b.

d - For the distance, √2.a, between neighbouring K ions we have:

$$tg(2\phi) = \frac{x}{L} = \frac{0,053}{0,1} \approx 0,53 \rightarrow a = \frac{\lambda}{2.\sin(\phi)} \approx \frac{0,15.10^{-9}}{2\ .\ 0,24} \approx 0,31\ nm$$

$$K{-}K \approx \sqrt{2}.0,31 \approx 0,44\ nm$$

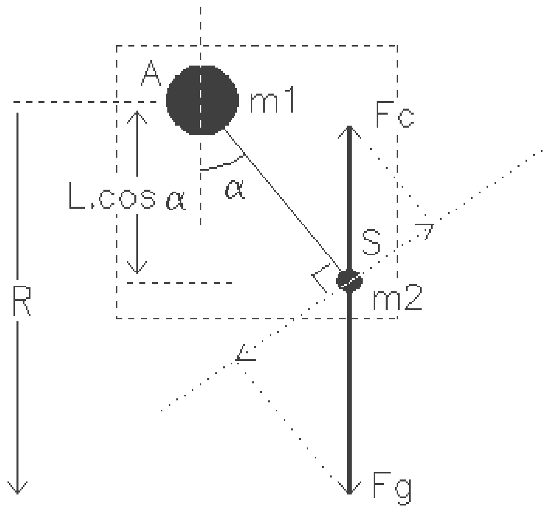
Marking Breakdown

a	position of principal maxima	:1
	width of principal maxima	:3
b	lattice constants	:1
	effect of thickness	:2
c	Bragg reflection	:2
d	Calculation of K-K spacing	:1

Solution of question 2.

a_1 - Since $m_2 << m_1$, the Atlantis travels around the earth with a constant speed. The motion of the satellite is composed of the circular motion of the Atlantis about the earth and (possibly) a circular motion of the satellite about the Atlantis.
For m_1 we have:

$$m_1.\Omega^2.R = \frac{G.m_1.m_a}{R^2} \rightarrow \Omega^2 = \frac{G.m_a}{R^3}$$



For m_2 we have:

$$m_2.L.\ddot{\alpha} = -(F_g - F_c).\sin(\alpha) = -\left(\frac{G.m_2.m_a}{(R - L.\cos(\alpha))^2} - m_2.\Omega^2.(R - L.\cos(\alpha))\right).\sin(\alpha)$$

Using the approximation:

$$\frac{1}{(R - L.\cos(\alpha))^2} \approx \frac{1}{R^2} + \frac{2.L.\cos(\alpha)}{R^3}$$

and equation (1), one finds:

$$L.\ddot{\alpha} = -\left(\frac{G.m_a}{R^2} + \frac{2.G.m_a}{R^3}.L.\cos(\alpha) - \frac{G.m_a}{R^3}.R + \frac{G.m_a}{R^3}.L.\cos(\alpha)\right).\sin(\alpha)$$

so:

$$\ddot{\alpha} + 3.\Omega^2.\sin(\alpha).\cos(\alpha) = 0 \qquad (2)$$

If α is constant: $\ddot{\alpha} = 0$ --> $\sin(\alpha) = 0$ --> $\alpha = 0$; $\alpha = \pi$
--> $\cos(\alpha) = 0$ --> $\alpha = \pi/2$; $\alpha = 3\pi/2$

a_2 - The situation is stable if the moment $M = m_2.L.\ddot{\alpha}.L = m_2.L^2.\ddot{\alpha}$ changes sign in a manner opposed to that in which the sign of α - α_0 changes:

sign(α - α_0)	- +	- +	- +	- +	- +
α	0	$\pi/2$	π	$3\pi/2$	2π
sign(M)	+ -	- +	+ -	- +	+ -
α	0	$\pi/2$	π	$3\pi/2$	2π

The equilibrium about the angles 0 en π is thus stable, whereas that around $\pi/2$ and $3\pi/2$ is unstable.

b - For small values of α equation (2) becomes:

$$\ddot{\alpha} + 3.\Omega^2.\alpha = 0$$

This is the equation of a simple harmonic motion.
The square of the angular frequency is:

$$\omega^2 = 3.\Omega^2$$

so:

$$\omega = \Omega.\sqrt{3} \quad \rightarrow \quad T_1 = \frac{2\pi}{\omega} = \frac{1}{3}\sqrt{3}.\left(\frac{2\pi}{\Omega}\right) \approx 0{,}58.T_0$$

c_1 - According to Lenz's law, there will be a current from the satellite (S) towards the shuttle (A).

c_2 - For the total energy of the system we have:

$$U = U_{kin} + U_{pot} = \frac{1}{2}.m.\Omega^2.R^2 - \frac{G.m.m_a}{R} = -\frac{1}{2}.\frac{G.m.m_a}{R}$$

A small change in the radius of the orbit corresponds to a change in the energy of:

$$\Delta U = \frac{1}{2}.\frac{G.m.m_a}{R^2}.\Delta R = \frac{1}{2}.m.\Omega^2.R.\Delta R$$

In the situation under c_1 energy is absorbed from the system as a consequence of which the radius of the orbit will decrease.
Is a current source inside the shuttle included in the circuit, which maintains a net current in the opposite direction, energy is absorbed by the system as a consequence of which the radius of the orbit will increase.
From the assumptions in c_2 we have:

$$\Delta U = F_l.v.t = B.I.L.\Omega.R.t = \frac{1}{2}.m.\Omega^2.R.\Delta R \quad \rightarrow \quad t = \frac{1}{2}.\frac{m.\Omega.\Delta R}{B.I.L}$$

Numerical application gives for the time: t ≈ 5,8 . 10^3 s; which is about the period of the system.

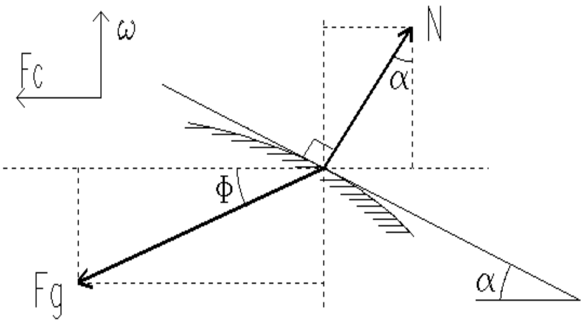
Marking breakdown:

a_1		: 1
a_2		: 1
b	- Atlantis in uniform circular motion	: 0,5
	- calculation of the period Ω	: 0,5
	- equation of motion of the satellite	: 2,5
	- equation of motion for small angles	: 0,5
	- period of oscillations	: 1
c_1	-	: 1
c_2	- calculation of the time the current has to be maintained	: 1,5
	- increase or decrease of the radius of the orbit	: 0,5

Solution of question 3.

a - *1st method*

For equilibrium we have $F_c = F_g + N$ where N is normal to the surface.



Resolving into horizontal and vertical components, we find:

$$\left.\begin{aligned} F_g.\cos(\phi) &= F_c + N.\sin(\alpha) \\ F_g.\sin(\phi) &= N.\cos(\alpha) \end{aligned}\right. \rightarrow F_g.\cos(\phi) = F_c + F_g.\sin(\phi).tg(\alpha)$$

From:

$$F_g = \frac{G.M}{r^2}\ ,\quad F_c = \omega^2.r\ ,\quad x = r.\cos(\phi)\ ,\quad y = r.\sin(\phi)\ \ en\ \ tg(\alpha) = \frac{dy}{dx}$$

we find:

$$y.dy + \left(1 - \frac{\omega^2.r^3}{G.M}\right).x.dx = 0$$

where:

$$\frac{\omega^2.r^3}{G.M} \approx 7.10^{-4}$$

This means that, although r depends on x and y, the change in the factor in front of xdx is so slight that we can take it to be constant. The solution of Eq. (1) is then an ellipse:

$$\frac{x^2}{r_e^2} + \frac{y^2}{r_p^2} = 1 \rightarrow \frac{r_p}{r_e} = \sqrt{1 - \frac{\omega^2.r^3}{G.M}} \approx 1 - \frac{\omega^2.r^3}{2.G.M}$$

and from this it follows that:

$$\epsilon = \frac{r_e - r_p}{r_e} = \frac{\omega^2.r^3}{2.G.M} \approx 3{,}7.10^{-4}$$

2nd method

For a point mass of 1 kg on the surface,

$$U_{pot} = -\frac{G.M}{r} \qquad U_{kin} = \frac{1}{2}.\omega^2.r^2.\cos^2(\phi)$$

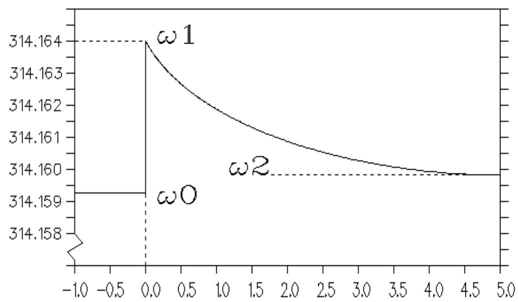
The form of the surface is such that U_{pot} - U_{kin} = constant. For the equator ($\Phi = 0$, $r = r_e$) and for the pole ($\Phi = \pi/2$, $r = r_p$) we have:

$$\frac{G.M}{r_p} = \frac{G.M}{r_e} + \frac{1}{2}.\omega^2.r_e^2 \rightarrow \frac{r_e}{r_p} = 1 + \frac{\omega^2.r_e^3}{2.G.M}$$

Thus:

$$\epsilon = \frac{r_e - r_p}{r_e} = \frac{1 + \frac{\omega^2.r_e^3}{2.G.M} - 1}{1 + \frac{\omega^2.r_e^3}{2.G.M}} \approx \frac{\omega^2.r_e^3}{2.G.M} \approx 3{,}7.10^{-4}$$

b - As a consequence of the star-quake, the moment of inertia of the crust I_m decreases by ΔI_m.



From the conservation of angular momentum, we have:

$$I_m.\omega_0 = (I_m - \Delta I_m).\omega_1 \rightarrow \Delta I_m = I_m.\frac{\omega_1 - \omega_0}{\omega_1}$$

After the internal friction has equalized the angular velocities of the crust and the core, we have:

$$(I_m + I_c).\omega_0 = (I_m + I_c - \Delta I_m).\omega_2 \rightarrow \Delta I_m = (I_m + I_c).\frac{\omega_2 - \omega_0}{\omega_2}$$

$$\frac{I_m}{I_m + I_c} = \frac{(\omega_2 - \omega_0).\omega_1}{(\omega_1 - \omega_0).\omega_2} \rightarrow 1 - \frac{I_c}{I_m + I_c} = \frac{(\omega_2 - \omega_0).\omega_1}{(\omega_1 - \omega_0).\omega_2}$$

$$I \ (:) \ R^2$$

$$\rightarrow \frac{I_c}{I_m + I_c} = \frac{r_c^2}{r^2} \rightarrow \frac{r_c}{r} = \sqrt{1 - \frac{(\omega_2 - \omega_0).\omega_1}{(\omega_1 - \omega_0).\omega_2}} \approx 0.95$$

Marking breakdown

a	1st method	- expressions for the forces	:1
		- equation for the surface	:2
		- equation of ellipse	:1
		- flattening factor	:1
	2nd method	- energy equation	:4
		- flattening factor	:1
b	- conservation of angular momentum for crust		:1.5
	- conservation of angular momentum for crust and core		:1.5
	- moment of inertia for a sphere		:1
	- ratio r_c/r		:1

Solution of question 4.

1. *The linearity of the photo-diode.*

The linearity of the photo-diode can be checked by using the inverse square law between distance and intensity. Suppose that the measured distance between the LED and the (box containing the) PD is x. The intensity of the light falling on the PD satisfies:

$$I(x) = \frac{I_0}{x^2}$$

If the intensity is indeed proportional to the current flowing through the PD, it will also be proportional to the voltage, V(x), measured across the resistor R3. From (1) it then follows that:

$$\frac{1}{\sqrt{V(x)}} \propto x$$

To obtain the correct value of V(x), one should subtract from the measured voltage V_1 the voltage V_2 that one measures when the LED is turned off (but the LED box is still in place in front of the PD).

x (cm)	V_1 (V)	V_2 (V)	i_1 (μA)	i_2 (μA)	$1/[i_1(x) - i_2(x)]^{½}$ (μA$^{-½}$)
1.0	5.66	.003	6.23	.003	0.40
2.0	4.07	.004	4.48	.005	0.47
3.0	3.03	.005	3.33	.005	0.55
4.0	2.32	.006	2.55	.006	0.63
5.0	1.83	.006	2.01	.006	0.71
6.0	1.48	.007	1.63	.007	0.79
7.0	1.23	.007	1.35	.007	0.86
8.0	1.006	.008	1.107	.008	0.95
9.0	0.859	.009	0.945	.009	1.03
10.0	0.744	.009	0.818	.009	1.11
11.0	0.648	.010	0.713	.010	1.19
12.0	0.570	.011	0.627	.011	1.27
13.0	0.507	.012	0.558	.012	1.35
14.0	0.456	.012	0.502	.012	1.43
15.0	0.414	.013	0.455	.013	1.50
16.0	0.373	.013	0.410	.014	1.59
17.0	0.341	.014	0.375	.014	1.66
18.0	0.312	.014	0.343	.014	1.74
19.0	0.291	.015	0.320	.015	1.81
20.0	0.272	.015	0.299	.015	1.88

Plotted on a graph, one finds a perfect straight line.

2. The light intensity as a function of the electrical power of the LED

The photo-current i_{PD} is determined from the voltage V over R3 = $1M\Omega$. The meter itself has an internal resistance of 100 $M\Omega$ in the 200 mV range and 10 $M\Omega$ in the other ranges. We have then: i_{PD} = 1.01 V resp. i_{PD} = 1.1 V where V is in volts and i_{PD} in μA. The current in ampères through the LED is the voltage over R1 in volts, divided by 100.

|---------- PD [x = 5 cm]	----------|		|---------------------- LED		------------------------|	
V_1 (V)	V_2(V)	$i_1 - i_2$ (μA)	i_{LED} (10^{-2} A)	V_{LED} (V)	P_{LED} (10^{-2} W)	$(i_1 - i_2)/P_{LED}$
1.806	.0061	1.98	2.70	1.752	4.73	0.419
1.637	.0061	1.79	2.30	1.742	4.01	0.446
1.511	.0061	1.66	2.08	1.735	3.61	0.460
1.225	.0061	1.34	1.606	1.722	2.77	0.484
1.117	.0061	1.22	1.433	1.718	2.46	0.496
0.903	.0061	0.99	1.123	1.705	1.91	0.518
0.711	.0061	0.78	0.889	1.708	1.52	0.513
0.448	.0061	0.49	0.555	1.673	0.93	0.527
0.315	.0061	0.34	0.410	1.659	0.68	0.5
0.192	.0061	0.21	0.258	1.637	0.42	0.2

The efficiency is proportional to $(i_1 - i_2)/P_{LED}$. In the graph of $(i_1 - i_2)/P_{LED}$ against i_{LED} the maximal efficiency corresponds to i_{LED} = 0,6 . 10^{-2} A. (See figure 2.)

3. Determination of the maximal efficiency.

The LED emits a conical beam with cylindrical symmetry. Suppose we measure the light intensity with a PD of sensitive area d^2 at a distance r_i from the axis of symmetry. Let the intensity of the light there be $\Phi(r_i)$, then we have:

$$i(r_i) = N_e.e = N_f q_f e = \frac{\Phi(r_i)}{h.\nu}.q_f e$$

$$\Phi = \sum_i \Phi(r_i).\frac{2.\pi.r_i.d}{d^2} = \frac{2.\pi}{d}.\sum_i \Phi(r_i).r_i = \frac{2.\pi}{d}.\frac{h.\nu}{q_f e}.\sum_i i(r_i).r_i$$

r_i (mm)	V_1 (V)	V_2 (V)	$(i_1 - i_2).r_i$ (x 10^{-9} Am)	r_i (mm)	V_1 (V)	V_2 (V)	$(i_1 - i_2).r_i$ (x 10^{-9} Am)
0	1.833	0.006	0	39	0.097	0.006	
3	1.906	0.006	6.27	42	0.089	0.006	4.16
6	1.846	0.006	12.54	45	0.082	0.006	3.86
9	1.750	0.006	17.28	48	0.071	0.006	3.79
12	1.347	0.006	17.76	51	0.066	0.006	3.48
15	0.997	0.006	16.20	54	0.050	0.006	3.39
18	0.643	0.006	12.60	57	0.045	0.006	2.52
21	0.313	0.006	7.14	60	0.037	0.006	2.45
24	0.343	0.006	8.88	63	0.032	0.006	2.08
27	0.637	0.006	18.90	66	0.023	0.006	1.83
30	0.681	0.006	22.20	69	0.017	0.006	1.27
33	0.266	0.006	9.57	72	0.014	0.006	0.88
36	0.119	0.006	4.48	75	0.011	0.006	0.68
							0.49

The efficiency = $\Phi/P_{LED} \approx 0.001$

Marking breakdown

1 linearity of the PD
- inverse square law :1.5
- number of measuring points [1,3>; [3,5>; [5,..> :0.5/1.0/1.5
- dark current :0.5
- correct graph :1

2 determination of current at maximal effiency
- principle :0.5
- number of measuring points [1,3>; [3,5>; [5,..> :0.5/1.0/1.5
- graph efficiency-current :0.5
- determination of current at maximal efficiency :0.5

3 determination of the maximal efficiency
- determination of the emitted light intensity :1.5
    - via estimation of the cone cross-section :0.5
    - via measurement of the intensity distribution :1.5
- determination of the maximum efficiency :1

Solution of question 5.

1. Theory Let
- the moment of inertia of the disk be : I
- the mass of the weight : m
- the moment of the frictional force : M_f
- magnetic field strength : B
- the radius of the axle : r
- the moment of the magnetic force : M_B

For the motion of the rotating disk we have:

$$I.\alpha = (m.g - m.a).r - M_f - M_B$$

We suppose that M_f is constant but not negligible. Because the disk moves in the magnetic field, eddy currents are set up in the disk. The magnitude of these currents is proportional to B and to the angular velocity. The Lorentz force as a result of the eddy currents and the magnetic field is thus proportional to the square of B and to the angular velocity, i.e.

$$M_B = c.B^2.\omega$$

Substituting this into Eq. (1), we find:

$$I.\alpha = (m.g - m.a).r - M_f - c.B^2.\omega$$

$$v_e = (\frac{g.r^2}{c.B^2}).\left(m - \frac{M_f}{g.r}\right)$$

After some time, the disk will reach its final constant angular velocity; the angular acceleration is now zero and for the final velocity v_e we find:

The final constant velocity is thus a linear function of m.

2. The experiment

The final constant speed is determined by measuring the time taken to fall the last 21 cm [this is the width of a sheet of paper].

In the first place it is necessary to check that the final speed has been reached. This is done by allowing the weight to fall over different heights. It is clear that, with the weaker magnet, the necessary height before the constant speed is attained will be larger. Measurements for the weak magnet system:

---------------- time taken to fall ----------------

height (m)	smaller weight	larger weight
0.30	5.04 ± 0.02 (s)	2.00 ± 0.01 (s)
0.40	4.67 ± 0.04 (s)	1.71 ± 0.02 (s)
0.50	4.59 ± 0.05 (s)	1.55 ± 0.02 (s)
0.60	4.44 ± 0.06 (s)	1.48 ± 0.01 (s)
0.70	4.49 ± 0.05 (s)	1.44 ± 0.04 (s)
0.80	4.43 ± 0.03 (s)	1.38 ± 0.03 (s)
0.90	4.43 ± 0.04 (s)	1.35 ± 0.02 (s)
1.10	- - -	1.34 ± 0.05 (s)
1.30	- - -	1.33 ± 0.04 (s)

3. *Final constant speed measurements for both magnet systems and for several choices of weight.*

Measurements for the weak magnet:

weight	T (s)	T (s)	T (s)	T (s)	<T> (s)	<v> (m/s)
small	4.42	4.23	4.24	4.33	4.31 ± 0.09	4.9 ± 0.1
large	1.89	1.91	1.98	1.92	1.93 ± 0.04	10.9 ± 0.2
both	1.29	1.32	1.23	1.30	1.29 ± 0.04	16.3 ± 0.5

Measurements for the strong magnet:

weight	T (s)	T (s)	T (s)	T (s)	<T> (s)	<v> (m/s)
small	8.93	9.01	9.17	8.91	9.0 ± 0.1	2.33 ± 0.03
large	4.03	3.92	4.03	3.95	3.98 ± 0.06	5.28 ± 0.08
both	2.53	2.52	2.53	2.48	2.52 ± 0.03	8.3 ± 0.1

4. *Discussion of the results:*

- A graph between v_e and the weight should be made.
- From Eq. (2) we observe that:
  - both straight lines should intersect on the horizontal axis.
  - from the square-root of the ratio of the slopes we have immediately the ratio of the magnetic field strengths.
  - For the above measurements we find:

$$\frac{B_1}{B_2} = \sqrt{\frac{7.22}{15}} \approx 0.69 \quad \rightarrow \quad \frac{\Delta\left(\frac{B_1}{B_2}\right)}{\left(\frac{B_1}{B_2}\right)} = \frac{1}{2}.\sqrt{\left(\frac{\Delta r_1}{r_1}\right)^2 + \left(\frac{\Delta r_2}{r_2}\right)^2} \approx 0.05$$

$$\frac{B_1}{B_2} = 0.69 \pm 0.03$$

Marking Breakdown

1	$M_B = c.B^2.\omega$		: 1
	Eq. (2)		: 1
2	Investigation of the range in which the speed is constant		: 2
3	Number of timing measurements [1,2,3,...]		: 0,1,2
		Error estimation	: 0.5
4	graph	- quality	: 0.5
		- the lines intersect each other on the mass-axis	: 1
		- calculation of B_1/B_2	: 1
		- Error calculation	: 1

T H E O R E T I C A L   P R O B L E M S

Problem 1

The figure 1.1 shows a solid, homogeneous ball radius *R*. Before falling to the floor its center of mass is at rest, but the ball is spinning with angular velocity ω_0 about a horizontal axis through its center. The lowest point of the ball is at a height *h* above the floor.

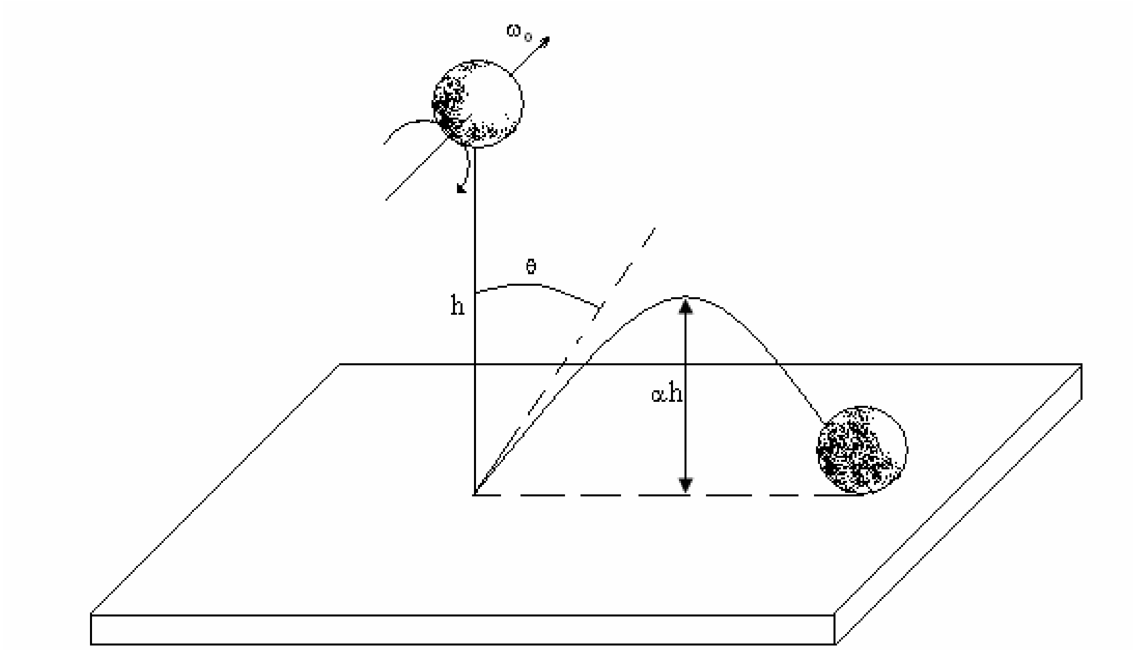


Figure 1.1

When released, the ball falls under gravity, and rebounds to a new height such that its lowest point is now *ah* above the floor. The deformation of the ball and the floor on impact may be considered negligible. Ignore the presence of the air. The impact time, although, is finite.

The mass of the ball is *m*, the acceleration due the gravity is *g*, the dynamic coefficient of friction between the ball and the floor is μ_k, and the moment of inertia of the ball about the given axis is:

$$I = \frac{2mR^2}{5}$$

You are required to consider two situations, in the first, the ball slips during the entire impact time, and in the second the slipping stops before the end of the impact time.

Situation I: slipping throughout the impact.

Find:

a) tan θ , where θ is the rebound angle indicated in the diagram;

b) the horizontal distance traveled in flight between the first and second impacts;

c) the minimum value of ω_0 for this situations.

Situation II: slipping for part of the impacts.

Find, again:

a) tan θ;

b) the horizontal distance traveled in flight between the first and second impacts.

Taking both of the above situations into account, sketch the variation of tan θ with ω_0.

Problem 2

In a square loop with a side length L, a large number of balls of negligible radius and each with a charge q are moving at a speed *u* with a constant separation *a* between them, as seen from a frame of reference that is fixed with respect to the loop. The balls are arranged on the loop like the beads on a necklace, L being much greater than *a*, as indicated in the figure 2.1. The no conducting wire forming the loop has a homogeneous charge density per unit length in the in the frame of the loop. Its total charge is equal and opposite to the total charge of the balls in that frame.

Consider the situation in which the loop moves with velocity v parallel to its side AB (fig. 2.1) through a homogeneous electric field of strength E which is perpendicular to the loop velocity and makes an angle θ with the plane of the loop.

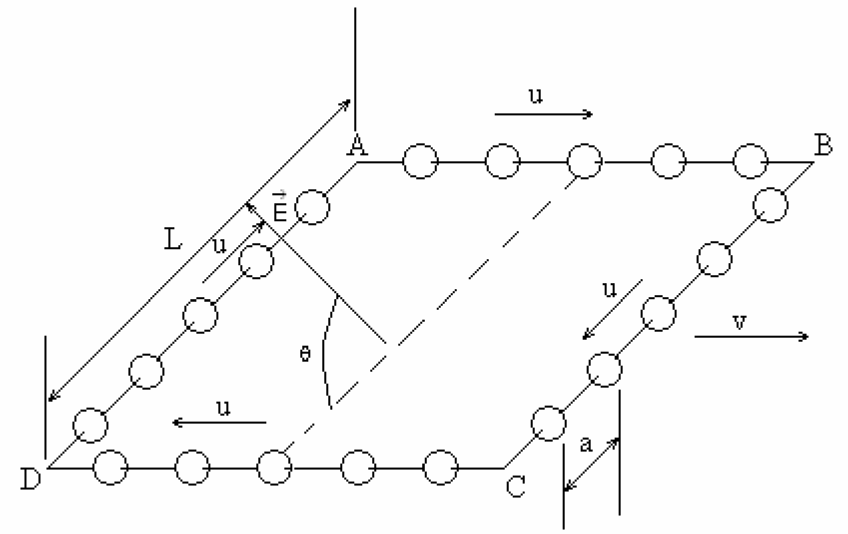


Figure 2.1

Taking into account relativistic effects, calculate the following magnitudes in the frame of reference of an observer who sees the loop moving with velocity v:

a) The spacing between the balls on each of the side of the loop, a_{AB} , a_{BC} , a_{CD} , y a_{DA}.

b) The value of the net charge of the loop plus balls on each of the side of the loop: Q_{AB}, Q_{BC}, Q_{CD} y, Q_{DA}

c) The modulus M of the electrically produced torque tending to rotate the system of the loop and the balls.

d) The energy W due to the interaction of the system, consisting of the loop and the balls with the electric field.

All the answers should be given in terms of quantities specified in the problem.

Note. The electric charge of an isolated object is independent of the frame of reference in which the measurements takes place. Any electromagnetic radiation effects should be ignored.

Some formulae of special relativity

Consider a reference frame S' moving with velocity V with reference to another reference frame S. The axes of the frames are parallel, and their origins coincide a $t = 0$. V is directed along the positive direction of the x axis.

Relativistic sum of velocities

If a particle is moving with velocity u' in the x' direction , as measured in S', the velocity of the particle measured in S is given by:

$$u = \frac{u' + V}{1 + \frac{u'V}{c^2}}$$

Relativistic Contraction

If an object at rest in frame S has length L_0 in the x-direction, an observer in frame S' (moving at velocity V in the x-direction} will measure its length to be:

$$L = L_0 \sqrt{1 - \frac{v^2}{c^2}}$$

Problem 3 Cooling Atoms by laser
To study the properties of isolated atoms with a high degree of precision they must be kept almost at rest for a length of time. A method has recently been developed to do this. It is called “laser cooling” and is illustrated by the problem below.

In a vacuum chamber a well collimated beam of Na^{23} atoms (coming from the evaporation of a sample at 10^3 K) is illuminated head-on with a high intensity laser beam (fig. 3.1). The frequency of laser is chosen so there will be resonant absorption of a photon by those atoms whose velocity is v_0. When the light is absorbed, these atoms are exited to the first energy level, which has a mean value E above the ground state and uncertainty of Γ (fig. 3.2).

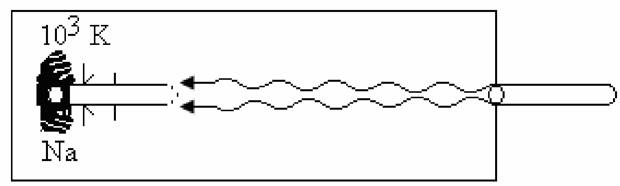


Figure 3.1

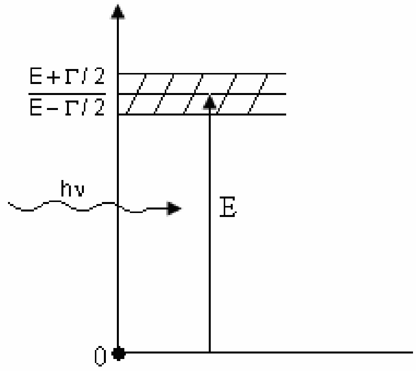


Figure 3.2

For this process, the atom’s decrease in velocity Δv_1 is given by $\Delta v_1 = v_1 - v_0$. Light is then emitted by the atom as it returns to its ground state. The atom’s velocity changes by $\Delta v' = v'_1 - v_1$ and its direction of motion changes by an angle φ (fig. 3.3).

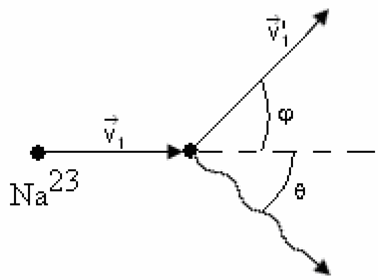


Figure 3.3

This sequence of absorption and emission takes place many times until the velocity of the atoms has decreased by a given amount Δv such that resonant absorption of light at frequency ν no longer occurs. It is then necessary to change the frequency of laser so as to maintain resonant absorption. The atoms moving at the new velocity are further slowed down until some of them have a velocity close to zero.

As first approximation we may ignore any atomic interaction processes apart from the spontaneous absorption and emission light described above.
Furthermore, we may assume the laser to be so intense that the atoms spend practically no time in the ground state.

Questions

a) Find the laser frequency needed ensure the resonant absorption of the light by those atoms whose kinetic energy of the atoms inside the region behind the collimator. Also find the reduction in the velocity of these atoms, Δv_1, after the absorption process.

b) Light of the frequency calculated in question a) is absorbed by atoms which velocities lie within a range Δv_0. Calculate this velocity range.

c) When an atom emits light, its direction of motion changes by φ from initial direction. Calculate φ.
d) Find the maximum possible velocity decrease Δv for a given frequency.
e) What is the approximate number N of absorption-emission events necessary to reduce the velocity of an atom from is initial value v_o
-found in question a) above- almost to zero? Assume the atom travels in a straight line.
f) Find the time t that the process in question e takes. Calculate the distance ΔS an atom travels in this time.

Data

$E = 3{,}36 \cdot 10^{-19}$ J
$\Gamma = 7{,}0 \cdot 10^{-27}$ J
$c = 3 \cdot 10^{8}$ ms^{-1}
$m_p = 1{,}67 \cdot 10^{-27}$ kg
$h = 6{,}62 \cdot 10^{-34}$ Js
$k = 1{,}38 \cdot 10^{-23}$ JK^{-1}

where c is speed of light, h is Planck's constant, k is the Boltzmann constant, and mp is the mass of proton.

THEORETICAL PROBLEMS. SOLUTIONS

Solution Problem 1

a) *Calculation of the velocity at the instant before impact*

Equating the potential gravitational energy to the kinetic energy at the instant before impact we can arrive at the pre-impact velocity v_0:

$$mgh = \frac{mv_0^2}{2} \quad (1)$$

from which we may solve for v_0 as follows:

$$V_0 = \sqrt{2gh} \quad (2)$$

b) *Calculation of the vertical component of the velocity at the instant after impact*

Let v_{2x} and v_{2y} be the horizontal and vertical components, respectively, of the velocity of the mass center an instant after impact. The height attained in the vertical direction will be αh and then:

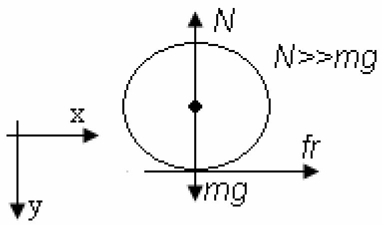
$$v_{2y}^2 = 2g\alpha h \quad (3)$$

from which, in terms of α (or the restitution coefficient $c = \sqrt{\alpha}$):

$$v_{2y} = \sqrt{2g\alpha h} = cv_0 \quad (4)$$

c) *General equations for the variations of linear and angular momenta in the time interval of the Impact*

Figure 1.2 shows the free body of the ball during impact



Considering that the linear impulse of the forces is equal to the variation of the linear momentum and that the angular impulse of the torques is equal to the variation of the angular momentum, we have:

$$I_y = \int_{t_1}^{t_2} N(t)dt = mv_0 + mv_{2y} = m(1+c)\sqrt{2gh} \quad (5)$$

$$I_x = \int_{t_1}^{t_2} f_r(t)dt = mv_{2x} \quad (6)$$

$$I_\theta = \int_{t_1}^{t_2} Rf_r(t)dt = R\int_{t_1}^{t_2} f_r(t)dt = \mathrm{I}\,(\omega_o - \omega_2) \qquad (7)$$

Where I_x, I_y and I_θ are the linear and angular impulses of the acting forces and ω_2 is the angular velocity after impact. The times t_1 and t_2 correspond to the beginning and end of impact.

Variants

At the beginning of the impact the ball will always be sliding because it has a certain angular velocity ω_0. There are, then, two possibilities:

I. The entire impact takes place without the friction being able to spin the ball enough for it to stop at the contact point and go into pure rolling motion.

II. For a certain time $t \in (t_1, t_2)$, the point that comes into contact with the floor has a velocity equal to zero and from that moment the friction is zero. Let us look at each case independently.

Case I

In this variant, during the entire moment of impact, the ball is sliding and the friction relates to the normal force as:

$$f_r = \mu_k N(t) \qquad (8)$$

Substituting (8) in relations (6) and (7), and using (5), we find that:

$$\mathrm{I}_x = \mu_k \int_{t_1}^{t_2} N(t)dt = \mu_k \mathrm{I}_y = \mu_k (1+c)\sqrt{2gh} = mv_{2x} \qquad (9)$$

and:

$$\mathrm{I}_\theta = \mathrm{R}\,\mu_k \int_{t_1}^{t_1} N(t)dt = \mathrm{R}\,\mu_k\, m(1+c)\sqrt{2gh} = \mathrm{I}(\mu_0 - \mu_2) \qquad (10)$$

which can give us the horizontal component of the velocity v_{2x} and the final angular velocity in the form:

$$V_{2x} = \mu_k (1+c)\sqrt{2gh} \qquad (11)$$

$$\omega_2 = \omega_0 - \frac{\mu_k mR(1+c)}{I}\sqrt{2gh} \qquad (12)$$

With this we have all the basic magnitudes in terms of data. The range of validity of the solution under consideration may be obtained from (11) and (12). This solution will be valid whenever at the end of the impact the contact point has a velocity in the direction of the negative x. That is, if:

$$\omega_2 \mathrm{R} > v_{2x}$$

$$\omega_0 - \frac{\mu_k mR(1+c)}{I}\sqrt{2gh} > \frac{\mu_k (1+c)}{R}\sqrt{2gh}$$

$$\omega_0 > \frac{\mu_k \sqrt{2gh}}{R}(1+c)\left(\frac{mR^2}{I}+1\right) \qquad (13)$$

so, for angular velocities below this value, the solution is not valid.

Case II

In this case, rolling is attained for a time t between the initial time t_1 and the final time t_2 of the impact. Then the following relationship should exist between the horizontal component of the velocity v_{2x} and the final angular velocity:

$$\omega_2 \mathrm{R} = v_{2x} \qquad (14)$$

Substituting (14) and (6) in (7), we get that:

$$mRv_{2x} = I\left(\omega_0 - \frac{v_{2x}}{R}\right) \qquad (15)$$

which can be solved for the final values:

$$V_{2x} = \frac{I\omega_0}{mR + \frac{I}{R}} = \frac{I\omega_0 R}{mR^2 + I} = \frac{2}{7}\,\omega_0 R \qquad (16)$$

and:

$$\omega_2 = \frac{I\omega_0}{mR^2 + I} = \frac{2}{7}\,\omega_0 \qquad (17)$$

Calculation of the tangents of the angles

Case I

For $\tan\theta$ we have, from (4) and (11), that:

$$\tan\theta = \frac{v_{2x}}{v_{2y}} = \frac{\mu_k(1+c)\sqrt{2gh}}{c\sqrt{2gh}} = \mu_k\frac{(1+c)}{c}$$

$$\tan\theta = \mu_k\frac{(1+c)}{c} \qquad (18)$$

i.e., the angle is independent of ω_0.

Case II

Here (4) and (16) determine for $\tan\theta$ that:

$$\tan\theta = \frac{v_{2x}}{v_{2y}} = \frac{I\omega_0 R}{I + mR^2}\;\frac{1}{c\sqrt{2gh}} = \frac{I\omega_0 R}{(I + mR^2)c\sqrt{2gh}}$$

$$\tan\theta = \frac{2\omega_0 R}{7c\sqrt{2gh}} \qquad (19)$$

then (18) and (19) give the solution (fig. 1.3).

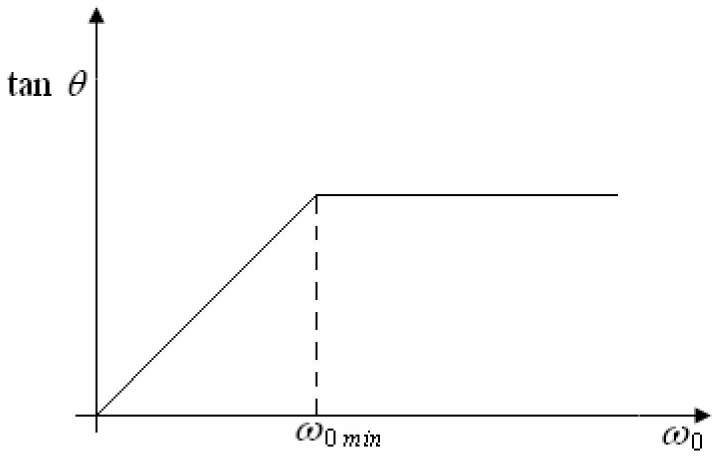


Figure 1.3

We see that θ does not depend on ω_o if $\omega_0 > \omega_{0\,min}$; where $\omega_{0\,min}$ is given as:

$$\omega_{o\,min} = \frac{\mu_k(1+c)\sqrt{2gh}\left(1 + \frac{mR^2}{I}\right)}{R}$$

$$\omega_{o\,min} = \frac{7\mu_k(1+c)\sqrt{2gh}}{2R} \qquad (20)$$

Calculation of the distance to the second point of impact

Case I

The rising and falling time of the ball is:

$$t_v = 2\frac{v_{2y}}{g} = \frac{2c\sqrt{2gh}}{g} = 2c\sqrt{\frac{2h}{g}} \qquad (21)$$

The distance to be found, then, is;

$$d_I = v_{2x}t_v = \mu_k(1+c)\sqrt{2gh}\,2c\sqrt{\frac{2h}{g}}$$

$$d_I = 4\mu_k(1+c)\,ch \qquad (22)$$

which is independent of ω_0.

Case II

In this case, the rising and falling time of the ball will be the one given in (21). Thus the distance we are trying to find may be calculated by multiplying t_v by the velocity v_{2x} so that:

$$d_{II} = v_{2x}t_v = \frac{I\omega_0}{mR^2+I}2c\sqrt{\frac{2h}{g}} = \frac{2\omega_0 Rc}{1+\frac{5}{2}}\sqrt{\frac{2h}{g}}$$

$$d_{II} = \frac{4}{7}c\sqrt{\frac{2h}{g}}R\omega_0$$

Thus, the distance to the second point of impact of the ball increases linearly with ω_0.

Marking Code

The point value of each of the sections is:

1.a	2 points
1.b	1.5 points
1.c	2 points
2.a	2 points
2.b	1.5 points
3	1 point

Solution Problem 2

Question a:

Let's call S the lab (observer) frame of reference associated with the observer that sees the loop moving with velocity v; S' to the loop frame of reference (the x' axis of this system will be taken in the same direction as $\vec{v}$; y' in the direction of side DA and z' axis, perpendicular to the plane of the loop). The axes of S are parallel to those of S' and the origins of both systems coincide at $t = 0$.

1. Side AB

S''_{AB} will be a reference frame where the moving balls of side AB are at rest. Its axes are parallel to those of S and S'. S'' has a velocity u with respect to S'.

According to the Lorentz contraction, the distance a, between adjacent balls of AB, measured in S'', is:

$$a_r = \frac{a}{\sqrt{1-\frac{u^2}{c^2}}} \tag{1}$$

(This result is valid for the distance between two adjacent balls that are in one of any sides, if a_r is measured in the frame of reference in which they are at rest).

Due to the relativistic sum of velocities, an observer in S sees the balls moving in AB with velocity:

$$u_{AB} = \frac{v+u}{1+\frac{uv}{c^2}} \tag{2}$$

So, because of Lorentz contraction, this observer will see the following distance between balls:

$$a_{AB} = \sqrt{1-\frac{u_{AB}^2}{c^2}}a_r \tag{3}$$

Substituting (1) and (2) in (3)

$$a_{AB} = \sqrt{\frac{1-\frac{v^2}{t^2}}{1+\frac{uv}{c^2}}}a \tag{4}$$

2. Side CD

For the observer in S, the speed of balls in CD is:

$$u_{CD} = \frac{v-u}{1-\frac{uv}{c^2}} \tag{5}$$

From the Lorentz contraction:

$$a_{CD} = \sqrt{1-\frac{u_{CD}^2}{c^2}}a_r \tag{6}$$

Substituting (1) and (5) in (6) we obtain:

$$a_{CD} = \frac{\sqrt{1-\frac{v^2}{c^2}}}{1-\frac{uv}{c^2}}a \tag{7}$$

3. Side DA

In system S', at time $t_o^{'}$, let a ball be at $x_1^{'} = y_1^{'} = z_1^{'} = 0$. At the same time the nearest neighbour to this ball will be in the position $x_2^{'} = 0, y_2^{'} = a$, $z_2^{'} = 0.$

The space-time coordinates of this balls, referred to system S, are given by the Lorentz transformation:

$$x = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}(x'+vt')$$

y=y'

z=z' (8)

$$t = \frac{1}{\sqrt{1+\frac{v^2}{c^2}}}\left(t'+\frac{x'v'}{c^2}\right)$$

Accordingly, we have for the first ball in S:

$$x_1 = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} v t_o^{'};\ y_1=0;\ z_1=0;\ t_1 = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} t_0^{'} \tag{9}$$

(9)

And for the second:

$$x_2 = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} v t_o^{'}; y_2 = a; z_2 = 0; t_2 = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} t_o^{'} \tag{10}$$

As $t_1 = t_2$, the distance between two balls in S will be given by:

$a_{DA}=(x_2-x_1)^2 + (y_2 - y_1)^2 + (z_2-z_1)^2$ (11)

So:

$a_{AD}=a$ (12)

4. Side BC

If we repeat the same procedure as above, we can obtain that:

$a_{BC}=a$

(13)

Question b:

The charge of the wire forming any of the sides, in the frame of reference associated with the loop can be calculated as:

$$Q_{wire} = -\frac{L}{a} q \tag{14}$$

Because L/a is the number of balls in that side. Due to the fact that the charge in invariant, the same charge can be measured in each side of the wire in the lab (observer) frame of reference.

1. Side AB

The charge corresponding to balls in side AB is, in the lab frame of reference:

$$Q_{AB,\,balls} = \frac{L\sqrt{1-\frac{v^2}{c^2}}}{a_{AB}} - q \tag{15}$$

This is obtained from the multiplication of the number of balls in that side multiplied by the (invariant) charge of one ball. The numerator of the first factor in the right side of equation (15) is the contracted distance measured by the observer and the denominator is the spacing between balls in that side.

Replacing in (15) equation (4), we obtain:

$$Q_{AB,\,balls} = \left(\frac{1+uv}{c^2}\right)\frac{Lq}{a} \tag{16}$$

Adding (14) and (16) we obtain for the total charge of this side:

$$Q_{AB} = \frac{uv}{c^2}\frac{L}{a} q \tag{17}$$

2. Side CD

Following the same procedure we have that:

$$Q_{CD,\,balls} = \frac{\sqrt{1-\frac{v^2}{c^2}}}{a_{CD}} - q = \left(1-\frac{uv}{c^2}\right)\frac{Lq}{a} \tag{18}$$

And adding (14) and (18) we obtain:

$$Q_{CD} = -\frac{uv}{c^2}\frac{L}{a} q \tag{19}$$

The length of these sides measured by the observer in S is L and the distance between balls is a, so:

$$Q_{BC,\,balls} = Q_{DA,\,balls} = \frac{Lq}{a} \tag{20}$$

Adding (14) and (20) we obtain:

$Q_{BC} = 0$ (21.1)
$Q_{DA}=0$ (21.2)
Question c:
There is electric force acting into the side AB equal to:

$$\vec{F}_{AB} = Q_{AB}\vec{E} = \left(\frac{uv}{c^2}\right)\frac{L}{a}q\vec{E} \tag{22}$$

and the electric force acting into the side CD is:

$$\vec{F}_{CD} = Q_{CD}\vec{E} = -\left(\frac{uv}{c^2}\right)\frac{L}{a}q\vec{E} \tag{23}$$

F_{CD} and F, form a force pair. So, from the expression for the torque for a force pair we have that (fig. 2.2):

$$M = \left|\vec{F}_{AB}\right| L\sin\theta \tag{24}$$

And finally:

$$M = \frac{uv}{c^2}\frac{L^2}{a}|q|\left|\vec{E}\right|\sin\theta \tag{25}$$

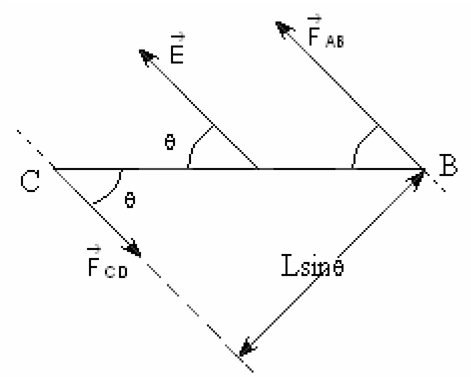


Fig 2.2

Question d:
Let's call V_{AB} and V_{CD} the electrostatic in the points of sides AB and CD respectively. Then:
$W=V_{AB}Q_{AB} + V_{CD}Q_{CD}$ (26)

Let's fix cero potential (V=0) in a plane perpendicular to $\vec{E}$ and in an arbitrary distance R from side AB (fig. 2.3).

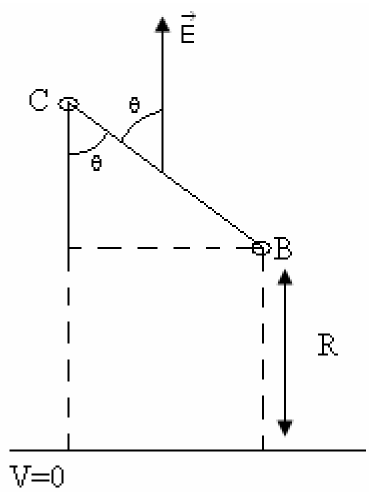


Figure 2.3

Then:
$W=-ERQ_{AB} – E(R+L\cos\theta)Q_{CD}$ (27)
But $Q_{CD}=-Q_{AB}$, so:
$W=-ELQ_{AB}\cos\theta$ (28)

Substituting (17) in (28) we obtain:

$$W = \frac{uvL^2qE}{c^2a}\cos\theta \qquad (29)$$

Marking Code

Grading for questions will be as follows:

a)4,5 points.

b)2,0 points.

c)1,5 points.

d)2,0 points.

These points are distributed in questions in the following way:

Question a:

1. Obtaining expressions (4) and (7) correctly: 3,0 points.
   Only one of them correct: 2,0 points.
2. Obtaining expressions (12) and (13) correctly including the necessary calculations to arrive to this results: 1,5 points.
   Only one of them correct: 1,0 points.

If the necessary calculations are not present: 0,8 point for both (12) and (13) correct; 0,5 points for only one of them correct.

Question b:

1. Obtaining expressions (17) and (19) correctly: 1,0 point.
   Only one of them correct: 1,0 point.
2. Obtaining expressions (21.1) and (21.2) correctly: 0,5 point.
   Only one them correct: 0,5 point.

Question d:

1. Obtaining expression (29) correctly: 2,0 points.

When the modulus of a vector is not present where necessary, the student will loose 0,2 points. When the modulus of q is not present where necessary the student will loose 0,1 points.

Solution Problem 3

Question a:

The velocity v_o of the atoms whose kinetic energy is the mean of the atoms on issuing from the collimator is given is given by:

$$\frac{1}{2}mv_o^2 = \frac{3}{2}kT \Rightarrow v_o = \sqrt{\frac{3kT}{m}} \qquad (1)$$

$$v_o = \sqrt{\frac{3\cdot 1{,}38\cdot 10^{-23}\cdot 10^3}{23\cdot 1{,}67\cdot 10^{-27}}}\ \ m/s$$

$v_o \approx 1{,}04\cdot 10^3$ m/s because:

$m \approx 23\ m_p$ (2)

Since this velocity is much smaller than c, $v_o << c$, we may disregard relativistic effects.

Light is made up of photons with energy $h\nu$ and momentum $h\nu/c$.

In the reference system of the laboratory, the energy and momentum conservation laws applied to the absorption process imply that:

$$\frac{1}{2}mv_o^2 + h\nu = \frac{1}{2}mv_1^2 + E; mv_o - \frac{h\nu}{c} = mv_1 \Rightarrow \Delta v_1 = v_1 - v_o = \frac{-h\nu}{mc}$$

$$\frac{1}{2}m(v_1^2 - v_o^2) = h\nu - E \Rightarrow \frac{1}{2}m(v_1 + v_o)(v_1 - v_o) = h\nu - E$$

$h\nu/c << mv_o$. Then $v_1 \approx v_o$ and this implies $mv_o\Delta v_1 = h\nu = E$, where we assume that $v_1 + v_o \approx 2v_o$

Combining these expressions:

$$\nu = \frac{\frac{E}{h}}{1+\frac{v_o}{c}} \tag{3}$$

and:

$$\Delta v_1 = -\frac{E}{mc}\frac{1}{1+\frac{v_o}{c}} \tag{4}$$

And substituting the numerical values:

$\nu \approx 5{,}0 \cdot 10^{14}$ Hz $\quad \Delta v_1 \approx -3{,}0 \cdot 10^{-2}$ m/s

If we had analyzed the problem in the reference system that moves with regard to the laboratory at a velocity v_o, we would have that:

$$\frac{1}{2}m(v_1 - v_2)^2 + E = h\nu$$

Where $\nu = \dfrac{\nu'}{1+\frac{v_o}{c}}$ is the frequency of the photons in the laboratory system. Disregarding Δv_1^2 we get the same two equations above.

The approximations are justifiable because:

$$-\frac{|\Delta v_1|}{v_o} \sim 10^{-4}$$

Then $v_1 + v_o = 2v_o - \Delta v_1 \approx 2v_o$

Question b:

For a fixed ν:

$$v_o = c\left(\frac{E}{h\nu} - 1\right) \tag{5}$$

if E has an uncertainty Γ, v_o would have an uncertainty:

$$\Delta v_o = \frac{c\Gamma}{h\nu} = \frac{c\Gamma\left(1+\frac{v_o}{c}\right)}{E} \approx \frac{c\Gamma}{E} = 6{,}25 \text{ m/s} \tag{6}$$

so the photons are absorbed by the atoms which velocities are in the interval $\left(v_o - \frac{\Delta v_o}{2}, v_o + \frac{\Delta v_o}{2}\right)$

Question c:

The energy and momentum conservation laws imply that:

$$\frac{1}{2}mv_1^2 + E = \frac{1}{2}mv'^2_1 + h\nu'$$

(ν' – is the frequency of emitted photon)

$$mv_1 = mv'_1\cos\varphi + \frac{h\nu'}{c}\cos\theta$$

$$0 = mv'_1\sin\varphi - \frac{h\nu'}{c}\sin\theta$$

The deviation φ of the atom will be greatest when $\theta = \frac{\pi}{2}$, then:

$$mv_1 = mv'_1 \cos\varphi_m ; \frac{h\nu'}{c} = mv'_1 \sin\varphi_m \Rightarrow \tan\varphi_m = \frac{h\nu'}{mv_1 c}$$

since $\nu' \approx \nu$:

$$\tan\varphi_m \approx \frac{E}{mv_1 c} \qquad (7)$$

$$\varphi_m = \mathrm{arctg}\frac{E}{mvc} \Rightarrow \varphi_m \approx 5\cdot 10^{-5}\ \mathrm{rad} \qquad (8)$$

Question d:

As the velocity of the atoms decreases, the frequency needed for resonant absorption increases according to:

$$\nu = \frac{\frac{E}{h}}{1+\frac{v_o}{c}}$$

When the velocity is $v_o = \Delta v$, absorption will still be possible in the lower part of the level if:

$$h\nu = \frac{E-\frac{\Gamma}{2}}{1+\frac{v_o-\Delta v}{c}} = \frac{E}{1+\frac{v_o}{c}} \Rightarrow \Delta v = \frac{c\Gamma}{2E}\left(1+\frac{v_o}{c}\right) \qquad (9)$$

Δv = 3,12 m/s

Question e:

If each absorption-emission event varies the velocity as $\Delta v_1 \approx \frac{E}{mc}$, decreasing velocity from v_o to almost zero would require N events, where:

$$N = \frac{v_o}{|\Delta v_1|} \approx \frac{mcv_o}{E} \Rightarrow N \approx 3{,}56\cdot 10^4$$

Question f:

If absorption is instantaneous, the elapsed time is determined by the spontaneous emission. The atom remains in the excited state for a certain time, $\tau = \frac{h}{\Gamma}$, then:

$$\Delta t = N\tau = \frac{Nh}{\Gamma} = \frac{mchv_o}{\Gamma E} \Rightarrow \Delta t \approx 3{,}37\cdot 10^{-9}\ \mathrm{s}$$

The distance covered in that time is $\Delta S = v_o\Delta t/2$. Assuming that the motion is uniformly slowed down:

$$\Delta S = \frac{1}{2} mchv_o^2 \Gamma E \Rightarrow \Delta S \approx 1{,}75\ \mathrm{m}$$

Marking Code

a)Finding	v_o	1 pt	Total 3 pt
“	ν	1 pt	
“	Δv_1	1 pt	
b) “	Δv_o	1,5 pt	Total 1,5 pt

c)	“	φ_m	1,5 pt	Total 1,5 pt
d)	“	Δv	1 pt	Total 1 pt
e)	“	N	1 pt	Total 1 pt
f)	“	Δt	1 pt	Total 2 pt
	“	ΔS	1 pt	
				Overall total 10 pts

We suggest in all cases: 0,75 for the formula; 0,25 for the numeral operations.

EXPERIMENTAL PROBLEM

Problem

Inside a black box provided with three terminals labeled A, B and C, there are three electric components of different nature. The components could be any of the following types: batteries, resistors larger then 100 ohm, capacitors larger than 1 microfarad and semiconductor diodes.

a)Determine what types of components are inside the black box and its relative position to terminal A, B and C. Draw the exploring circuits used in the determination, including those used to discard circuits with similar behaviour

b)If a battery was present, determine its electromotive force. Draw the experimental circuit used.

c)If a resistor was present, determine its value. Draw the experimental circuit used.

d)If a capacitor was present, determine its value. Draw the experimental circuit used.

e)If a diode was present, determine V_o and V_r, where V_o the forward bias threshold voltage and V_r is the reverse bias breakdown voltage.

f)Estimate, for each measured value, the error limits.

The following equipments and devices are available for your use:

1 back box with three terminals labeled A, B and C;
1 variable DC power supply;
2 Polytest 1 W multimeters;
10 connection cables;
2 patching boards;
1 100 kΩ, 5 % value resistor;
1 10 kΩ, 5 % value resistor;
1 1 kΩ, 5 % value resistor;
1 100 μF, 20 % value capacitor;
1 chronometer;
2 paper sheets;
1 square ruler;
1 interruptor.

Voltmeter internal resistance.

Scale	Value in kΩ	
0-1 V	3,2	1 %
0-3 V	10	1 %
0-10 V	32	1 %
0-20 V	64	1 %
0-60 V	200	1 %

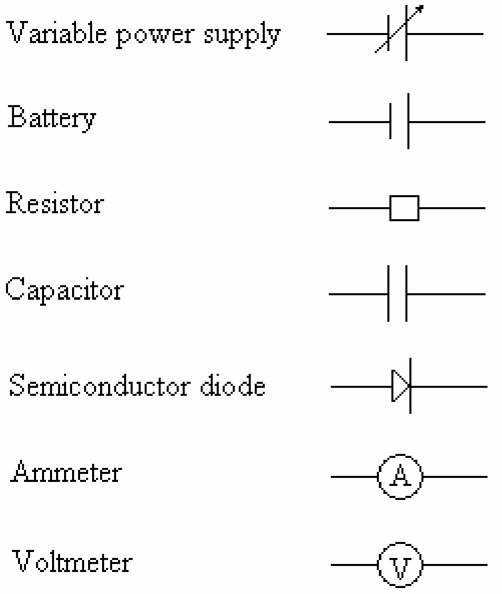
Ammeter internal resistance.

Scale	Value in Ω	
0-0,3 mA	1 000	1 %
0-1 mA	263	1 %
0-3 mA	94	1 %
0-20 mA	30,4	1 %
0-30 mA	9,84	1 %
0-100 mA	3,09	1 %
0-300 mA	0,99	1 %
0-1 mA	0,31	1 %

Notice: Do not use the Polystes 1 W as an ohmmeter. Protect your circuit against large currents, and do not use currents larger than 20 mA.

Give your results by means of tables or plots.

When drawing the circuits, use the following symbols:



E X P E R I M E N T A L PR O B L E M. S O L U T I O N

Solution Problem
Since a battery could be present, the first test should be intended to detect it. To do that, the voltage drops V_{ab}, V_{ac} and V_{bc} should be measured using a voltmeter. This test will show that no batteries are present.

Next, a testing circuit as shown in figure 4.1 should be used.

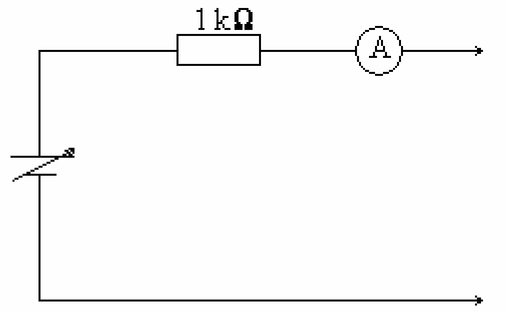


Figure 4.1

By means of this circuit, the electric conduction between a pair of terminals should be tested, marking all permutations and reversing the polarity. Resistor R_1 is included to prevent a large current across the diode. One conclusion is that between A and C there is a diode and a resistor in series, although its current position is still unknown. The other conclusion is that a capacitor is tighted to terminal B. To determine the actual circuit topology, further transient experiments have to be conducted.

In this way, it is concluded that the actual circuit inside the black box is that shown in figure 4.2.

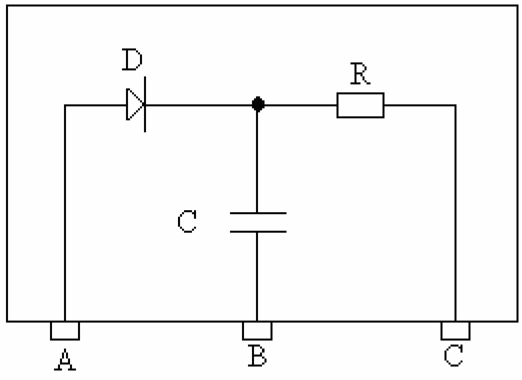


Figure 4.2

The best procedure for the resistor value determination is to plot a set of voltage and current values measured between A and C. Figure 4.3 shows the resulting plot. Extrapolating both linear regions, the values of V_o and V_z are obtained and the resistor value equals the reciprocal of the slope.

Similar, the best method to measure the capacitor value is to build a testing circuit as shown in figure 4.4. The current is adjusted to full scale and then, the switch is opened.

The time needed by the current to drop to its half value is measured. Applying the formulae t = RCIn(2), the value of C is obtained.

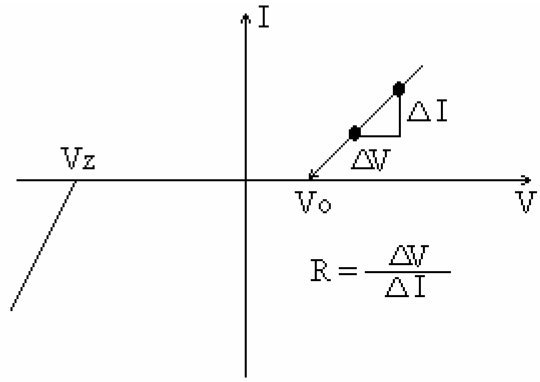


Figure 4.3

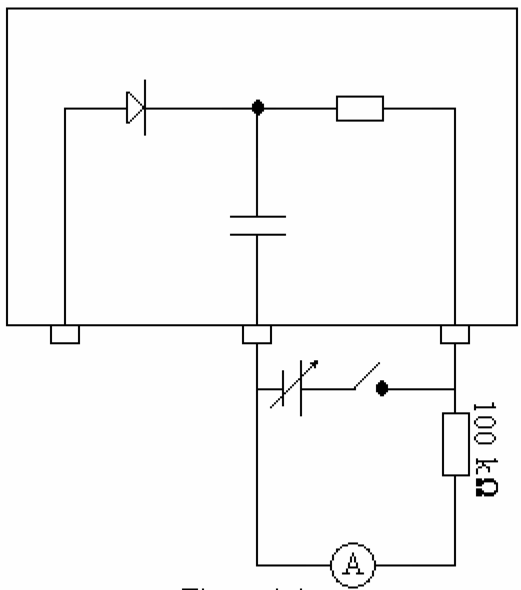


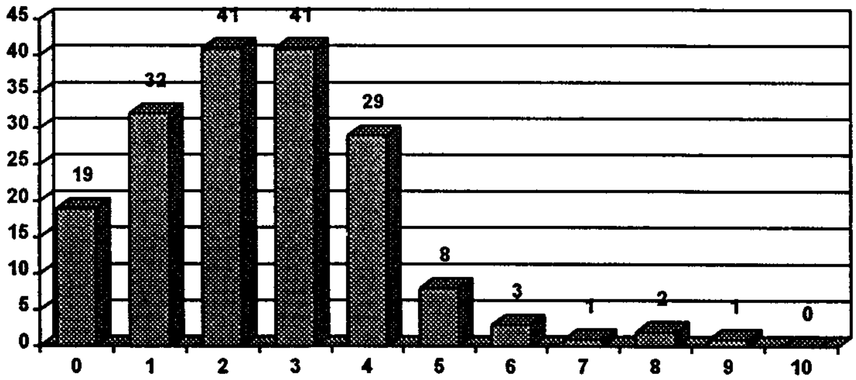
Figure 4.4

Marking Code

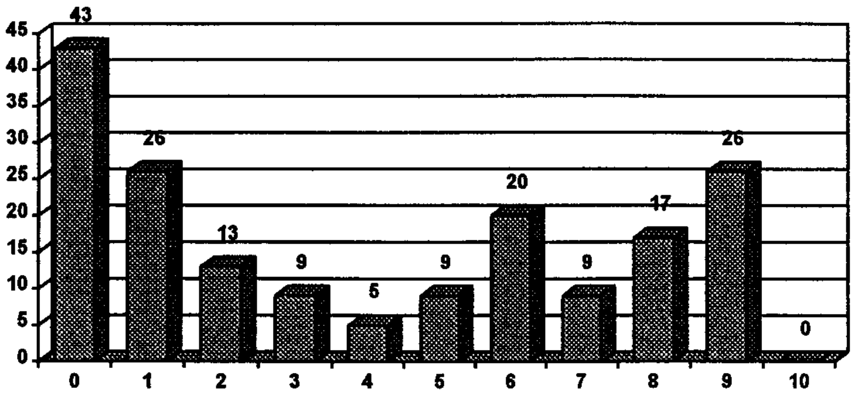
1. Determination of circuit topology: 8 points.
   1.1 For discarding the presence of a battery: 1 point.
   1.2 For drawing the exploring circuit which determine the circuit topology in a unique way: 7 points.
2. Resistor and diode parameters value measurement: 8 points.
   2.1 For drawing the measuring circuit: 2 points.
   2.2 Error limits calculation: 3 points.
   2.3 Result: 3 points.
   2.3.1 Coarse method: 2 points.
   2.3.2 Graphic method: 3 points.
3. Capacitor value measurement: 4 points.
   3.1 For drawing the measuring circuit: 2 points.
   3.2 Error limits calculations: 2 points.

Distribution of the results of the XXIII IPhO

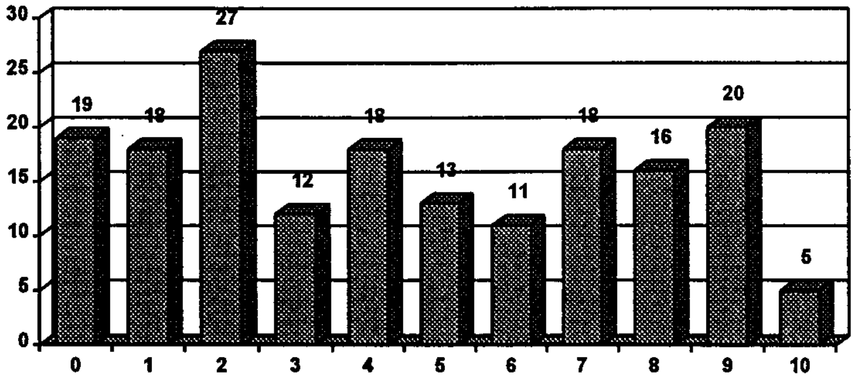
Theoretical #1



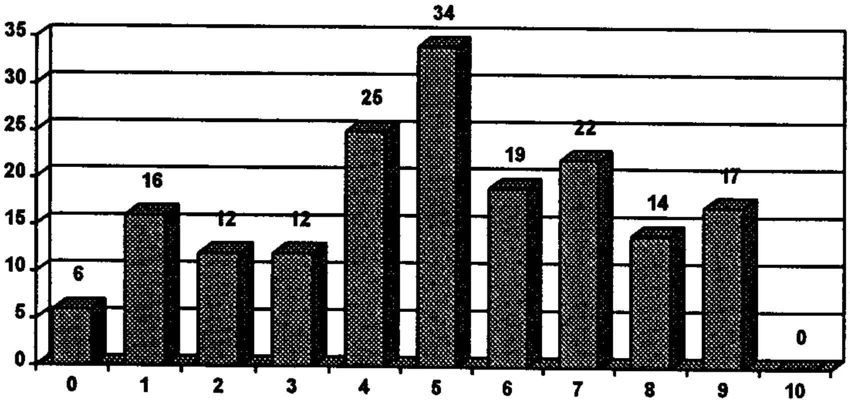
Theoretical #2



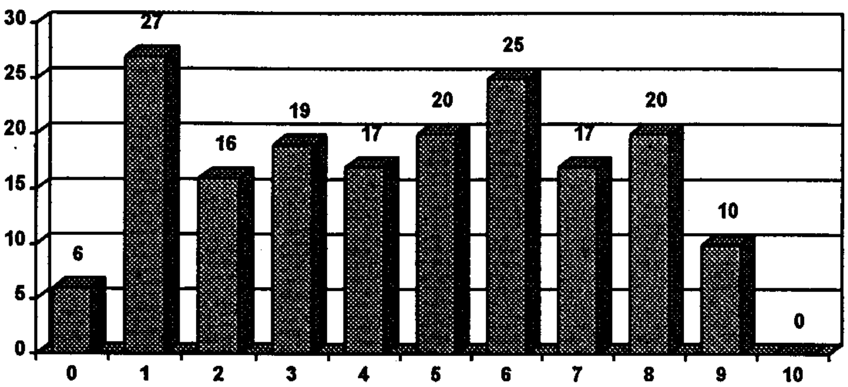
Theoretical #3



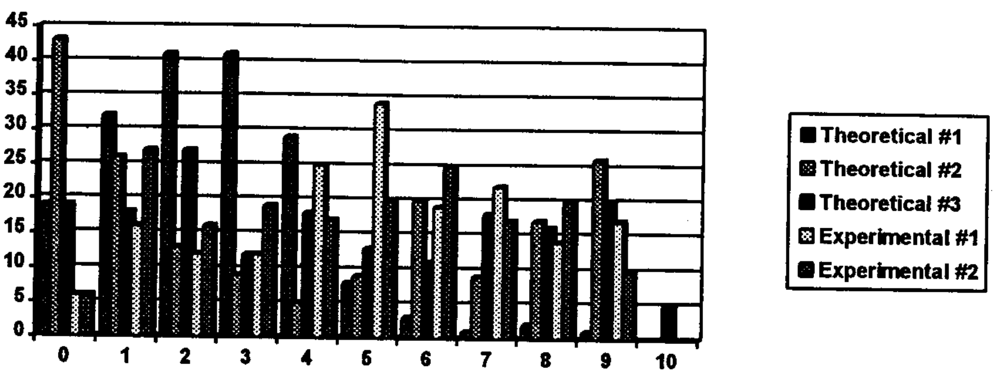
Experimental #1



Experimental #2

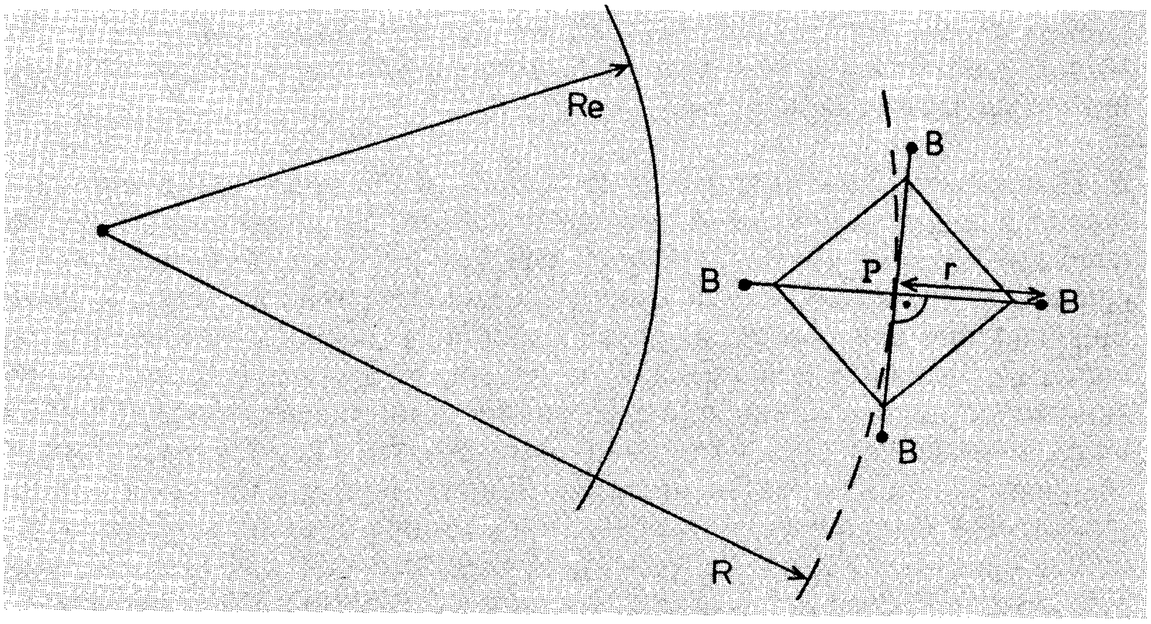


All problems



PROBLEM 1 : A ROTATING SATELLITE.

The figure shows a satellite which is circling the Earth in an approximately circular orbit in the Earth's equatorial plane. The satellite consists of a massless central body P and four small peripheral bodies B. The four bodies B each have mass m: they are fastened to P by means of long thin wires of length r that do not stretch. All these five bodies, P and the four bodies B, are coplanar with the equatorial plane, and they can rotate within this plane. The four radial wires are linked to each other by further thin wires which keep the angles between the radial wires constant at 90°.



The link wires are included in the system in order to prevent oscillatory movement of the individual bodies B which would otherwise make the analysis of the movements extremely complicated. All the bodies B rotate around P at the same angular velocity, which is ω with respect to the fixed stars. Thus, the satellite behaves as a rigid body.

Analyze the following questions for the general case, considering all possible situations, including both senses of rotation of the bodies B. Also obtain numerical values for certain of the quantities found in questions (1) and (2)—the quantities required and the necessary numerical data are listed at the end of the problem.

1) The drawing shows the satellite in the position where for the various wires, **r** is parallel, anti-parallel or perpendicular to **R**. (The vector **r** runs from body P to a body B and has length r; the vector **R** runs from the centre of mass of the Earth to the body P.)

   Determine the force exerted by a radial wire on one of the bodies B in each of these four positions. These positions correspond approximately to the maximum and minimum forces.

2) Inside the four bodies B there are four identical machines, powered by solar energy, connected to the radial wires. Each machine pulls the wire in, towards B, for a short time whenever there is near maximum force in the wire (as indicated in the previous question), and lets the same length of wire out again when the tension is at a minimum. The length of wire that is pulled in and let out is 1% of the mean length of the radial wire. The mean length does not change with time.

   What is the net power converted by one machine, averaged over one rotation of the satellite?

   The net power is defined as $\dfrac{W_1 - W_2}{T}$, where W_1 is the work that the machine performs on the wire when pulling it in, W_2 is the work that the wire performs on the machine when it is reeled out and T is the period of rotation.

3) Discuss the changes in the motion of the satellite that are caused by the action of the machines. In particular, analyze any changes that may occur in each of the situations listed in the table overleaf.

   Fill in the table with your results and comments, and don't forget to hand it in.

Data:

Numerical answers are required in the following situation:

The radius of the orbit of the central body is given by $R = R_E + 500\,\text{km}$.

The mean length of the radial wires is $r = 100\,\text{km}$.
Thus the diameter of the satellite system is 200 km.

The bodies B have masses $m = 1000\,\text{kg}$.

Initially the four bodies B rotate, as referred to the stars, around the central body P at 10 revolutions/hour.

The masses of the wires are negligible, and the central body P is massless.

Advice:

Consider both senses of rotation for ω.
Exact solutions are not expected. Results with 5% accuracy are fully acceptable.
Ignore the gravitational effect of the moon and the sun.

Useful data:

Mass of Earth	$M_E = 5.97 \times 10^{24}\,\text{kg}$
Gravitational constant	$G = 6.673 \times 10^{-11}\,\text{m}^3\,\text{kg}^{-1}\,\text{s}^{-2}$
Radius of Earth at equator	$R_E = 6378\,\text{km}$
Denote the product $M_E G$ by K.	$K = 3.983 \times 10^{14}\,\text{m}^3\,\text{s}^{-2}$

Country code:

ANSWER TABLE

Fill in this table as part of your answer. Write down equalities or inequalities and/or short explanations where necessary.

The quantity indicated below ...	increases if ...	decreases if ...	stays unchanged if ...	stays unchanged in all situations.
orbital **velocity** of the satellite				Yes ☐ No ☐
radius R of the orbit of the satellite				Yes ☐ No ☐
angular velocity ω of the satellite				Yes ☐ No ☐
gravitational **potential energy** of the satellite				Yes ☐ No ☐

Could the satellite reach a higher orbit as a result of the work done by the machines? Yes ☐ No ☐
Could the satellite reach an arbitrarily high orbit, practically leaving the gravitational influence of Earth? Why? Answer:..

SOLUTION : PROBLEM $\boxed{1}$

Notation. Vectors are denoted as $\vec{r}$, $\vec{R}$. Without vector symbol, r and R mean the lengths of these vectors.
Unit vectors of specified direction are indicated by indicating the direction vector in brackets: $\vec{e}(\vec{R})$ is a vector of unit length, directed from the centre of earth to point P. The vector pointing from B to the centre of earth is $-\vec{e}(\vec{R}+\vec{r})$ and $\vec{a} = -\vec{e}(\vec{R})K/R^2 = -K\vec{R}/R^3$ represents the gravitational acceleration at $\vec{R}$.
Different cases are denoted as follows: in the first section, the word *parallel* means that $\vec{R}$ and $\vec{r}$ are parallel, i.e. that the periferal body B is highest up in its orbit. In the same way, *antiparallel* means the position of B nearest to the earth. In later sections, the different senses of rotation of the satellite are denoted as *parallel* and *antiparallel*: When the angular velocity vectors $\vec{\omega}$ and $\vec{\Omega}$ (the angular velocity of P with respect to the centre of the earth) are parallel, it means that the satellite rotates in the direction of its orbital motion.

Determination of the tensional forces

Determination of the tensional forces requires certain approximations to be done:
1. The centre of the satellite is on a circular Kepler orbit, i.e.

$$\Omega^2 R = K/R^2.$$

The problem formulation indicates that the initial orbit is intended to be circular. Physically, the orbit could also be elliptical.
2. ω and Ω are constant.
3. $r \ll R$ so that higher powers of r/R can be neglected.
4. As shown in Figure 1, the extreme end of each radial wire of the satellite is free to swing back and forth according to the resultant acceleration of the body B. The force acting on a body B is generally not directed towards the centre P of the satellite. However, the end section of the wire between P and B is directed along the direction of the force. It is assumed that these free end sections are so short that their swinging doesn't affect significantly the motion of the bodies B of the satellite, i.e. that $\vec{\omega}$ and $\vec{r}$ are valid for describing the motion of B around P.
5. The side position was defined in the problem as the position where $\vec{r}$ and $\vec{R}$ are perpendicular. The distance between this point and the centre of the earth is $\sqrt{r^2+R^2} = R\sqrt{1+(r^2/R^2)} \approx 1.00011\,R \approx R + 0.008r$. Thus a good approximation for the side position is the point whose distance to the centre of earth equals R. We can equally well estimate the force for this approximate side point, the error of approximation will certainly be less than 5 %.

None of the assumptions 1, 2, 3, and 5 holds if r, the radius of the satellite, is thousands of kilometers and if the satellite is near the earth. With the numerical values given in the problem, $r/R \approx 0.0145$ so that the approximations are better than the expected accuracy of solution. Rigorous proof of these approximations

might be more demanding than solving the problem itself. It was not expected from the competitors and it will not be given here.
The vectors $\vec{r}$ and $\vec{R}$, and the angular velocities $\vec{\omega}$ and $\vec{\Omega}$ are defined with respect to an inertial (non-rotating) frame of reference.

The location vector for one body B is $\vec{R}+\vec{r}$. Thus we get for the velocity and acceleration of B

$$\vec{v} = \vec{\Omega}\times\vec{R}+\vec{\omega}\times\vec{r}, \tag{1}$$
$$\vec{a} = -\Omega^2\vec{R}-\omega^2\vec{r} \tag{2}$$

With less formalism: the motion of a body B is a superposition (or sum) of two circular motions, one around the earth and the other around the centre of the satellite. Thus also the acceleration of B is the sum of the two accelerations: one directed towards the centre of earth, and of magnitude $\Omega^2 R$, and the other directed towards the centre of the satellite, and of magnitude $\omega^2 r$.
The gravitational force acting on the body B depends on the distance of B from the centre of earth, i.e. on the length of the sum of vectors $\vec{R}+\vec{r}$:

$$F_{gravity} = m\frac{K}{|\vec{R}+\vec{r}|^2}. \tag{3}$$

and is directed towards the centre of earth, i.e. the direction is opposite to the sum of vectors $\vec{R}+\vec{r}$. In vector notation this can be written as

$$\vec{F}_{gravity} = -m\frac{K\vec{e}(\vec{R}+\vec{r})}{|\vec{R}+\vec{r}|^2} \tag{4}$$
$$= -m\frac{K(\vec{R}+\vec{r})}{|\vec{R}+\vec{r}|^3}. \tag{5}$$

Quite often, the second form is simpler in computations.
The total force acting on B corresponds to the acceleration:

$$\vec{F} = \vec{F}_{wire}+\vec{F}_{gravity} \tag{6}$$
$$= m\vec{a} = m\left(-\Omega^2\vec{R}-\omega^2\vec{r}\right) \tag{7}$$

This gives the tensional force:

$$\vec{F}_{wire}/m = -\Omega^2\vec{R}-\omega^2\vec{r}+\frac{K(\vec{R}+\vec{r})}{|\vec{R}+\vec{r}|^3}. \tag{8}$$

This is an exact result. Numerical answers can be calculated with this expression. One example is given in the section for numerical results. However, a better understanding is possible if we find an approximation so that such higher order terms are neglected which don't have a significant influence on the results. Indicate the

SOLUTION : PROBLEM 1

position of the body B by defining the distance of B from the centre of earth as $|\vec{R}+\vec{r}| = R+\rho$ where $-r \le \rho \le r$. Express the denominator in powers of ρ/R:

$$(R+\rho)^{-3} = R^{-3}(1 - 3\rho/R + O(r/R)^2) \tag{9}$$

Physically this approximation means that the change of gravitational acceleration is assumed to be linearly proportional to ρ, the change of radius. Substituting this approximation in the expression of $\vec{F}_{wire}$ gives the tensional force in arbitrary rotational position of the satellite as

$$\begin{aligned}
\vec{F}_{wire}/m &= -\Omega^2\vec{R} - \omega^2\vec{r} + \frac{(\vec{R}+\vec{r})(1-3\rho/R+O(r/R)^2)}{R}\frac{K}{R^2} && (10)\\
&= -\Omega^2\vec{R} - \omega^2\vec{r} + \Omega^2 R\left(\vec{R}/R + \vec{r}/R - 3\rho\vec{R}/R^2 + \vec{R}O(r/R)^2\right) && (11)\\
&\approx -\omega^2\vec{r} + \Omega^2\vec{r} - 3\Omega^2\rho\frac{\vec{R}}{R} && (12)
\end{aligned}$$

Analyze the contribution of the last term in the positions up ($\vec{r}$ and $\vec{R}$ parallel), down ($\vec{r}$ and $\vec{R}$ antiparallel) , and sideways (instead of the exact definition, use the approximate definition which corresponds to the value $\rho = 0$):

Up, $\rho = +r$: $\quad \vec{R} = \vec{r}\frac{R}{r}$, giving $\rho\vec{R} = r\vec{R} = r\vec{r}\frac{R}{r} = \vec{r}R$.
The last term equals $-3\Omega^2\vec{r}$.

Down, $\rho = -r$: $\vec{R} = -\vec{r}\frac{R}{r}$, giving $\rho\vec{R} = -r\vec{R} = -r(-\vec{r}\frac{R}{r}) = \vec{r}R$.
Again, the last term equals $-3\Omega^2\vec{r}$.

Sideways, $\rho = 0$: The last term is zero.

Substituting for the last term gives the force in the three different cases:

$$\begin{aligned}
\vec{F}_{min} &= -(\omega^2-\Omega^2)\,\vec{r}\,m \quad \text{if } \rho = 0 && (13)\\
\vec{F}_{max} &= -(\omega^2+2\Omega^2)\,\vec{r}\,m \quad \text{if } \rho = \pm r && (14)\\
\Delta F &= F_{max} - F_{min} = 3\Omega^2\,r\,m && (15)
\end{aligned}$$

In all three cases the force is parallel or antiparallel to the direction of vector $\vec{r}$. The expression $(\omega^2 + 2\Omega^2)$ is always positive. Thus $\vec{F}_{max}$ is always directed to the direction of $-\vec{r}$, i.e. the wire is pulling the body B at 'up' and 'down' positions. However, the expression $(\omega^2 - \Omega^2)$ becomes negative if $\omega < \Omega$. This would mean that $\vec{F}_{min}$ is directed parallel to $\vec{r}$, i.e. that the wire is pushing the body B. This is impossible, however: it would cause the collapse of the satellite because the structure consisting of thin wires can only withstand pulling forces. Thus we must require $\omega > \Omega$ in the following analysis. The expression given for ΔF is based on this assumption.

PROBLEM $\boxed{1}$: A ROTATING SATELLITE.

Work done by the machines

The maximum force affecting any selected body B is present when B is in 'up' position and when B is in 'down' position, i.e. twice during one revolution of the satellite with respect to the vertical axis. Similarly the minimum force is present twice during one relative revolution, in the 'left' and 'right' side positions. The vertical direction rotates with the angular velocity Ω of the orbital motion. If the satellite rotates in the direction of its orbital motion ($\vec{\omega}$ and $\vec{\Omega}$ are parallel) then the satellite must rotate slightly more than one full revolution in order to make one revolution with respect to the vertical axis. Then the angular velocity of the satellite with respect to the vertical axis is $\omega - \Omega$. In the other case (satellite rotates in the direction opposite to the direction of orbital motion) the angular velocity of the satellite with respect to the vertical axis is $\omega + \Omega$: less than one absolute revolution is needed for one relative revolution. In vector notation both cases are given by the expression $\vec{\omega} - \vec{\Omega}$.

The machines perform two work cycles (one cycle: pulling the wire + releasing it) during one relative revolution. Thus the work per one relative revolution is

$$\Delta E = 2\ \Delta r\ (F_{max} - F_{min}) = 6\ m\ \Delta r\ r\ \Omega^2 = 0.06\ m\ r^2\ \Omega^2$$

The period of one relative revolution is

$$\Delta T = 2\pi\ /\ (\omega \pm \Omega)$$

where plus sign corresponds to the antiparallel case. The mean power is given by

$$P = \Delta E\ /\ \Delta T = 2\ \Delta r\ (F_{max} - F_{min})\ /\ (2\pi/(\omega \pm \Omega)) = \Delta r\ (F_{max} - F_{min})\ (\omega \pm \Omega)/\pi$$

Numerical results

From $K/R^2 = \Omega^2 R$ one gets $KR = 25.4\ 10^{20} m^4 s^{-2}$ and
$\Omega = \sqrt{(KR^{-3})}\ =\ 0.001106\ rad/s$. The orbital period is 5678 s.
The angular velocity of the satellite is $\omega = 2\pi/360s = 0.01745\ rad/s$.
The relative angular velocities are
$\omega - \Omega = 0.01634$ rad/s (parallel case) and
$\omega + \Omega = 0.01856$ rad/s (antiparallel case).

$$\begin{aligned} F_{min} &= (\omega^2 - \Omega^2)\,rm = 100km\ 1000kg\ 303.08\ 10^{-6}s^{-2} = 30339\ N && (16)\\ F_{max} &= (\omega^2 + 2\Omega^2)\,rm = 100km\ 1000kg\ 307.68\ 10^{-6}s^{-2} = 30706\ N && (17)\\ F_{max} - F_{min} &= 3\ r\ m\ \Omega^2 = 367\ N. && (18)\end{aligned}$$

$$\begin{aligned} P &= \Delta r\ (F_{max} - F_{min})\ (\omega \pm \Omega)/\pi, && (19)\\ P_{antiparallel} &= 1\ km\ 367\ N\ 0.01856\ rad/s\ \pi^{-1}\ =\ 2168\ W. && (20)\\ P_{parallel} &= 1\ km\ 367\ N\ 0.01634\ rad/s\ \pi^{-1}\ =\ 1909\ W. && (21)\end{aligned}$$

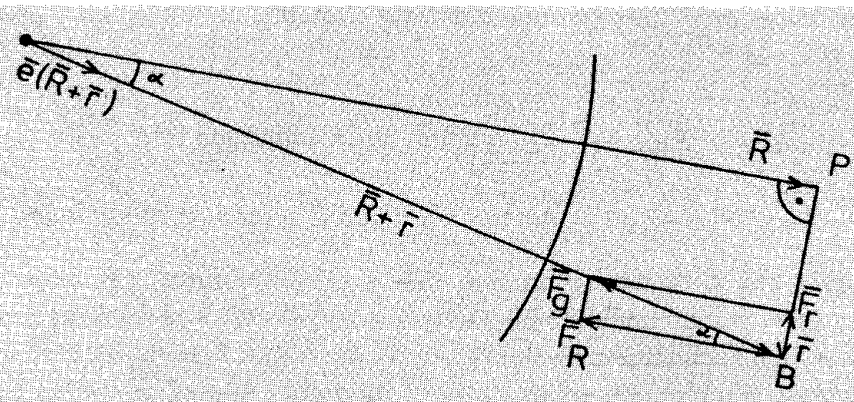


Figure 2: Analysis of the forces acting on body B in the sideways position. The angle α is given by $\tan\alpha = r/R$. The tension is $F_r = \sin\alpha\, F_g \approx \tan\alpha\, F_g = F_g\, r/R$.

These values should be reported rounded to two significant figures because approximations have been used:

$$P_{antiparallel} = 2200\ W, \tag{22}$$
$$P_{parallel} = 1900\ W. \tag{23}$$

Example of the exact expression. As an example, we evaluate here the force for the exact side position directly from the expression

$$\vec{F}_{wire}/m = -\Omega^2\vec{R} - \omega^2\vec{r} + \frac{K\,(\vec{R}+\vec{r})}{|\vec{R}+\vec{r}|^3} \tag{24}$$
$$= -\Omega^2\vec{R} - \omega^2\vec{r} + \frac{K\ \vec{e}(\vec{R}+\vec{r})}{|\vec{R}+\vec{r}|^2}. \tag{25}$$

Taking into account the Kepler equation and the radius value for the side point, we get

$$\vec{F}_{wire}/m = -\Omega^2\vec{R} - \omega^2\vec{r} + \frac{K}{1.00011^2\ R^2}\ \vec{e}(\vec{R}+\vec{r}) \tag{26}$$
$$= -\Omega^2\vec{R} - \omega^2\vec{r} + 0.99979\ \Omega^2 R\,\vec{e}(\vec{R}+\vec{r}) \tag{27}$$

Now the unit vector $\vec{e}(\vec{R}+\vec{r})$ must be expressed with the vectors $\vec{R}$ and $\vec{r}$:

$$\vec{e}(\vec{R}+\vec{r}) = \frac{\vec{R}+\vec{r}}{|\vec{R}+\vec{r}|} = 0.99989\left(\frac{\vec{R}}{R} + \frac{\vec{r}}{R}\right).$$

The same result may be obtained from a triangular construction, see Figure 2. Thus we get

$$\vec{F}_{wire}/m = -\Omega^2\vec{R} - \omega^2\vec{r} + 0.99968\,\Omega^2 R\left(\frac{\vec{R}}{R} + \frac{\vec{r}}{R}\right) \quad (28)$$

$$= -0.00032\,\Omega^2\vec{R} - (\omega^2 - 0.99968\,\Omega^2)\vec{r}. \quad (29)$$

Within the numerical accuracy, this is an exact result. It is easily seen that the first term can be neglected because it is small and also because it represents a force which is perpendicular to the radial wire. In fact, it could be used for estimating the direction of the free-swinging end of the radial wire. The second term is practically identical with our earlier result for F_{min}.

Change of orbit

In principle, the work done by the machines in the satellite could transform the orbit into a non-circular shape. In this problem, however, it is given that the machines work four times during *each* rotational cycle of the satellite. Thus the effect of the machines is distributed more or less symmetrically all around the orbit. Because of the circular symmetry, we may safely assume that the orbit stays circular even while the machines are operating. Thus the change of orbit, as caused by the action of the machines, is from one circular Kepler orbit to another circular Kepler orbit.
It would be possible to analyze the change of orbit by analyzing the resultant of the graviational forces which are acting on the four bodies B. This is, however, a difficult and laborious route. Full analysis of the situation is extremely difficult by that route. However, that might be the only possible method for analyzing how an intermittent use of the machines leads to an elliptic Kepler orbit. But in the present case it is not needed. Conservation laws are the all-important technique for analyzing many physical situations, and if it is possible to identify a sufficient number of conserved quantities, the problem can be transformed to solving the conservation equations.
When rotational motion is considered, typical conserved quantities are: energy and angular momentum. The difficulty is often how to define the system correctly so that it includes all the energies which together are conserved, or all the angular momenta. First consider the angular momentum. As is explained elsewhere, the angular momentum of the rotational motion of the satellite need not be conserved. However, the total angular momentum I_{tot} of the satellite with respect to the centre of earth is conserved because the only external forces acting on the satellite are gravitational and directed towards the centre of earth. (This would not be true if the satellite were in a polar orbit and the non-spherical shape of the earth were considered). The I_{tot} consists of two parts: the internal angular momentum, due to the rotation of the satellite, and the orbital angular momentum, due to the motion of the centre-of-mass of the satellite around the earth.
Another conserved quantity is the energy. To be more precise, the total energy E_{tot} of the satellite is increased by the net work done by the machines. The following terms are included in E_{tot}:

SOLUTION : PROBLEM 1

- The rotational energy of the satellite, $1/2\ 4m\ \omega^2 r^2$
- The orbital kinetic energy of the satellite, i.e. the kinetic energy of the motion of the centre-of-mass, $1/2\ 4m\ \Omega^2 R^2$
- The potential energy of the satellite in the gravitational fiel of earth, $-4mK/R$. In the first order approximation which we are using, this can be calculated as if the total mass of the satellite were concentrated in the centre point P.

Thus we have for the total energy the equation

$$\begin{aligned} E_{tot} &= 4m(-K/R + 1/2\ \Omega^2 R^2 + 1/2\ \omega^2 r^2) && (30) \\ &= 2m(-\Omega^2 R^2 + \omega^2 r^2) \qquad \text{(because of Kepler).} && (31) \end{aligned}$$

A third equation for solving the system is obtained from the Kepler law, connecting the radius of orbit and the orbital velocity of the satellite. Thus we have the system of three equations

$$\begin{aligned} \frac{K}{R^2} &= \Omega^2 R \qquad \text{(Kepler law)} && (32) \\ I_{tot} &= 4m(\omega r^2 \pm \Omega R^2) = \text{Constant} && (33) \\ E_2 - E_1 &= E_m, && (34) \end{aligned}$$

where E_1 and E_2 are the total energies before and after the machines have done the net work E_m. The upper sign corresponds to the *parallel case*: the satellite rotates in the sense of the orbital motion, and the lower sign to the *antiparallel case*: the senses of the rotations are opposite.

These three independent equations are sufficient for solving the three unknowns Ω, R, and ω. As such, the equations do not give a clear picture of the change. The total angular momentum depends on three variables which all can vary when the orbit changes. Analysis of equations is best started by solving the orbital angular momentum as a function of R from the Kepler equation:

$$I_{orbit} = 4m\Omega R^2 = 4m\sqrt{\Omega^2 R^4} = 4m\sqrt{KR}.$$

This shows that the orbital angular momentum increases whenever R increases. Also, this gives for the total angular momentum the equation

$$I_{tot} = 4m(\omega r^2 \pm \sqrt{(KR)}) = \text{Const} .$$

Because r and K are constants, this equation defines a connection between ω and R.

- Parallel case: if the satellite rotates faster, i.e. ω increases, then R must decrease in order that I_{tot} be conserved. And if ω decreases, R must increase.

- Antiparallel case: if the satellite rotates faster, i.e. ω increases, then R must increase in order that I_{tot} be conserved. Similarly, if ω decreases in the antiparallel case, then R also must decrease.

Intuitively one would expect that the increase of the total energy of the satellite (because of the positive work done by the machines) would lead to increase of ω, then the whole problem would be fully analyzed. However, it is necessary to analyze the energy equations in order to make certain that this really is true. The orbital energy as a function of R is

$$E_{orbit}/4m = -K/R + 1/2\ \Omega^2 R^2 = -1/2\ K/R\ ,$$

and the total energy:

$$E_{total}/2m = -K/R + \omega^2 r^2\ .$$

As shown above, in the antiparallel case the conservation of angular momentum requires that ω and R either both increase or both decrease. The first alternative is valid because then both terms of the total energy expression increase which correctly corresponds to the increase of total energy.
The parallel case requires a more detailed analysis. We form the differential change of I_{tot}:

$$d\omega\ r^2 + 1/2\ K\ (KR)^{-1/2}\ dR\ =\ 0\ .$$

This is substituted in the expression of total energy,

$$\begin{aligned}
dE_{total}/2m &= d(-K/R + \omega^2 r^2) && (35)\\
&= K/R^2\ dR + 2\omega\ d\omega\ r^2 && (36)\\
&= K/R^2\ dR - 2\omega\ 1/2\ K\ (KR)^{-1/2}\ dR && (37)\\
&= K\ dR\ (1/R^2 - \omega\ (KR)^{-1/2}) && (38)\\
&= K\ dR\ (1/R^2 - \omega/(\Omega\ R^2)) && (39)\\
&= K\ dR\ R^{-2}(1 - \omega/\Omega) && (40)
\end{aligned}$$

Because $\omega > \Omega$, an increase of total energy corresponds to a decrease of R and further to an increase of ω. This confirms that ω is increasing in both cases, as intuitively expected.

Answers to the tabulated questions.

The radius R decreases in the parallel case and increases in the antiparallel case.
The change of the orbital velocity is opposite to the change of R: increase in the parallel and decrease in the antiparallel case.
The angular velocity ω increases.
The potential energy increases with increasing R, thus increasing in the antiparallel and decreasing in the parallel case.
As seen from earlier answers, it is possible that the satellite gets in a higher orbit.

SOLUTION : PROBLEM $\boxed{1}$

It happens in the antiparallel case.

The last question was not quite clear. It was hoped that this question might bring forward the contrast with ordinary rocket propulsion: it is possible for a rocket to practically leave the gravitational field of earth by using a *finite* amount of energy. However, a rotating satellite would need an infinite amount of energy if R grows without limit: The equation for I_{tot} shows that ω must increase without limit, proportional to the square root of the radius of orbit. Tensional forces in the satellite would then also increase without limit, proportional to the radius R. Thus there would be a maximum value for R, corresponding to the strength of the radial wires. With larger values of R, the wires would break. Rather few participants were able to analyze this aspect of the problem.
In a few answers, the last question was seen in a different perspective. When the Kepler equation is taken into account, the work per one revolution can be written as

$$\Delta E = 2\ dr\ (F_{max} - F_{min}) = 6\ m\ dr\ r\ \Omega^2 = 6K\ m\ dr\ r\ R^{-3}$$

showing that the mean power decreases proportionally to $R^{-2.5}$. Thus the increase of R gets slower and slower when time goes on. Strictly speaking, this alone would not prevent R from reaching any predetermined value, given enough time.

Grading

The credit points for this problem were split to two parts of five points each:

- Correct results for the forces 'up' and 'down' were given one and half points. Another 1.5 points were given for the correct force in the 'sideways' position. Small numerical errors were forgiven. If there was an essential error in the equations, then no credit was given for such a result.

- Two points were given if the mean power was correctly obtained as based on the results of the first part. This merit was given even if the forces were wrong. This part of the problem was very easy. (In fact, it was difficult enough because of the need to use the relative angular velocity, but this was only recognized after the competition.)

- The second half of points were given for the analysis of the changes of orbit. One point was given for the answer which correctly related changes in the orbital velocity and radius R, although it did not help in understanding the mechanism of orbit change. Half a point was given for each one of the conservation equations of energy and angular momentum even if there was no further analysis of the situation. One point was given if the conservation equations were correctly analyzed for one rotational sense, and another point if the other rotational sense was also covered. No credit was given for a few correct stray answers in the table if they did not reflect an understanding of the situation.

PROBLEM $\boxed{1}$: A ROTATING SATELLITE.

- One point was given for the last question of the table, concerning the ability of the satellite to leave the gravitational field of earth.

Remarks

1. Coriolis force? There was a difficulty in this problem which luckily was not affecting the competitors. When the problem was scrutinized, several people thought that it would be necessary to include the Coriolis term in the solution of the problem. Of course, it would be possible to use a rotating coordinate system. Either, one could use a system which rotates with Ω, so that one coordinate axis points 'down', towards centre of earth. Or, one could use a system which rotates with the satellite, with angular velocity ω, so that the bodies B of the satellite would be in fixed positions in this coordinate system. But both cases generate unnecessary complications without any useful simplifications. There are no relative movements which would need to be defined with respect to a rotating frame of reference.

> The only thing that is relative to another coordinate system is the relative angular velocity of the satellite with respect to the vertical axis (needed for the computation of the mean power). It is obtained simply as a sum or difference of the angular velocities ω and Ω.

Thus it is better to work in one inertial coordinate system. Only a few of the competitors used or attempted to use Coriolis formalism!

2. The reason for varying forces. The vector presentation which we used for the solution does not explain the 'reason' for the variation of tension: The extra tension in up and down positions is caused by the variation of the gravitational force as a function of radius: higher up, the pull of earth is less, thus more tension is needed for keeping the body in orbit. And deeper down, the gravitational pull is stronger, needing more tension for supporting the body. The smaller tension in the side positions is explained by the direction of the gravitational pull: there is an angle between the pull directions at the centre of the satellite and at the body B. The whole phenomenon is well known as *the tide*: the gravitational forces of the sun and the moon create a change of apparent gravity on earth which is exactly the same phenomenon as the varying tensional forces of our rotational satellite.

3. Consistency check. The resultant of the calculated four forces acting on the four bodies B is zero (the opposing forces are of same magnitude but point in opposite directions). This is correct, it is consistent with the assumption that the central structures (wires and centre point) of the satellite are massless.
If the analysis were carried out to second order, then the resultant would not be zero, which would indicate a contradiction. This means that the original assumptions (centre point of satellite on circular Kepler orbit, constant ω and Ω) would need to be revised in second order calculations. It would turn out that the centre point oscillates around the circular orbit with a frequency $2(\omega \pm \Omega)/\pi$.

4. Difficulty of the problem. From the outset it was estimated that this is a difficult problem. However, the problem turned out to be even more difficult than we estimated. Only about 10 % of participants were able to analyze the change of orbit. One detail of the solution fooled both the participants, the team leaders, the grading team, and the author of the problem: we all calculated the mean power on the basis of the absolute angular velocity ω of the satellite. Only during the writing of this final report it was recognized that the relative angular velocity $\omega \pm \Omega$ must be used when calculating the mean power. The natural meaning of 'mean power' of a periodic process is the work done during one period divided by the length of that period. In one period there are the four positions of any single body B, thus the length of the period must be the time of one relative revolution of the satellite (relative with respect to the local vertical direction). Question 2 says ambiguously: 'averaged over one rotation of the satellite', but the only sensible interpretation of this is 'averaged over one rotation with respect to the local vertical'!

5. Usual mistakes in the solutions. In many solutions the decrease of tension in the side position was not recognized at all, it was assumed that the tension in side position is $\omega^2 rm$. (The author of the problem first made this error, too. Only two days before the competition he got this part of the solution right.) If the vector formalism is used, then this decrease appears automatically. It can also be obtained by means of a geometrical diagram where the difference of the 'vertical' directions at P and at B is taken into account.

In a surprising number of solutions the tensions in up and down positions were wrong because of the following mistake: it was asssumed that the body B was performing one circular motion with angular velocity ω and radius r and the second circular motion with Ω and $R + r$ (when considering the up position). This results in an excessive tension for the up position. Often there was also another error which caused the tension in the down position to be too small. Such a situation is not consistent, but the competitors did not make a consistency check.

In many solutions it was erroneously assumed that the angular momentum $4m\omega r^2$ of the rotation of the satellite about its centre point P would be conserved. If this were true, then also ω would not be changed by the work done by the machines. The gravitational forces acting on B are not directed towards the centre of the satellite, they are not *central forces* with respect to the centre of the satellite. Thus there is no reason for assuming that the angular momentum or ω would remain constant.

It seems that a few competitors remembered the classical example of conservation of angular momentum: a skater accelerates his/her pirouette by pulling arms close to the body. It was thought that the work done by the machines goes for increasing the angular velocity of the satellite. In small scale, this would seem to be true: pulling one body B closer to P would indeed speed up ω. The effect would not be cumulative, however: later the same B would recede back to the original distance and there would be a slowing down of ω back to the original value. Without the inhomogeneous gravitational field, there would be no cumulative change of ω. Furthermore, the problem was by purpose formulated so that while two bodies get closer to P, another

two recede from P. Thus the moment of inertia of the satellite does not change and there is no fluctuation of the value of ω.
In a few solutions the numerical values for maximum and minimum forces were rounded to two significant figures before calculating the difference. But then the error in the difference of two nearly equal forces may be nearly 100 %!. It is essential to maintain full accuracy in the intermediate results.

6. Experience with the fill-in table The fill-in table was introduced in the hope of achieving the following:

- Eliminating unnecessary explanations from the answers, thus making it possible to grade the answers with a minimum of language translations. It was thought that by asking sufficiently many details one can get a complete picture of whether the competitor does or doesn't understand the situation.
- Making the grading process fast, objective, and straightforward, treating all the participants justly and on equal basis.

These goals were only partly fulfilled:
It was possible to see if a competitor had a good understanding of the situation. Then all or almost all entries of the table were well answered. However, it was somewhat problematic how to deal with partly filled tables. Many answers contained such relations which are trivially true for all circular Kepler orbits. This had not been expected. It was necessary to formulate a policy about how to deal with true answers which did not address the intended matters.
The last question was unfortunately formulated so that it could be understood in two different ways. Also, this question was not supported by other related questions. Thus it was difficult to decide how to grade half-correct answers to the last question.
Our experience indicates that a fill-in table may be a good device in making the grading process easier and more objective. However, it requires a good deal of careful planning and also test filling by a number of persons in order to eliminate multiple meanings of the questions.

PROBLEM 2 : THE LONGITUDINAL MOTION OF A LINEAR MOLECULE

In this problem you will analyze the longitudinal motion of a linear molecule, i.e., the motion along the molecular axis. The rotational motion and the bending of the molecule are not considered. The molecule is assumed to consist of N atoms of mass m_1, m_2, .., m_N, respectively. Each atom is assumed to be connected to its neighbors by a chemical bond. Each bond is approximated by a massless spring which obeys Hooke's law with spring constants k_1, k_2, ... k_{N-1}. The molecule is shown in Fig. 1.

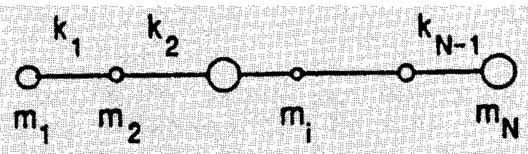


Fig.1. A linear molecule with N atoms.

Use the following facts when solving this problem: The longitudinal vibrational motion of a linear molecule consists of a superposition of separate vibrational motions called normal vibrations, or normal modes. In a normal mode all atoms vibrate in simple harmonic motion with the same frequency and pass through their equilibrium positions simultaneously.

Questions

1) Let x_i be the displacement of atom i from its equilibrium position. Express the force F_i acting on each atom i as a function of the displacements x_1, x_2 ,..., x_N and the spring constants k_1, k_2, ..., k_{N-1}. What relationship is there among the forces F_1, F_2, ..., F_N? Using this relationship, derive a relationship between the displacements x_1, x_2, ..., x_N and give a physical interpretation of this relationship.

2) Analyze the motion of a diatomic molecule AB (Fig. 2). The value of the spring constant is k. Derive an expression for the forces acting on atoms A and B. Determine the possible types of motion of the molecule. Determine the corresponding vibrational frequencies and interpret the result. In particular, how is it possible for the atoms to vibrate with the same frequency even though their masses are not the same?

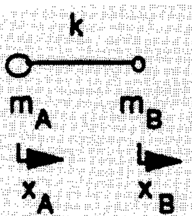


Fig. 2. The diatomic molecule AB

3) Analyze the motion of the triatomic molecule BA_2 (Fig. 3)

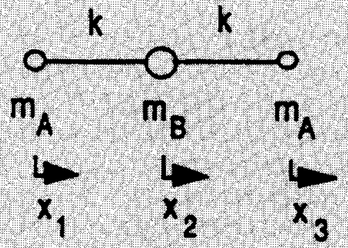


Fig. 3. The triatomic molecule BA_2.

Express the net force on each atom as a function of its displacement only. Deduce the possible motions of the molecule and the corresponding vibrational frequencies.

4) The frequencies of the two longitudinal modes of vibration of the CO_2 molecule are 3.998×10^{13} Hz and 7.042×10^{13} Hz, respectively. Determine a numerical value for the spring constant of the CO bond.

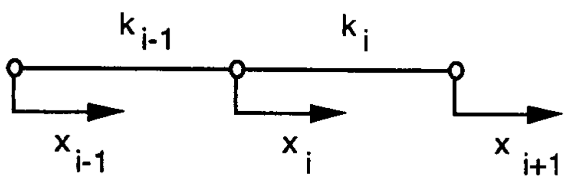
How well do you think this approximation for the bond structure of the molecule describes the vibrational motion of the real molecule?

The atomic mass of the carbon atom = 12 amu and that of the oxygen atom = 16 amu. The atomic mass unit = 1.660×10^{-27} kg.

SOLUTION : PROBLEM 2

The solution is given as several basically equivalent versions. The problem is formulated in such a way that no knowledge of matrix theory as applied to problems of this kind is assumed. However, as many of the participants produced elegant and balanced solutions using matrix theory, a brief sketch of this kind of solution is also presented below.

1) The force on atom i can be deduced from Fig. 1. below.



SOLUTION : PROBLEM $\boxed{2}$

A positive displacement x_{i-1} of atom i-1 causes a shortening of the spring k_{i-1}. That causes a force $k_{i-1}x_{i-1}$ (acting to the right) on atom i. Correspondingly, a displacement x_i of atom i causes a force $-k_{i-1}x_{i-1} - k_i x_i$ acting to the left on atom i. Finally, a displacement x_{i+1} on atom i causes a force $k_{i+1}x_{i+1}$ acting to the right on atom i. The forces on atom i add up to

$$F_i = -k_{i-1}(x_i - x_{i-1}) - k_i(x_i - x_{i+1}) \tag{1}$$

Taking into account that atom 1 has no left neighbor and atom N no right neighbor, the forces can be written

$$\begin{aligned}
F_1 &= -k_1(x_1 - x_2) \\
F_2 &= -k_1(x_2 - x_1) - k_2(x_2 - x_3) \\
&\dots\dots \\
F_i &= \qquad -k_{i-1}(x_i - x_{i-1}) - k_i(x_i - x_{i+1}) \\
&\dots\dots \\
F_N &= \qquad\qquad -k_{N-1}(x_N - x_{N-1})
\end{aligned} \tag{2}$$

Adding up the forces gives the total force F acting on the molecule:

$$F = F_1 + F_2 + \ldots + F_N = 0 \tag{3}$$

According to Newton's second law, this force equals the mass of the molecule multiplied by the acceleration of its center of mass:

$$F = Ma = 0 \tag{4}$$

Each separate force equals the mass of the corresponding atom multiplied by the acceleration of that atom:

$$F_i = M_i a_i \tag{5}$$

(3) and (5) together give

$$m_1a_1 + m_2a_2 + \ldots. + m_Na_N = 0 \tag{6}$$

Relation (6) gives

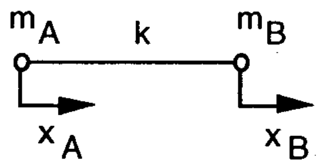
$$m_1v_1 + m_2v_2 + \ldots + m_Nv_N = Mv_0 = \text{constant} \tag{7}$$

where v_0 denotes the velocity of the center of mass. If the molecule is observed in a coordinate system moving with the center of mass, this velocity equals zero. Thus, we find the following relation between the displacements of the separate atoms:

$$m_1x_1 + m_2x_2 + ... + m_Nx_N = Mx_0 = \text{constant} \tag{8}$$

This constant can be set equal to zero, meaning that the origin coincides with the center of mass of the molecule and that the motion of the center of mass is not influenced upon by the internal forces of the molecule .

2) The molecule and the pertinent quantities are shown in the figure below:



The forces on the atoms can be expressed as

$$\begin{aligned} F_A &= -k(x_A - x_B) = m_A a_A \\ F_B &= -k(x_B - x_A) = m_B a_B \end{aligned} \tag{9}$$

Again,

$$F_A + F_B = m_A + m_B = 0 \tag{10}$$

In the center - of - mass system there correspondingly holds

$$m_Ax_A + m_Bx_B = 0 \tag{11}$$

and further

$$x_B = -\frac{m_A}{m_B}x_A \tag{12}$$

SOLUTION : PROBLEM 2

Relations (9) can then be written

$$F_A = -k(x_A + \frac{m_A}{m_B} x_A) = -k(\frac{m_A + m_B}{m_A})x_A$$

$$F_B = -k(x_B + \frac{m_B}{m_A} x_B) = -k(\frac{m_A + m_B}{m_B})x_B \quad (13)$$

According to the formulation of the problem, the force on each atom is proportional to its displacement. This can be expressed as

$$F_A = -r_A x_A$$

$$F_B = -r_B x_B \quad (14)$$

The proportionality constants r_A and r_B are obtained by comparing (13) and (14):

$$r_A = k(\frac{m_A + m_B}{m_B}); \ r_B = k(\frac{m_A + m_B}{m_A}) \quad (15)$$

The crucial point in the solution is now to utilize the fact given in the formulation of the problem that the atoms vibrate with equal frequencies:

$$\omega_A = \sqrt{\frac{r_A}{m_A}} = \sqrt{\frac{m_A + m_B}{m_A m_B}} = \omega_B \quad (16)$$

The other solution to be deduced from Eqns. (9) and (11) is the trivial one corresponding to

$$x_A = x_B \quad (17)$$

giving w = 0, which corresponds to a uniform translation of the molecule without vibrational motion, or in the center-of-mass system, to a molecule at rest.
Another possible solution is obtained by assuming that x_A and x_B are proportional to each other, as can be inferred from the solution to Part 1 of the problem. Thus, we set

$$x_B = cx_A \quad (18)$$

Inserting (18) into (13) gives

$$F_A = -k(x_A - cx_A) = -k(1 - c_B)x_A = -r_A x_A$$

$$F_B = -k(\frac{1}{C} x_B - x_B) = -k(\frac{1}{c_B} - 1)x_B = -r_B x_B \quad (19)$$

PROBLEM $\boxed{2}$: THE LONGITUDINAL MOTION OF A LINEAR MOLECULE

The vibrational angular frequencies are

$$\omega_A = \sqrt{\frac{r_A}{m_A}} = \sqrt{\frac{k(1-c)}{m_A}} = \omega_B = \sqrt{\frac{r_B}{m_B}} = \sqrt{\frac{k(\frac{1}{c}-1)}{m_B}} \tag{20}$$

Solving the resulting second-degree equation for c gives the earlier derived results

$$c_1 = 1,\ c_2 = -\frac{m_A}{m_B} \tag{21}$$

The solution $c_1 = 1$ directly gives $F_A = F_B = 0$ without any further conditions on x_A and x_B.
The solution $c_2 = -m_A/m_B$ corresponds to the genuine vibrational motion.
A third way of obtaining the solution is, of course, to use the full equations of motion

$$\begin{aligned} F_A &= m_A\ddot{x}_A = -k(x_A - x_B) \\ F_B &= m_B\ddot{x}_B = -k(x_B - x_A) \end{aligned} \tag{22}$$

and assuming harmonic solutions of the form

$$x_A = x_{A0}e^{i\omega t};\ x_B = x_{B0}e^{i\omega t} \tag{23}$$

(23) inserted in (22) leads to the linear system of equations

$$\begin{aligned} (k - m_A\omega^2)x_{A0} - kx_{B0} &= 0 \\ -kx_{A0} + (k - m_B\omega^2)x_{B0} &= 0 \end{aligned} \tag{24}$$

Surprisingly many of the participants obtained the solution in this way, correctly utilizing the fact that the condition for a non-trivial solution is that the determinant of the coefficients of the unknowns equal zero:

$$\begin{vmatrix} k - m_A\omega^2 & -k \\ -k & k - m_B\omega^2 \end{vmatrix} = 0 \tag{25}$$

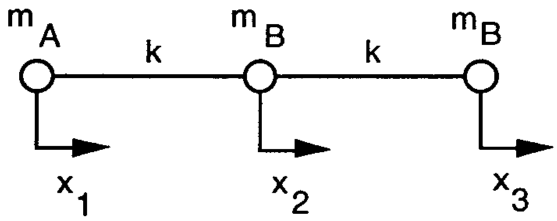
SOLUTION : PROBLEM 2

The solution to this equation again retrieves the earlier results:

$$\omega_1 = 0; \ \omega_2 = \sqrt{\frac{k(m_A + m_B)}{m_A m_B}} \tag{26}$$

with the amplitudes x_A and x_B obtained as before.

3) The molecule to be analyzed in the third part of the problem is illustrated in the following figure together with the pertinent quantities defined:



The forces on the atoms are

$$\begin{aligned} F_1 &= -k(x_1 - x_2) \\ F_2 &= -k(x_2 - x_1) - k(x_2 - x_3) = -k(-x_1 + 2x_2 - x_3) \\ F_3 &= -k(x_3 - x_2) \end{aligned} \tag{27}$$

Again we the displacements can be assumed proportional to each other, as the sum of the mass-weighted displacements is a constant:

$$x_2 = c_2 x_1; \ x_3 = c_3 x_1 \tag{28}$$

where c_2 and c_3 are constants to be determined. According to the formulation of the problem, the participants were supposed to proceed by trying to express the force acting on each atom as a function of the displacement of that particular atom only. Inserting (28) in (27) then gives

$$\begin{aligned} F_1 &= -k(1 - c_2)x_1 = -r_1 x_1 \\ F_2 &= -k(-\frac{1}{c_2} + 2 - c_3)x_2 = -r_2 x_2 \\ F_3 &= -k(1 - \frac{c_2}{c_3})x_3 = -r_3 x_3 \end{aligned} \tag{29}$$

PROBLEM 2 : THE LONGITUDINAL MOTION OF A LINEAR MOLECULE

The constants c_2 and c_3 can now be determined from the condition that the atoms vibrate with equal angular frequencies:

$$\omega_1 = \sqrt{\frac{r_1}{m_A}} = \omega_2 = \sqrt{\frac{r_2}{m_B}} = \omega_3 = \sqrt{\frac{r_3}{m_A}} \tag{30}$$

Squaring the roots and using (29) gives the equations

$$\frac{1-c_2}{m_A} = \frac{-\frac{1}{c_2}+2-\frac{c_3}{c_2}}{m_B} = \frac{1-\frac{c_2}{c_3}}{m_A} \tag{31}$$

These equations must hold simultaneously, so that there hold the relations

$$1-c_2 = 1-\frac{c_2}{c_3} \tag{32}$$

$$\frac{1-c_2}{m_A} = (2-(1+c_3)\frac{1}{c_2})\frac{1}{m_B} \tag{33}$$

The first of these equations has two different solutions:

$$\begin{aligned} &1)\ c_2 = 0\ \&\ c_3 \neq 0 \\ &2)\ c_3 = 1\ \&\ c_2 \neq 0 \end{aligned} \tag{34}$$

The first solution inserted in (33) gives the result

$$\frac{1}{m_A} = (2-\frac{1+c_3}{c_2})\frac{1}{m_B} \tag{35}$$

If c_2 is directly set = 0 in the right-hand member, the expression diverges. For that not to occur, the expression $1+c_3$ must vanish, impying the result

$$c_3 = -1 \tag{36}$$

Thus, we have

$$x_2 = 0,\ x_3 = -x_1 \tag{37}$$

SOLUTION : PROBLEM 2

From (29) and (37) we obtain

$$r_1 = k;\ \omega_1 = \sqrt{\frac{k}{m_A}} \tag{38}$$

The angular frequency w_3 is equal to w_1, because the solution actually was obtained on that condition. An additional complication is that the frequency w_2 comes out indeterminate, as atom 2 does not move at all in this particular vibrational mode. The participants were not supposed to analyze that fact any further; obtaining the result that the central atom does not move was enough.
The second solution in (34), i.e, c_3=1 and $c_2 \neq 0$ gives inserted in (33)

$$\frac{1-c_2}{m_A} = 2(1-\frac{1}{c_2})\frac{1}{m_B} \tag{39}$$

This gives a second-degree equation for c_2:

$$c_2^{\ 2} + (\frac{2m_A}{m_B} - 1)c_2 - \frac{2m_A}{m_B} = 0 \tag{40}$$

The roots of this equation are

$$c_{2,1} = 1;\ c_{2,2} = -\frac{2m_A}{m_B} \tag{41}$$

The first solution corresponds to equal amplitudes for all atoms, again implying that no bonds are stretched and no vibrational motion occurs. The second root gives

$$F_1 = -k(1+\frac{2m_A}{m_B}) = -r_1x_1 \tag{41}$$

with the corresponding vibrational angular frequency

$$\omega_1 = \sqrt{\frac{r_1}{m_A}} = \sqrt{k(\frac{2}{m_B}+\frac{1}{m_A})} \tag{42}$$

PROBLEM $\boxed{2}$: THE LONGITUDINAL MOTION OF A LINEAR MOLECULE

As in part 2 of this problem, the solution can also be obtained from the vanishing of the determinant formed from the equations of motion. They are

$$
\begin{aligned}
F_1 &= m_A\ddot{x}_1 = -k(x_1 - x_2) \\
F_2 &= m_B\ddot{x}_2 = -k(-x_1 + 2x_2 - x_3) \\
F_3 &= m_A\ddot{x}_3 = -k(x_3 - x_2)
\end{aligned}
\tag{43}
$$

Again assuming an complex exponential solution

$$x_i = x_{i0}e^{i\omega t} \tag{44}$$

a linear system of equations is obtained by factoring out the exponential:

$$
\begin{aligned}
&(k - m_A\omega^2)x_{10} - kx_{20} = 0 \\
&-kx_{20} + (2k - m_B\omega^2)x_{20} - kx_{30} = 0 \\
&kx_{20} + (k - m_A\omega^2)x_{30} = 0
\end{aligned}
\tag{45}
$$

The condition for the existence of a non-vanishing solution is again

$$
\begin{vmatrix} k - m_A\omega^2 & -k & 0 \\ -k & 2k - m_B\omega^2 & -k \\ 0 & -k & k - m_A\omega^2 \end{vmatrix} = 0 \tag{46}
$$

The roots for the determinant are obtained as

$$\omega_1 = 0;\ \omega_2 = \sqrt{\frac{k}{m_A}};\ \omega_3 = \sqrt{k(\frac{2}{m_B} + \frac{1}{m_A})} \tag{47}$$

thus reproducing the earlier results. The amplitudes are trivially solved by inserting the roots in the equation system one at a time. This method of solution is, of course, much faster than the one suggested in the text, but it was not assumed that the participants would have to master the more advanced techniques. On the other hand, those who did it were rewarded for a correct solution, even though they took a shorter route demanding less physical reasoning than that suggested in the formulation of the problem.

SOLUTION : PROBLEM 2

4) Within the realm of the model adopted, we note that $w_3 > w_2$, so that the higher vibrational frequency, i.e $7.042*10^{13}$ Hz, should be set to correspond to w_3 and the lower one, $3.998*10^{13}$ Hz, should be set to correspond to w_2. First the correspondence between the angular frequency and the frequency is noted:

$$\omega = 2\pi\nu \tag{48}$$

Thus, there holds

$$\omega_2 = 2\pi\nu_2;\ \omega_3 = 2\pi\nu_3 \tag{49}$$

The estimates for k come out as

$$\begin{aligned} k_2 &\approx m_A\omega_2{}^2 \approx 1670\ \mathrm{N/m} \\ k_3 &\approx \left(\frac{m_A mB}{2m_A + m_B}\right)\omega_3{}^2 \approx 1420\ \mathrm{N/m} \end{aligned} \tag{50}$$

The agreement is reasonable. The participants were not expected to produce any further speculations as to the reasons for the discrepancy. This part of the problem was rather meant as an illustration of the degree of accuracy inherent in a simple model of the kind presented here.

PROBLEM 3 : A SATELLITE IN SUNSHINE

In this problem you will calculate the temperature of a space satellite. The satellite is assumed to be a sphere with a diameter of 1 m. All of the satellite remains at a uniform temperature. All of the spherical surface of the satellite is coated with the same kind of coating. The satellite is located near the earth but is not in the earth's shadow.

The surface temperature of the sun (its blackbody temperature) $T_{sun} = 6000$ K and its radius is 6.96×10^8 m. The distance between the sun and the earth is 1.5×10^{11} m. The sunlight heats the satellite to a temperature at which the blackbody emission from the satellite equals the power absorbed from the sunlight. The power per unit area emitted by a black body is given by Stefan-Boltzmann's law $P = \sigma T^4$ where σ is the universal constant 5.67×10^{-8} $W \cdot m^{-2} K^{-4}$. In the first approximation, you can assume that both the sun and the satellite absorb all electromagnetic radiation incident upon them.

1) Find an expression for the temperature T of the satellite. What is the numerical value of this temperature?

2) The blackbody radiation spectrum $u(T,f)$ of a body at temperature T obeys Planck's radiation law

$$u(T,f)\,df = \frac{8\pi k^4 T^4}{c^3 h^3} \frac{\eta^3\,d\eta}{e^\eta - 1}$$

where $\eta = hf/kT$ and $u(T,f)df$ is the energy density of the electromagnetic radiation in a frequency interval $[f, f + df]$. In the equation $h = 6.6 \times 10^{-34}$ $J \cdot s$ is Planck's constant, $k = 1.4 \times 10^{-23}$ $J \cdot K^{-1}$ is Boltzmann's constant, and $c = 3.0 \times 10^8$ $m \cdot s^{-1}$ is the speed of light.

The blackbody spectrum, integrated over all frequencies f and directions of emission, gives the total radiated power per unit area $P = \sigma T^4$ as expressed in the Stefan-Boltzmann law given above.

$$\sigma = \frac{2\pi^5 k^4}{15\,c^2 h^3}$$

The figure shows the normalized spectrum

$$\frac{c^3 h^3}{8\pi k^4} \frac{u(T,f)}{T^4}$$

as a function of η.

In many applications it is necessary to keep the satellite as cool as possible. To cool the satellite, engineers use a reflective coating that reflects light above a cut-off frequency but does not prevent heat radiation at lower frequency from escaping. Assume that this (sharp) cut-off frequency corresponds to $hf/k = 1200$ K.

What is the new equilibrium temperature of the satellite? The exact answer is not needed. Therefore, do not perform any tedious integrations; make approximations where necessary. The integral over the entire range is

$$\int_0^\infty \frac{\eta^3 \, d\eta}{e^\eta - 1} = \frac{\pi^4}{15}$$

and the maximum of $\eta^3/(e^\eta - 1)$ occurs at $\eta \approx 2.82$. For small η you can expand the exponential function as $e^\eta \approx 1 + \eta$.

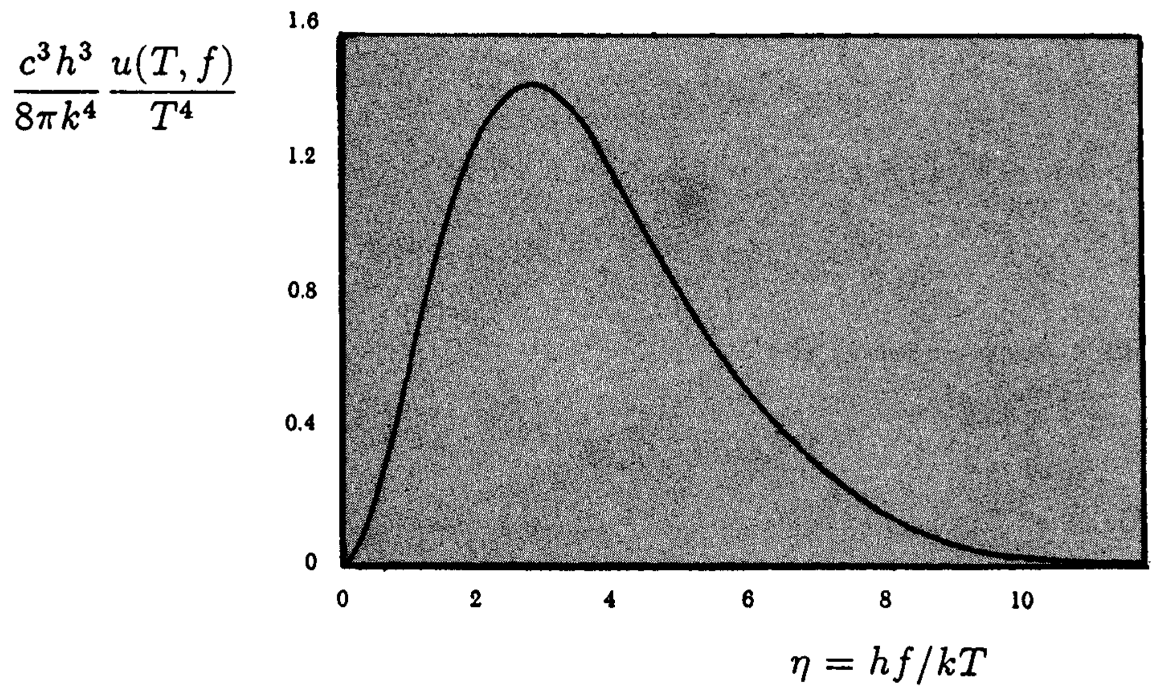
3) If we now have a real satellite, with extending solar panels that generate electricity, the dissipated heat in the electronics inside the satellite acts as an extra source of heat. Assuming that the power of the internal heat source is 1 kW, **what is the equilibrium temperature of the satellite in case 2 above?**

4) A manufacturer advertises a special paint in the following way:

"This paint will reflect more than 90% of all incoming radiation (both visible light and infrared) but it will radiate at all frequencies (visible light and infrared) as a black body, thus removing lots of heat from the satellite. Thus the paint will help keep the satellite as cool as possible."

Can such paint exist? Why or why not?

5) What properties should a coating have in order to raise the temperature of a spherical body similar to that of the satellite considered here above the temperature calculated in 1?



SOLUTION : PROBLEM 3

1. Over the whole surface of the sun, the emitted energy is $4\pi R_{Sun}^2 \cdot \sigma T_{Sun}^4$. All this energy passes through a spherical shell at earth's distance R, where the intensity now is $4\pi R_{Sun}^2 \cdot \sigma T_{Sun}^4 / 4\pi R^2$.
   The satellite is a circular object absorbing

$$\pi r_{sat}^2 \cdot \frac{4\pi R_{Sun}^2 \cdot \sigma T_{Sun}^4}{4\pi R^2}$$

   but a spherical object emitting $4\pi r_{sat}^2 \cdot \sigma T_{sat}^4$.
   Equating this absorption and emission we get $T_{sat} = T_{Sun}\sqrt{\frac{R_{Sun}}{2R}}$ giving $T_{sat} = 289 \text{ K} = 16\ °\text{C}$.

2. We have to calculate what part of the absorbed power comes from the part of the spectrum below 1200 K.

$$\eta_{cutoff} = \frac{1200 \text{ K}}{6000 \text{ K}} = 0.2 \ll 1$$

   This fraction of power is

$$\delta = \int_0^{\eta_{cutoff}} \frac{\eta^3\, d\eta}{e^\eta - 1} \Big/ \int_0^{\eta_\infty} \frac{\eta^3\, d\eta}{e^\eta - 1}$$
$$\approx \int_0^{\eta_{cutoff}} \eta^2\, d\eta \Big/ \frac{\pi^4}{15} = \frac{\eta_{cutoff}^3}{3} \Big/ \frac{\pi^4}{15} = 4.1 \cdot 10^{-4}$$

   Now, the satellite is cold with respect to 1200 K so we ignore that a small part of the satellite blackbody emission will be retained. The energy balance is now

$$4\pi r_{sat}^2 \cdot \sigma T_{sat}^4 = \delta \cdot \pi r_{sat}^2 \cdot \frac{4\pi R_{Sun}^2 \cdot \sigma T_{Sun}^4}{4\pi R^2}$$

   by which the new satellite temperature is the previous corrected by a factor $\delta^{1/4}$

$$T_{sat} = (4.1 \cdot 10^{-4})^{1/4} \cdot 289 \text{ K} = 41 \text{ K}$$

3. The whole absorbed energy is

$$\delta \cdot \pi r_{sat}^2 \cdot \frac{4\pi R_{Sun}^2 \cdot \sigma T_{Sun}^4}{4\pi R^2} = 0.5 \text{ W}$$

   which is small (ignorable) compared to $P_{internal} = 1$ kW. Thus the energy balance becomes

$$P_{internal} = 4\pi r_{sat}^2 \cdot \sigma T_{sat}^4$$

giving $T_{sat} = 274$ K ($\eta = 4.38$). Note: strictly speaking, this is not accurate, because for a blackbody radiation of 274 K, some 33 % of the emitted power lies above the 1200 K cutoff ! This means that the satellite has to be hotter, to emit all of the 1 kW in frequencies below the cutoff. The resulting integral equation is

$$(\frac{\eta}{4.38})^4 = \int_0^{\eta} \frac{\eta^3\, d\eta}{e^{\eta} - 1} / \frac{\pi^4}{15}$$

which can be solved numerically by iteration. The true solution is $\eta = 3.80$ corresponding to a temperature of 316 K.

4. The paint cannot exist, because it would violate the second law of thermodynamics. The physics textbook explanation is the principle of detailed balance, which means that for equilibrium to exist, the emission and absorption in a given frequency interval must match exactly. This should not be confused with the fact that reflection and absorption can be quite different. If the manufacturer's paint existed, one could create a temperature difference between two bodies in a closed system, and hence a perpetum mobile.

5. The coating should be transparent for high frequencies (in the range of the peak or tail of the sun's radiation), but reflective and hence insulating at low frequencies (the satellite temperature).

Theoretical Problem 1

ATMOSPHERIC ELECTRICITY

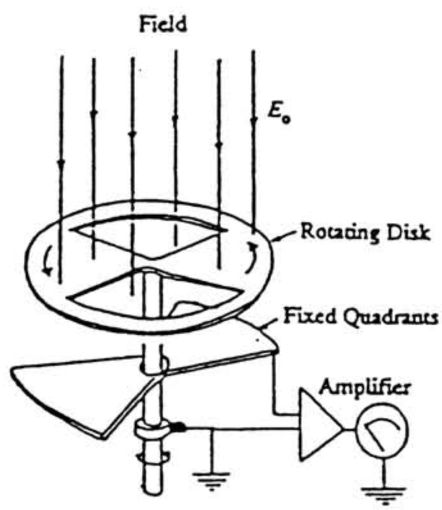
From the standpoint of electrostatics, the surface of the Earth can be considered to be a good conductor. It carries a certain total charge Q_o and an average surface charge density σ_o.

1) Under fair-weather conditions, there is a downward electric field, E_o, at the Earth's surface equal to about 150 V/m. Deduce the magnitude of the Earth's surface charge density and the total charge carried on the Earth's surface.

2) The magnitude of the downward electric field decreases with height, and is about 100 V/m at a height of 100 m. Calculate the average amount of net charge per m^3 of the atmosphere between the Earth's surface and 100 m altitude.

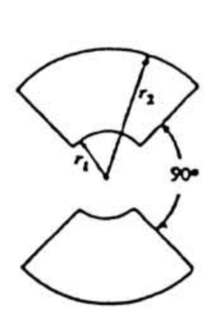
3) The net charge density you have calculated in (2) is actually the result of having almost equal numbers of positive and negative singly-charged ions per unit volume (n_+ and n_-). Near the Earth's surface, under fair-weather conditions, $n_+ \approx n_- \approx 6 \times 10^8$ m^{-3}. These ions move under the action of the vertical electric field. Their speed is proportional to the field strength:

$$v \approx 1.5 \times 10^{-4} \cdot E,$$



where v is in m/s and E in V/m. How long would it take for the motion of the atmospheric ions to neutralize half of the Earth's surface charge, if no other processes (e.g. lightning) occurred to maintain it?

4) One way of measuring the atmospheric electric field, and hence σ_o, is with the system shown in the diagram. A pair of metal quadrants, insulated from ground but connected to each other, are mounted just underneath a grounded uniformly rotating disk with two quadrant-shaped holes cut in it. (In the diagram, the spacing has been exaggerated in order to show the arrangement.) Twice in each revolution the insulated quadrants are completely exposed to the field, and then (1/4 of a period later) are completely shielded from it. Let T be the period of revolution, and let the inner and outer radii of the insulated quadrants be r_1 and r_2 as shown.



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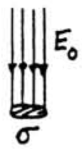
Theoretical Problem 1 -- Solution

1) By Gauss' law, $\sigma = \varepsilon_0 E_0$.

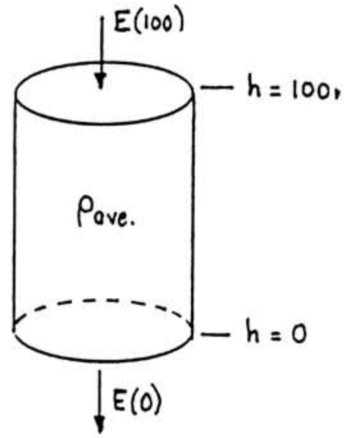
$$\therefore \quad \sigma = -8.85\ 10^{-12} \times 150$$

$$\approx -1.3 \times 10^{-9}\ \mathrm{C/m^2}.$$



$$Q = 4\pi R^2 \sigma = 4\pi \times (6.4 \cdot 10^6)^2 \times 1.3 \cdot 10^{-9} = -6.7 \cdot 10^5\ \mathrm{C}.$$

2) Consider a cylinder of cross-section A with faces at heights of 0 and 100 m.



By Gauss' law, $E(0)A - E(100)A = q_{\text{enclosed}}/\varepsilon_0$

$$= \rho_{\text{ave.}} \times (100A)/\varepsilon_0.$$

$$\therefore \quad \rho_{\text{ave.}} = \frac{\varepsilon_0[E(0) - E(100)]}{100}$$

$$= \frac{8.85\ 10^{-12} \times 50}{100} \approx 4.4 \times 10^{-12}\ \mathrm{C/m^3}.$$

3) If a conductor contains n charges per unit volume, each with charge q and travelling with speed v, the current per unit area (j) is given by:

$$j = nqv.$$

Here, we have both positive and negative charges ($\pm e$). Clearly, with a downward electric field, the positive charges move downward and the negative charges move upward.
In the situation as described, only the positive charges can contribute to neutralization of the Earth's surface charge. Hence we have (taking downward as the positive direction for this purpose):

$$j = n_+ e\, v$$

$$\approx (6 \cdot 10^8) \times (1.6 \cdot 10^{-19}) \times (1.5 \cdot 10^{-4}\, E)$$

$$= 1.44 \times 10^{-14}\, E.$$

Now j is the rate of change ($d\sigma/dt$) of the surface charge density σ, and E (if defined as positive downward) is equal to $-\sigma/\varepsilon_0$. Thus the above equation can be written:

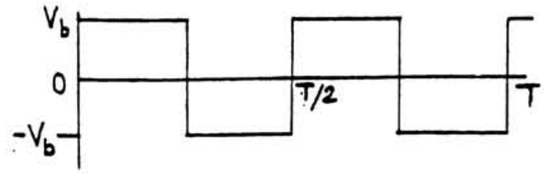
$$\frac{d\sigma}{dt} = -1.44 \cdot 10^{-14} \frac{\sigma}{\varepsilon_0} = -\frac{1.44 \cdot 10^{-14}}{8.85 \cdot 10^{-12}} \sigma = -1.63 \cdot 10^{-3} \sigma \approx -\frac{1}{600} \sigma.$$

This is just like the equation of radioactive decay. Its solution is an exponential decrease of σ with time:

$$\sigma(t) = \sigma_0\, e^{-t/\tau}, \quad \text{with } \tau \approx 600 \text{ sec.}$$

of this current is approximately equal to $|q_{max.}|/(T_b/4)$. The resulting voltage across R is approximately constant during each quarter-period, and is alternately positive and negative. In this case,

$$V_{max.} = V_b \approx \frac{4\, q_{max.} R}{T_b}.$$



Putting these results together, we see that:

$$\frac{V_a}{V_b} \approx \frac{T_b}{4CR}.$$

6) We have $CR = 10^{-8} \times 2\ 10^{7} = 0.2$ s, and $T = 1/50 = 0.02$ s.
Thus $CR = 10 \times T$, which satisfies the criterion $CR \gg T$.
Therefore we can use the solution 5(a) above.
We have $A_{max.} = \frac{\pi}{2}(7^2 - 1^2) = 75\ \text{cm}^2 = 7.5 \times 10^{-3}\ \text{m}^2$.
$E_o = 150$ V/m $\rightarrow \sigma = \varepsilon_o E_o \approx 1.33 \times 10^{-9}$ C/m^2 (as in part 1).
$\therefore\ q_{max.} = 1.33 \cdot 10^{-9} \times 7.5\ 10^{-3} \approx 1.0 \times 10^{-11}$ C,

and so $V_{max.} = \dfrac{q_{max.}}{C} = \dfrac{1.0 \times 10^{-11}}{1.0 \times 10^{-8}} = 10^{-3}\ \text{V} = 1\ \text{mV}.$

<u>Theoretical Problem 1: Grading Scheme</u>

Part 1.	1 point	(1/2 point for σ_o, 1/2 point for Q)
Part 2.	1 point	
Part 3.	2 points	(1/2 point for recognizing $j = nev$; 1/2 point for recognizing $j = d\sigma/dt$; 1/2 point for getting $\sigma(t) = \sigma_o\, e^{-t/\tau}$; 1/2 point for final numerical answer.) [1 point maximum for using $t = \sigma_o/2j_o$.]
Part 4.	1-1/2 points	(1/2 point for each equation; 1/2 point for graph.)
Part 5.	3-1/2 points	(1 point for correct graphical form of (a); 1 point for correct graphical form of (b); 1-1/2 points for correct evaluation of V_a/V_b.)
Part 6.	1 point	(1/2 point for recognizing that $T \ll CR$; 1/2 point for final answer)

2) Express, in terms of I_0, θ, h, w and y_0, the x and y components of the net force exerted on the prism by the laser light when the apex of the prism is displaced a distance y_0 from the x axis where $|y_0| \leq 3h$.

Plot graphs of the values of the horizontal and vertical components of force as functions of vertical displacement y_0.

3) Suppose that the laser beam is 1 mm wide in the z direction and 80 μm thick (in the y direction). The prism has $\alpha = 30°$, $h = 10$ μm, $n = 1.5$, $w = 1$ mm and $\rho = 2.5$ g cm^{-3}. How many watts of laser power would be required to balance this prism against the pull of gravity (in the $-y$ direction) when the apex of the prism is at a distance $y_0 = -h/2$ (= -5 μm) below the axis of the laser beam?

4) Suppose that this experiment is done in the absence of gravity with the same prism and a laser beam with the same dimensions as in (3), but with $I_0 = 10^8$ W m^{-2}.

What would be the period of oscillations that occur when the prism is displaced and released a distance $y = h/20$ from the center line of the laser beam?

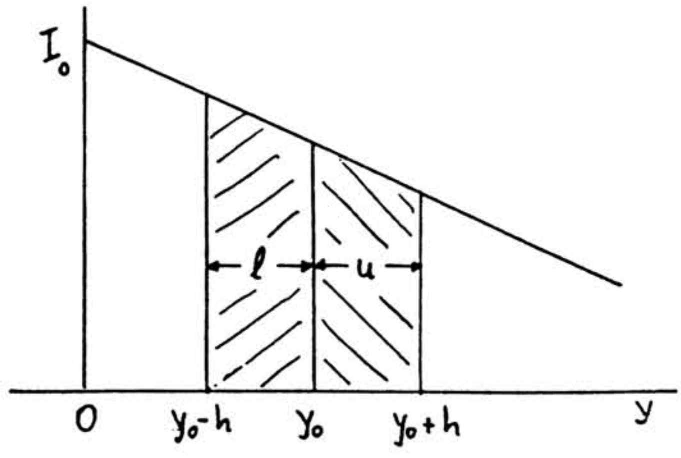


Figure 2: $\bar{I}_u$ and $\bar{I}_\ell$ when $y_0 \geq h$

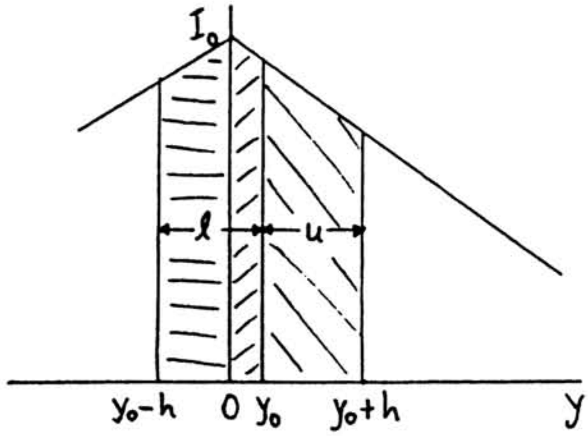


Figure 3: $\bar{I}_u$ and $\bar{I}_\ell$ when $0 < y_0 < h$

(b) When $0 < y_0 < h$, the lower half of the prism is partly in the lower half of the laser beam as shown in Fig. 3. Then the part of the lower half of the prism between 0 and y_0 has a fraction y_0/h of the area of the lower half of the prism and sees an average intensity

$$\bar{I}_{\ell_1} = I(y_0/2) = I_0\left(1 - \frac{y_0}{8h}\right).$$

The part between 0 and $y_0 - h$ has a fraction $1 - y_0/h$ of the area and sees an average intensity of

$$\bar{I}_{\ell_2} = I(\frac{h - y_0}{2}) = I_0\left(\frac{7}{8} + \frac{y_0}{8h}\right).$$

Putting these together we get

$$\begin{aligned} P_\ell &= hw\,\frac{y_0}{h}\bar{I}_{\ell_1} + hw\,(1 - \frac{y_0}{h})\bar{I}_{\ell_2} \\ &= hwI_0\left(\frac{7}{8} + \frac{y_0}{4h} - \frac{y_0^2}{4h^2}\right). \end{aligned}$$

The average intensity on the upper face has the same functional dependence on y_0 as in the first case. Therefore, $P_u = hwI_0\left(\frac{7}{8} - \frac{y_0}{4h}\right)$ as before.

Putting these together gives

$$\begin{aligned} P_u + P_\ell &= hwI_0\left(\frac{7}{4} - \frac{y_0^2}{4h^2}\right) \\ P_u - P_\ell &= -hwI_0\frac{y_0}{2h}\left(1 - \frac{y_0}{2h}\right) \end{aligned}$$

from which it follows that

$$\begin{aligned} F_x &= \frac{hwI_0}{c}\left(\frac{7}{4} - \frac{y_0^2}{4h^2}\right)(1 - \cos\theta) \\ F_y &= -\frac{hwI_0}{c}\frac{y_0}{2h}\left(1 - \frac{y_0}{2h}\right)\sin\theta. \end{aligned}$$

Because the intensity distribution is symmetric about the axis of the laser beam, the solutions for $y_0 < 0$ will mirror the solutions for $y_0 > 0$. Graphs of the F_x and F_y as functions of y_0 are shown in Fig. 4.

3. Both the equation and the graph of F_y show that to have $F_y > 0$ and opposite the force of gravity, y_0 must be < 0. Then to find the force necessary to support the prism against gravity, find the prism's mass, and equate the expression for the vertical component of force from the laser beam to the weight of the prism, and find I_0 for the parameters given. Use that result to find the total power in the laser beam. This

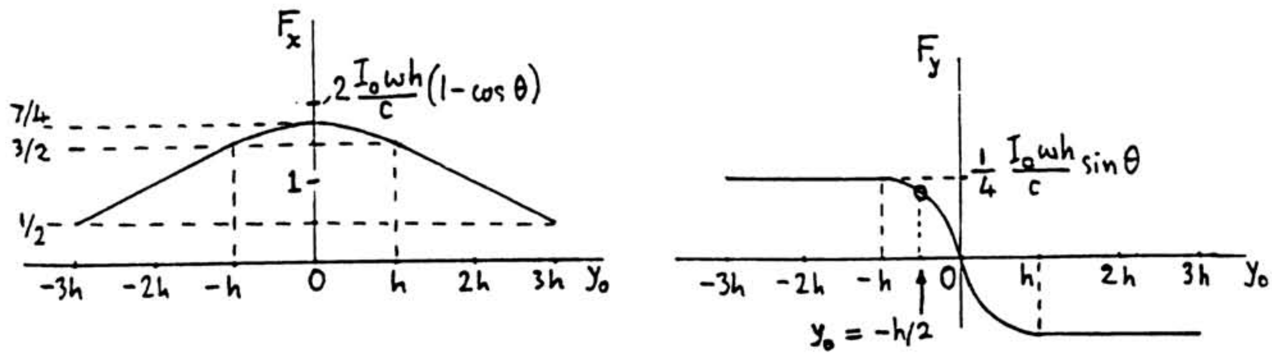


Figure 4: (a) F_x vs y_0; (b) F_y vs y_0

can be done by finding the average value $\bar{I}$ over the specified cross sectional area of the laser beam.

To find the mass of the prism first find its volume $= \tan\alpha\, h^2 w$ then

$$\begin{aligned} m &= \frac{1}{\sqrt{3}} \times (10^{-3})^2 \times .1 \times 2.5 \\ &= 1.44 \times 10^{-7} \text{ g} \\ &= 1.44 \times 10^{-10} \text{ kg}; \\ mg &= 1.42 \times 10^{-9} \text{ N} \end{aligned}$$

The solution to (2) assumed a displacement in the $y > 0$ direction, but the problem is symmetric so we can use that solution. We want the value of I_0 that satisfies

$$\frac{I_0 h w}{c} \frac{y_0}{2h}\left(1 - \frac{y_0}{2h}\right) \sin\theta = mg = 1.42 \times 10^{-9}$$

when

$$\begin{aligned} \theta &= 15.9^\circ \\ y_0 &= \frac{h}{2} \\ h &= 10 \times 10^{-6} \text{ m} \\ w &= 10^{-3} \text{ m} \end{aligned}$$

$$I_0 = \frac{3 \times 10^8 \times 1.42 \times 10^{-9}}{10^{-5} \times 10^{-3} \times .274 \times \frac{3}{16}} = 8.30 \times 10^8 \text{ W/m}^2$$

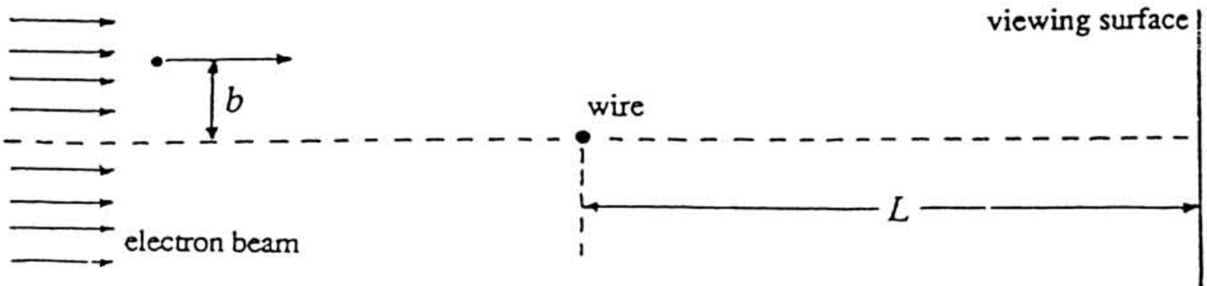
since the power P is given by $P = \bar{I} \times$ area of laser beam where $\bar{I} = \frac{I_0}{2}$. This yields

$$P = \frac{1}{2} \times 8.30 \times 10^8 \times 10^{-3} \times 80 \times 10^{-6} = 33.2 \text{ W}.$$

Theoretical Problem 3

ELECTRON BEAM

An accelerating voltage V_0 produces a uniform, parallel beam of energetic electrons. The electrons pass a thin, long, positively charged copper wire stretched at right angles to the original direction of the beam, as shown in the figure. The symbol b denotes the distance at which an electron would pass the wire if the wire were uncharged. The electrons then proceed to a screen (viewing surface) a distance L ($\gg b$) beyond the wire, as shown. The beam initially extends to distances $\pm b_{max}$ with respect to the axis of the wire. Both the width of the beam and the length of the wire may be considered infinite in the direction perpendicular to the paper.



The charged wire extends perpendicularly to the plane of the paper. The sketch is not to scale.

Some numerical data are provided here; you will find other numerical data in the table at the front of the examination:

radius of wire = r_0 = 10^{-6} m

maximum value of b = b_{max} = 10^{-4} m

electric charge per unit length of wire = q_{linear} = 4.4×10^{-11} C m^{-1}

accelerating voltage = V_0 = 2×10^4 V

length from wire to observing screen = L = 0.3 m.

Note: For parts 2 - 4, make reasonable approximations that lead to analytical and numerical solutions.

1) Calculate the electric field **E** produced by the wire. Sketch the magnitude of **E** as a function of distance from the axis of the wire.

(Continued on next page)

2) Use classical physics to calculate the angular deflection of an electron. Do this for values of the parameter b such that the electron does not strike the wire. Let θ_{final} denote the (small) angle between the initial velocity of the electron and the velocity when the electron reaches the viewing surface. Hence, calculate θ_{final}.

3) Calculate and sketch the pattern of impacts (i.e., the intensity distribution) on the viewing screen that classical physics predicts.

4) Quantum physics predicts a major difference in the intensity distribution (relative to what classical physics predicts). Sketch the pattern for the quantum prediction and provide quantitative detail.

COUNTRY : ____________________

XXIV INTERNATIONAL PHYSICS OLYMPIAD

WILLIAMSBURG, VIRGINIA, U.S.A.

PRACTICAL COMPETITION

Experiment No. 1

July 14, 1993

Time available: 2.5 hours

READ THIS FIRST!

INSTRUCTIONS:

1. Use only the pen provided.
2. Use only the marked side of the paper.
3. Write at the top of each and every page:

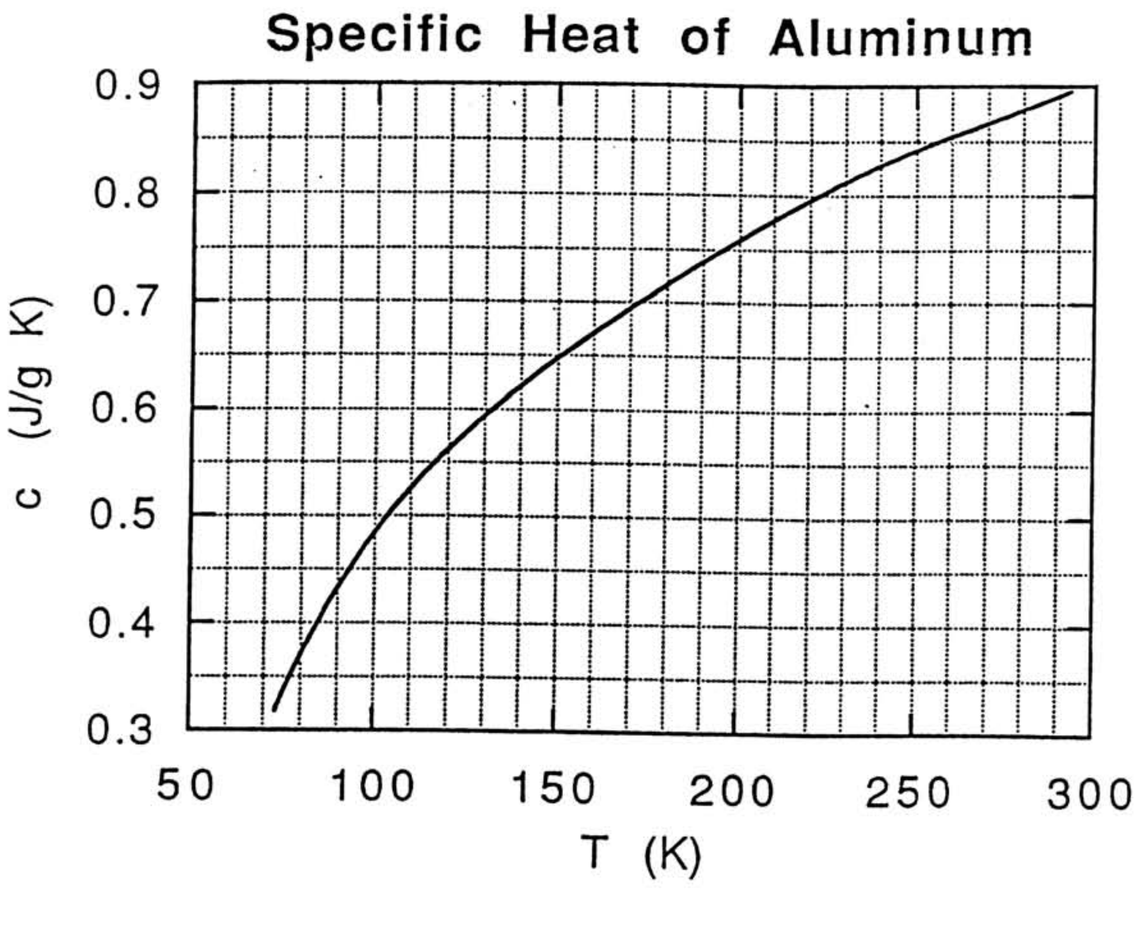
   - The number of the problem
   - The number of the page of your report
   - The total number of pages in your report.

   Example (for Problem 1): 1 1/4; 1 2/4; 1 3/4; 1 4/4.

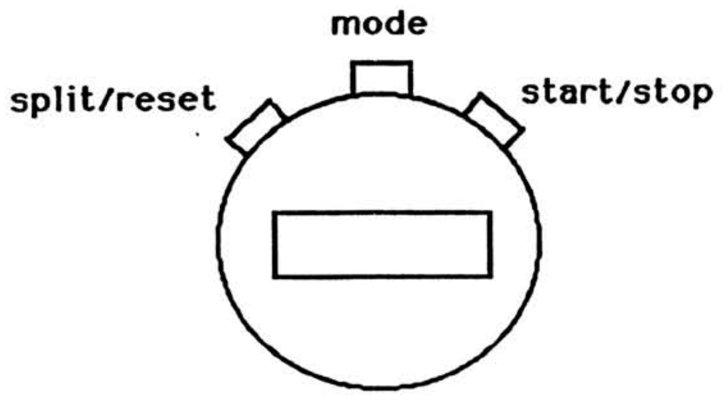
capacitor installed across its terminals. Use this result to determine the heat of vaporization per unit mass of nitrogen. Be sure to provide a quantitative estimate of the accuracy of your result.

Notes:

(1) Please include sketches, schematic diagrams, properly labelled tables, numbers with the proper units, etc. so the graders can determine exactly what you did.
(2) Ask for assistance if any piece of equipment is not working properly.



Digital Stopwatch



To Perform Timing Operations

1. Press "Mode" until 0 00 00 appears
   (You may have to press "Mode" several times to get the 0 00 00 to appear)

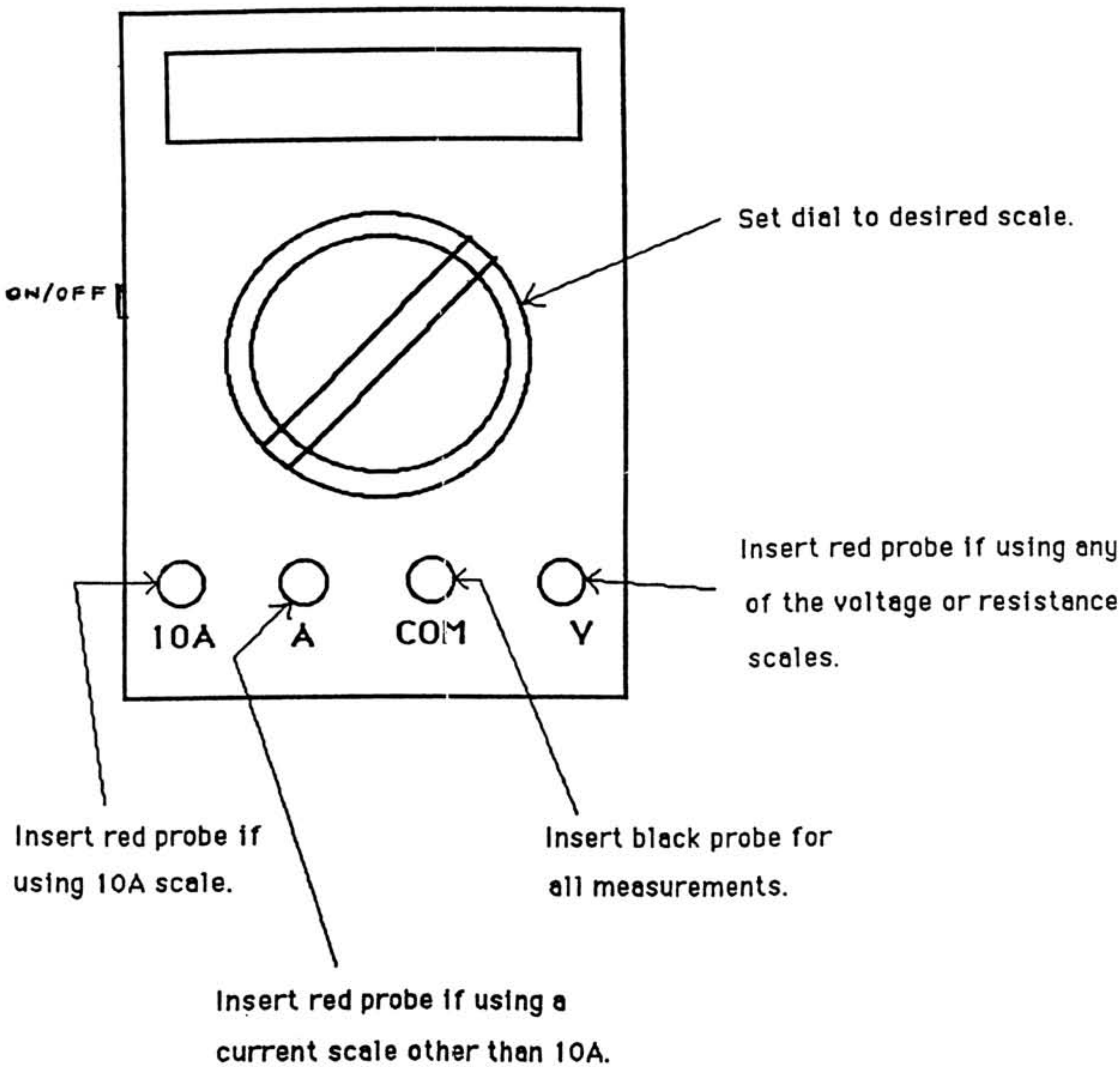
To Time a Single Interval

1. Press "Start/Stop" to start stopwatch.
2. Press "Start/Stop" to stop stopwatch.
3. Press "Split/Reset" to reset stopwatch to zero.

To Time Multiple Events Without Stopping the Stopwatch

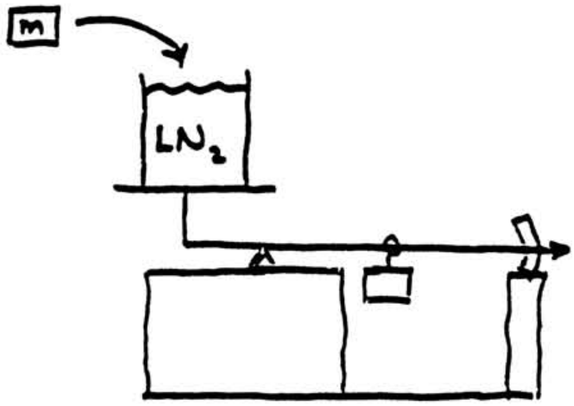
1. Press "Start/Stop" to start stopwatch.
2. Press "Split/Reset" to stop the display while stopwatch keeps running.
3. Press "Split/Reset" to reset display to actual time.
4. Press "Start/Stop" to stop stopwatch after last event.
5. Press "Split/Reset" to reset stopwatch to zero.

Multimeter


Set dial to desired scale.
ON/OFF
10A
A
COM
V
Insert red probe if using any of the voltage or resistance scales.
Insert red probe if using 10A scale.
Insert black probe for all measurements.
Insert red probe if using a current scale other than 10A.

Experimental Problem 1 -- Solutions

Method #1

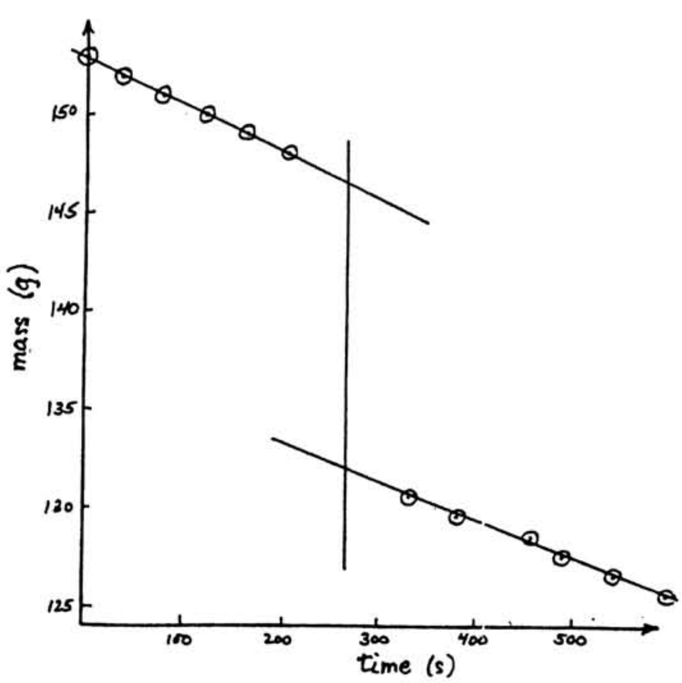


$Q = mc\Delta T = m\int c\, dT$

$Q = L\ \Delta M_{LN_2}$

$m = 19.4 \pm 0.1$ g

	total mass	clock time	time
	153 g	0:00.0	0
	152	0:36.8	36.8
	151	1:19.1	79.1
	150	2:00.7	120.7
	149	2:40.5	160.5
	148	3.23.1	203.1
Add Al mass			
	150 (130.6)	5:31.8	331.8
	149 (129.6)	6:21.6	381.6
	148 (128.6)	7:17.3	457.3
	147 (127.6)	8:08.6	488.6
	146 (126.6)	9:00.9	540.9
	145 (125.6)	9:54.6	594.6



$\Delta M_{LN_2} = 146.5 - 132.0$
$= 14.5 \pm 0.3$ g

Experimental Problem 1: Grading Scheme

Method No. 1 (5 points maximum)

1) 0.5 Uses $Q = mc\Delta T$ or $Q = m \int c\, dT$

2) 0.5. Uses $Q = L\Delta M_{LN_2}$

3) 0.5 Measures mass of aluminum correctly

4) 0.5 Measures ΔM_{LN_2} in some way

5) 0.5 Takes into account "thermal leakage" in some way and corrects for aluminum added to container

6) 0.5 Takes into account "thermal leakage" not being constant in time

7) 0.5 Uses reasonable values for c and ΔT or does $\int c\, dT$ integral in a reasonable way

8) 0.5 No mistakes made in computing L

9) 0.5 Error estimate is reasonable for methods used

10) 0.5 Value for L is within bounds set by grading team using good procedures

Method No. 2 (5 points maximum)

1) 0.5 Uses $P = \Delta Q/\Delta t$

2) 0.5 Uses $P = IV = I^2R = V^2/R$

3) 0.5 Uses $Q = LM_{LN_2}$

4) 0.5 Measures two parameters (to get P) correctly

5) 0.5 Measures M_{LN_2} in some way

6) 0.5 Takes into account "thermal leakage" in some way

7) 0.5 Takes into account "thermal leakage" not being constant in time

8) 0.5 No mistakes made in computing L

9) 0.5 Error estimate is reasonable for methods used

10) 0.5 Value for L is within bounds set by grading team using good procedures

COUNTRY:________________

XXIV INTERNATIONAL PHYSICS OLYMPIAD
WILLIAMSBURG, VIRGINIA, U.S.A.

PRACTICAL COMPETITION

Experiment No. 2

July 14, 1993

Time available: 2.5 hours

READ THIS FIRST!

INSTRUCTIONS:

1. Use only the pen provided, and only the equipment supplied.
2. Use only the marked side of the paper.
3. Write at the top of each page:
   - The number of the problem
   - The number of the page of your report
   - The total number of pages in your report.

   Example (for problem 1): 1 1/4; 1 2/4; 1 3/4; 1 4/4 to the pair from envelope X, though its magnetic moment (μ_A) cannot be assumed equal to μ_X. A given pair of magnets can be "split" and placed around the bronze disk attached to the thread, forming a "compass" whose torsional oscillation period may be measured. (The value I_X given on envelope X includes the effects of the bronze disk.)

One magnet-pair, centered in the hole in the wooden holder, can be used to influence the "compass" pair, possibly affecting its period and its angular equilibrium position. The angular position is best studied by placing the copper plate a few millimeters below the "compass" so as to provide electromagnetic damping. **Please do not mark or write on the copper plate.**

You will need to use more than one arrangement of the magnets. **Draw clearly labelled diagrams showing each experimental arrangement used. Also, write equations to show how you will combine your different observations to obtain the value of μ_X.**

Keep all magnets in the same horizontal plane. Note for the main stand that the top knob can be rotated, and the thread length adjusted. The position of each shelf can also be adjusted.

Practical Details (IMPORTANT!)

1) COMPASS ASSEMBLY AND USE: Hold one magnet from a given pair between the thumb and forefinger of one hand. Center the bronze disk over one end. Then, carefully, and without pulling on the thread, slowly bring in the second magnet. This forms the compass pair ("X" or "A"). Also, avoid pulling on the thread in taking the compass apart.
   Warning: Rapid snapping of magnets or magnet pairs together can break the thread or chip the magnets. The tiny loop can be threaded again if thread breakage occurs. (Consult the organizers if necessary.)

2) Study the torsional mode of oscillation. To prevent excitation of the "pendulum" mode, a small assembly made of copper wire is mounted on the lower shelf of the main stand. Rotate this assembly so that the horizontal piece is up against the thread at a point about 2 mm above where the thread is tied. With a slight additional rotation in the same direction, move the wire a few mm further .
   Warning: If this is not done, the two modes can "couple," causing a periodic variation in the amplitude of the torsional oscillations, and affecting their period.
   Use the nail (see diagrams at end) to start the torsional oscillations in a controlled way.

3) Keep magnetic or magnetizable objects stationary, and as far as possible from the experimental area. Consider such items as the nail, wrist watches, pens, etc. The table has some steel support parts; if you want to change the position of the apparatus, consider this fact.

Suggestions

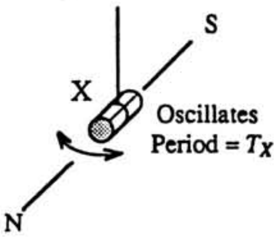
(i) The torsion constant of the thread is quite small. It turns out that you can neglect its effect in the analysis provided the thread is reasonably long, e.g. around 15 cm.

Second Solution : Dynamic Method with 3 Unknowns

The experience from our tests was that the "Turn-Around" method did not occur naturally to most students. They were much more comfortable with the idea of using one magnet to influence the period of another. Since the magnetic moments are not necessarily equal, it is clear that two measurements will no longer suffice. Our guess is that the following 3-measurement scheme will be the most common student choice.

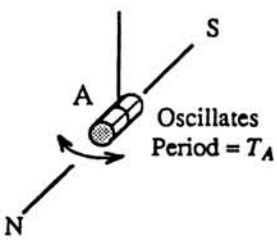
Experimental Arrangement | Equation

#1:



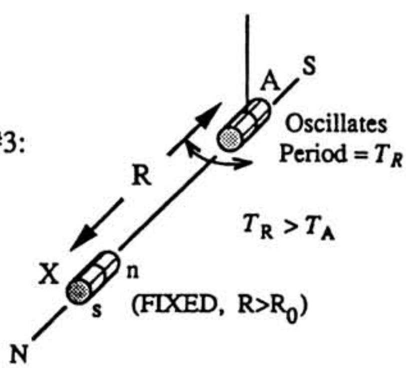
$$\mu_X B_h = I_X (2\pi/T_X)^2 \quad (1)$$

#2:



$$\mu_A B_h = I_A (2\pi/T_A)^2 \quad (2)$$

#3:



$$\mu_A \left[B_h - \mu_X \frac{2K}{R^3} \right] = I_A (2\pi/T_R)^2 \quad (3)$$

Note that the X magnet (positioned at a distance R which is somewhat larger than the turn-around distance R_0) is being used here to slow the oscillations of the A magnet on the compass.

One worries at first that there are actually 4 unknowns, since the inertial moment of A need not equal that of X. Inspection of equations (2) and (3) shows, however, that the ratio μ_X/B_h can be expresed in terms of experimentally known quantities. Since 1 gives the <u>product</u> $\mu_X B_h$, the calculational strategy is clear. One easily finds:

$$\boxed{\mu_X = \frac{R^{3/2}}{(2K)^{1/2}}\ \frac{2\pi}{T_X}\ (I_X)^{1/2}\ \left[1 - (T_A/T_R)^2\right]^{1/2}} \qquad (4)$$

Alternatively, by reversing its poles, one can use the X magnet to <u>speed-up</u> the oscillations of the A magnet. Then, of course we have $T_R < T_A$. In this case (which is formally equivalent to the first case, with a reversal of the sign of K), one finds:

$$\boxed{\mu_X = \frac{R^{3/2}}{(2K)^{1/2}}\ \frac{2\pi}{T_X}\ (I_X)^{1/2}\ \left[(T_A/T_R)^2 - 1\right]^{1/2}} \qquad (4')$$

SAMPLE EXPERIMENT

The Dynamic Method just outlined was used (in the case where the X magnet was used to slow down the oscillations of the A magnet in Arrangement #3). In all cases 20 oscillations were timed. The distance R was (17.0±0.1) cm. The X moment of inertia was $I_X = (4.95 \pm 0.1) \times 10^{-8}$ kg m^2. Using the notation given previously, the data were as follows:

Measurements (in seconds) of $20T_X$: 10.83, 10.99, 10.91, 10.94. [Arrangement #1]

Measurements (in seconds) of $20T_A$: 10.95, 11.10, 11.01, 10.92. [Arrangement #2]

Measurements (in seconds) of $20T_R$: 21.70, 21.65, 21.78, 21.59. [Arrangement #3]

Using a pocket calculator (HP32S) to obtain the averages and statistical errors gives:

$$T_X = (0.546 \pm .003)\ \text{sec}$$

$$T_A = (0.550 \pm .004)\ \text{sec}$$

$$T_R = (1.084 \pm .004)\ \text{sec}$$

The "statistical errors"" here are naively based on what the calculator gave for the estimated standard deviation <u>around</u> the sample mean. More carefully, one should divide this by the square root of the number of observations to give the estimated standard error <u>of</u> the sample mean. [Still more carefully, for such a <u>small</u> sample, one should apply the appropriate statistical correction factor]. For simplicity we will use the naively calculated results. This will suffice for our purposes.

Write 4 as $\mu_X = G\,F$, where

$$G = \frac{R^{3/2}}{(2K)^{1/2}} \frac{2\pi}{T_X} (I_X)^{1/2} \qquad \text{and} \qquad F = \left[1 - (T_A/T_R)^2\right]^{1/2}$$

The expression for G is identical for that for μ_X in the "turnaround method" when $R=R_0$. This must be true, since in that case T_R goes to infinity.

Numerically

$$G = \frac{[(0.170\pm0.001)\ \text{m}]^{3/2}}{[2\times10^{-7}\ \text{N/A}^2]^{1/2}} \frac{2\pi}{(0.546\pm.003)\ \text{sec}} \left[(4.95\pm0.1)\times10^{-8}\ \text{kg m}^2\right]^{1/2}$$

then standard error propagation and reduction of the units give

$$G = (0.401 \pm 0.006)\ \text{Am}^2$$

which is a 1.5% uncertainty. For F we find numerically :

$$F = \left\{1.000 - \left[\frac{(0.550\pm.004)\ \text{sec}}{(1.084\pm.004)\ \text{sec}}\right]^2\right\}^{1/2}$$

The central value here is 0.862. One can easily use a pocket calculator to see the effects of the permitted statistical variations in each of the two places above. This shows that the effect of the numerator uncertainty is essentially ± 0.0022, while that of the denominator is ± 0.0013. Combining these statistically gives an net uncertainty in F of 0.0026, so that the fractional uncertainty in F is 0.0033. [An analysis of this by calculus is straightforward, but cumbersome.] Then the fractional uncertainty in μ_X is practically that in G. We find:

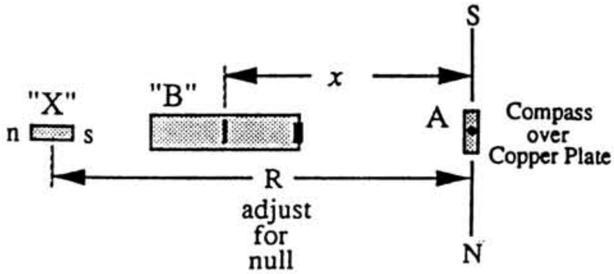
$$\mu_X = (0.862 \pm 0.0026)(0.401 \pm 0.006)\ \text{Am} = (0.346\pm.005)\ \text{A m}^2.$$

By way of comparison, measurement of the same magnet X using Fluxgate Magnetometry (at a distance of around 16 cm) gave $\mu_X = (0.345 \pm .003)\ \text{A m}^2$.

PART 2 : DISTANCE DEPENDENCE OF FIELD OF "B" UNKOWN

Method I (Close Distances) : Nulling of Transverse Static Deflection

Arrangement (top view)



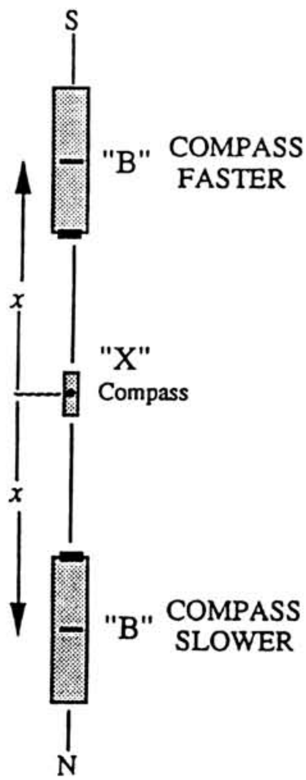
Equation

$$B_X(x) = \frac{2K\mu_X}{R^3}$$

Method II (Intermediate Distances) : Differential $1/T^2$ Technique

General Relation : $T = T_X$; B_h =local (horiz.) field $\left\{ (2\pi/T)^2 = \frac{\mu_X B_h}{I_X} \right.$

Arrangement (top view)



DEFINE:

$$\Delta(1/T^2) \equiv (1/T^2)_{\text{faster}} - (1/T^2)_{\text{slower}}$$

THEN:

$$\Delta(1/T^2) = \frac{\mu_X \Delta B_h}{4\pi^2 I_X} \quad \text{where} \quad \Delta B_h = 2B_x(x)$$

$$\boxed{B_x(x) = \frac{2\pi^2 I_X}{\mu_X}\,\Delta(1/T^2)}$$

"Master Equation"

Sample Experiment

Method I $\quad B_x(x) = \frac{2K\mu_x}{R^3} = \frac{[(2\times10^{-7})\text{T m/A}][0.346\pm0.005]\text{Am}^2}{[\text{R (m)}]^3}$

DATA TABLE FOR METHOD I

measured data		calculated	standard error propagation	see below	
x (m)	R (m)	$B_x(x)$ (10^{-7}T)	ΔB/B	$4\Delta x/x$	$(\Delta B/B)_{eff}$
.062±.001	.112±0.112	493.	.031	.065	.072
.0705±.0015	.133±.0015	294	.019	.085	.087
.0845±.0015	.167±.002	149	.039	.071	.081
.102±.0015	.206±.005	79	.074	.059	.095

The uncertainty in R includes the ruler reading error, together with the imprecision in locating the null position, the latter effect becoming predominant at larger x. The R uncertainty, together with the small uncertainty in μ_x define the ΔB/B values listed in the 4th column.

Of course there are also the uncertainties in the x values, which we could represent graphically by horizontal error bars. Since this is technically awkward, we choose instead to define an effective vertical uncertainty. Since it turns out that the log-log plot slope is about -4, a given fractional error in x corresponds to 4 times as much in B(x). These fractional errors have been tabulated in the 5th column. From this it is clear that we should take the effective ΔB/B as the square root of the sum of the squares of thecontributions in colums 4 and 5. These values, listed in column 6, form the basis for the error bars used. Though we would certainly not expect a student to do this, we would expect him to be aware of the horizontal uncertainties.

Method II $\quad B_x(x) = \frac{2\pi^2 I_x}{\mu_x}\Delta(1/T^2) = (28.2\pm.51)\times10^{-7}\text{ Tesla sec}^2\cdot\Delta(1/T^2)$

- x = (.120±.001)m :

Data in seconds for 20 oscillations: Pocket Calculator Results:

$20\,T_{slow}$: 14.56, 14.50, 14.52, 14.58 $\quad T_{slow}$ = (.727±.0018)sec

$20\,T_{fast}$: 11.32, 11.34, 11.31, 11.28 $\quad T_{fast}$ = (.5656±.0013)sec

$\Delta(1/T^2) = [(3.1257\pm.0138)-(1.892\pm.0095)]\text{ sec}^{-2} = (1.23\pm.017)\text{sec}^{-2}$

$\longrightarrow B_x(x) = (34.7\pm0.8)\times10^{-7}$ Tesla

DATA TABLE FOR METHODS II and III

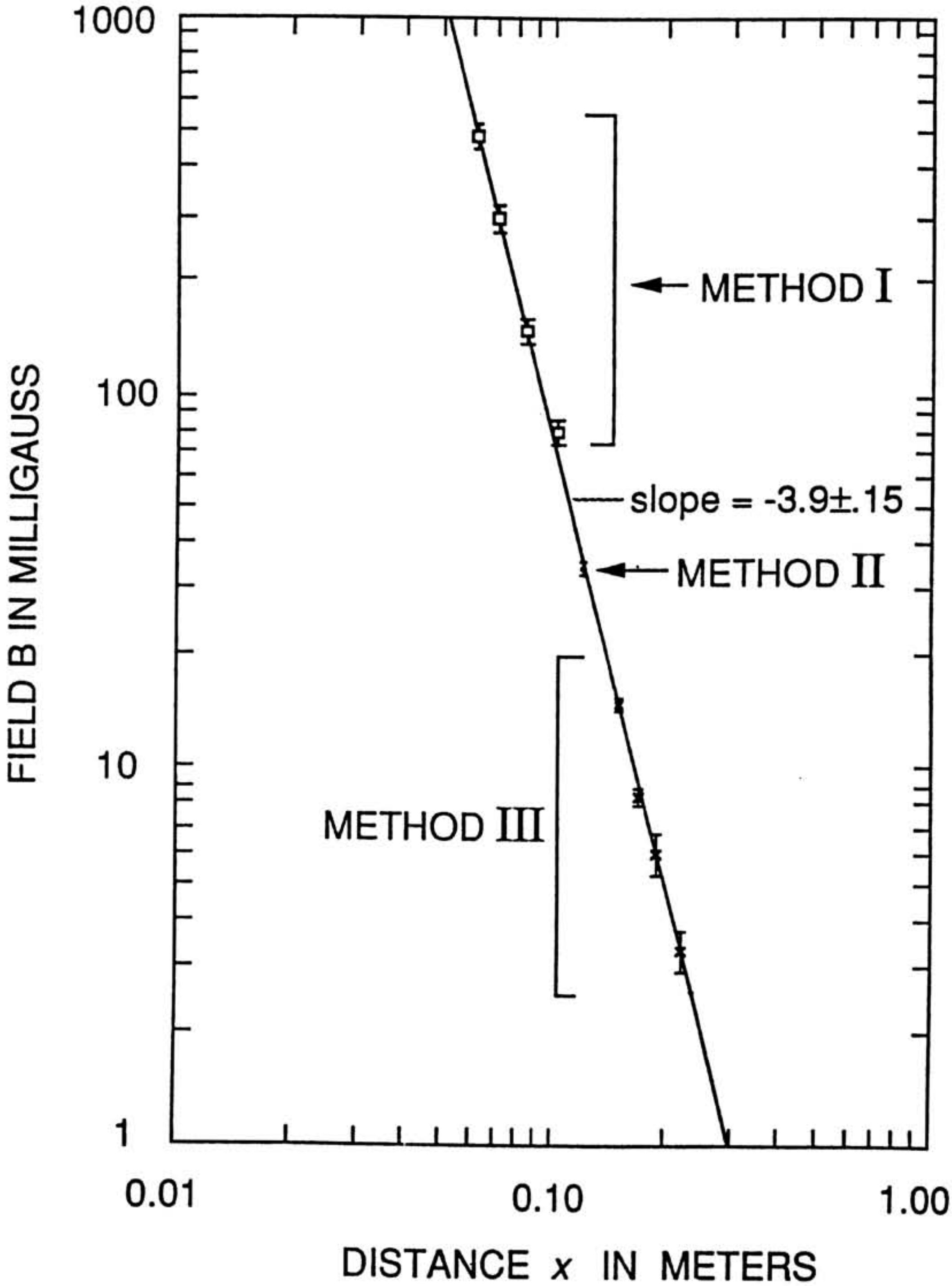
x (m)	Method	calculated $B_x(x)$ $(10^{-7}$ T)	standard error propagation $\Delta B/B$	see above $4\Delta x/x$	see above $(\Delta B/B)_{eff}$
.120±.001	II	34.7	.023	.033	.040
.150±.001	III	14.8	.024	.027	.036
.170±.001	III	8.4	.05	.024	.055
.190±.001	III	6.0	.13	.021	.13
.220±.001	III	3.3	.12	.018	.12

The equivalent vertical uncertainties have calculated as before and tabulated in the last column above. These give the error bars on the log-log plot shown on the next page. The three different methods are nicely consistent, and the whole data set well fits the power law indicated by the drawn line. When this is done on the regular log paper (as provided), the easiest way in this case to get the slope is to use a pocket calculator to find the ratio of the log of the vertical rise ratio to that of the horizontal run ratio for the possible lines consistent with the errors. Since the line has to drop vertically through three decades in total, this is roughly

$$\text{slope} = \frac{-3}{\log_{10}\left[\dfrac{(0.30\pm.02)}{(.051\pm.03)}\right]} = -3.9 \pm 0.15$$

For this particular unknown, the fluxgate magnetometer data gave an effective exponent of -3.92 over the range from 0.07m to 0.22m. A more detailed absolute comparison with those measurements is shown on the second graph. Here the drawn line corresponds to the actual magnetometer data. The student experiment is clearly doing an excellent job. Of particular interest is the next to the lowest point (x=0.19m). For this point, the "buckout" magnet had been moved out a little bit so that the natural compass period in the Earth's field was about 0.89 sec., which was close to the period of the "pendulum mode". This was done deliberately to test the effectiveness of the copper wire "mode-decoupler". The point at x=0.19 m which is <u>on</u> the line was taken <u>using</u> the decoupler. The point at the same x value which is almost a factor of 3 higher than the line was taken <u>without</u> the decoupler.

This shows that the decoupler is both effective and important. Without it, the "fast" and "slow" measurements are effected <u>differently</u> by the coupling to the pendulum mode. Then the small difference between them can be <u>very</u> poorly determined.



1000
100
10
1
FIELD B IN MILLIGAUSS
METHOD I
slope = -3.9±.15
METHOD II
METHOD III
0.01
0.10
1.00
DISTANCE x IN METERS

THE EXAMINATION

XXV INTERNATIONAL PHYSICS OLYMPIAD

BEIJING, PERPLE'S REPUBLIC CHINA

THEORETICAL COMPETITION

July 13, 1994

Time available: 5 hours

READ THIS FIRST!

INSTRUCTIONS:

1. Use only the ball pen provided.
2. Your graphs should be drawn on the answer sheets attached to the problem.
3. Your solutions should be written on the sheets of paper attached to the problems.
4. Write at the top of the first page of each problem:
   - The total number of pages in your solution to the problem
   - Your name and code number

Theoretical Problem 1

RELATIVISTIC PARTICLE

In the theory of special relativity the relation between energy E and momentum P or a free particle with rest mass m_0 is

$$E = \sqrt{p^2c^2 + m_0^2c^4} = mc^2$$

When such a particle is subject to a conservative force, the total energy of the particle, which is the sum of $\sqrt{p^2c^2 + m_0^2c^4}$ and the potential energy, is conserved. If the energy of the particle is very high, the rest energy of the particle can be ignored (such a particle is called an ultra relativistic particle).

1) consider the one dimensional motion of a very high energy particle (in which rest energy can be neglected) subject to an attractive central force of constant magnitude f. Suppose the particle is located at the centre of force with initial momentum p_0 at time t=0. Describe the motion of the particle by separately plotting, for at least one period of the motion: x against time t, and momentum p against space coordinate x. Specify the coordinates of the "turning points" in terms of given parameters p_0 and f. Indicate, with arrows, the direction of the progress of the mothon in the (p, x) diagram. There may be short intervals of time during which the particle is not ultrarelativistic. However, these should be neglected.

   Use Answer Sheet 1.

2) A meson is a particle made up of two quarks. The rest mass M of the meson is equal to the total energy of the two-quark system divided by c^2.

   Consider a one--dimensional model for a meson at rest, in which the two quarks are assumed to move along the x-axis and attract each other with a force of constant magnitude f It is assumed they can pass through each other freely. For analysis of the high energy motion of the quarks the rest mass of the quarks can be neglected. At time t=0 the two quarks are both at x=0. Show separately the motion of the two quarks graphically by a (x, t) diagram and a (p, x) diagram, specify the coordinates of the "turning points" in terms of M and f, indicate the direction of the process in your (p, x) diagram, and determine the maximum distance between the two quarks.

   Use Answer Sheet 2.

3) The reference frame used in part 2 will be referred to as frame S, the Lab frame, referred to as S, moves in the negative x-direction with a constant velocity v=0.6c. the coordinates in the two reference frames are so chosen that the point

x=0 in S coincides with the point $x' = 0$ in S'' at time $t = t' = 0$. Plot the motion of the two quarks graphically in a (x' , t') diagram. Specify the coordinates of the turning points in terms of M, f and c, and determine the maximum distance between the two quarks observed in Lab frame S'.

Use Answer Sheet 3.

The coordinates of particle observed in reference frames S and S'' are related by the Lorentz transformation

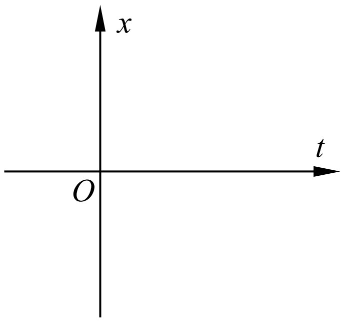
$$\begin{cases} x' = \gamma(x + \beta ct) \\ t' = \gamma(t + \beta \dfrac{x}{c}) \end{cases}$$

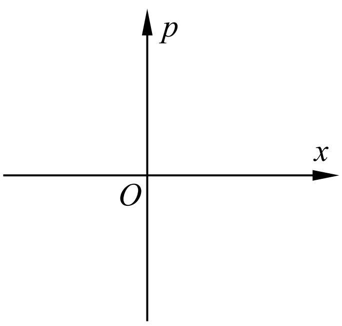
where $\beta = v/c$, $\gamma = 1/\sqrt{1-\beta^2}$ and v is the velocity of frame S moving relative to the frame S''.

4) For a meson with rest energy Mc^2=140 MeV and velocity 0.60c relative to the Lab frame S'', determine its energy E' in the Lab Frame S''.

ANSWER SHEET 1

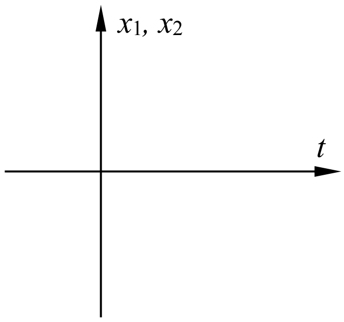
1)

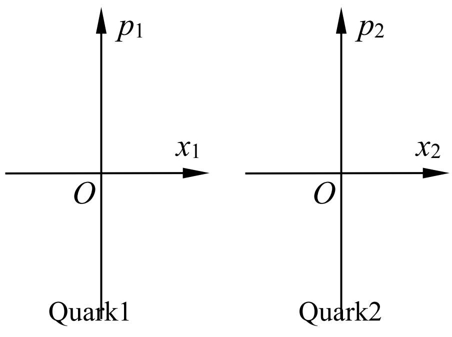




ANSWER SHEET 2

2)

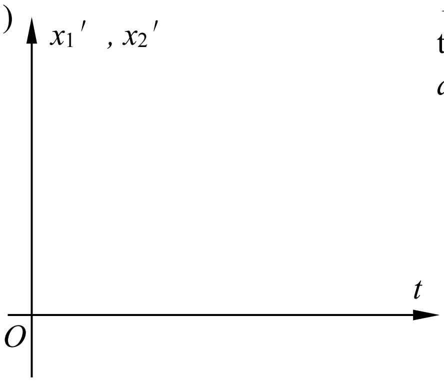




The maximum distance between the two quarks is d=

ANSWER SHEET 3

3)



The maximum distance between the two quarks observed in S′ frame is
$d' =$

Theoretical Problem 1—Solution

1) **1a. Taking the force center as the origin of the space coordinate** x and the zero potential point, the potential energy of the particle is

$$U(x) = f\,|\,x\,| \tag{1}$$

The total energy is

$$W = \sqrt{p^2c^2 + m_0^2c^4} + f\,|\,x\,|.$$

1b. Neglecting the rest energy, we get

$$W = |\,p\,|\,c + f\,|\,x\,|, \tag{2}$$

Since W is conserved throughout the motion, so we have

$$W = |\,p\,|\,c + f\,|\,x\,| = p_0c, \tag{3}$$

Let the x axis be in the direction of the initial momentum of the particle,

$$\left.\begin{array}{lll} pc + fx = p_0c & \text{when} & x > 0, \quad p > 0; \\ -pc + fx = p_0c & \text{when} & x > 0, \quad p < 0; \\ pc - fx = p_0c & \text{when} & x < 0, \quad p > 0; \\ -pc - fx = p_0c & \text{when} & x < 0, \quad p < 0. \end{array}\right\} \tag{4}$$

The maximum distance of the particle from the origin, let it be L, corresponds to p=0. It is

$$L = p_0c/f.$$

1c. From Eq. 3 and Newton’s law

$$\frac{dp}{dt} = F = \begin{cases} -f, & x > 0; \\ f, & x < 0; \end{cases} \tag{5}$$

we can get the speed of the particle as

$$\left|\frac{dx}{dt}\right| = \frac{c}{f}\left|\frac{dp}{dt}\right| = c, \tag{6}$$

i.e. the particle with very high energy always moves with the speed of light except that it is in the region extremely close to the points $x = \pm L$. The time for the particle to move from origin to the point $x = L$, let it be denoted by τ , is

$$\tau = L / c = p_0 / f .$$

So the particle moves to and for between $x = L$ and $x = -L$ with speed c and period $4\tau = 4p_0 / f$. The relation between x and t is

$$\left.\begin{array}{ll} x = ct, & 0 \le t \le \tau \\ x = 2L - ct, & \tau \le t \le 2\tau, \\ x = 2L - ct, & 2\tau \le t \le 3\tau, \\ x = ct - 4L, & 3\tau \le t \le 4\tau, \end{array}\right\} \tag{7}$$

The required answer is thus as given in Fig. 1 and Fig. 2.

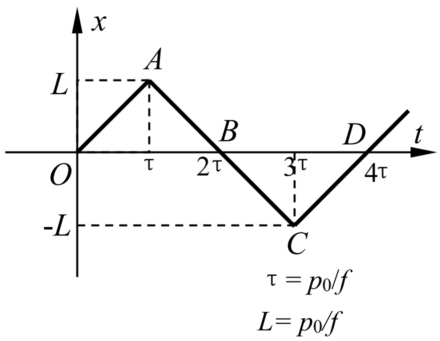


Fig. 1

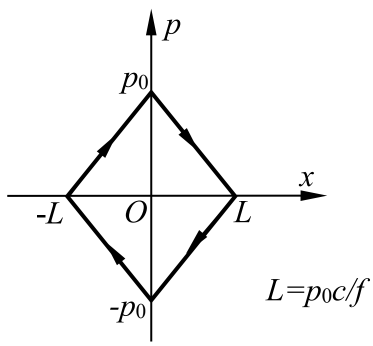


Fig. 2

2) The total energy of the two-quark system can be expressed as

$$Mc^2 = | p_1 | c + | p_2 | c + f | x_1 - x_2 |, \tag{8}$$

where x_1, x_2 are the position coordinates and p_1, p_2 are the momenta of quark 1 and quark 2 respectively. For the rest meson, the total momentum of the two quarks is zero and the two quarks move symmetrically in opposite directions, we have

$$p = p_1 + p_2 = 0 , \qquad p_1 = -p_2 , \quad x_1 = -x_2 . \tag{9}$$

Let p_0 denote the momentum of the quark 1 when it is at x=0, then we have

$$Mc^2 = 2p_0 c \qquad \text{or} \qquad p_0 = Mc / 2 \tag{10}$$

From Eq. 8, 9 and 10, the half of the total energy can be expressed in terms of p_1 and x_1 of quark 1:

$$p_0 c = | p_1 | c + f | x_1 |, \tag{11}$$

just as though it is a one particle problem as in part 1 (Eq. 3) with initial momentum

$p_0 = Mc/2$. From the answer in part 1 we get the (x, t) diagram and (p, x) diagram of the motion of quark 1 as shown in Figs. 3 and 4. For quark 2 the situation is similar except that the signs are reversed for both x and p; its (x, t) and (p, x) diagrams are shown in Figs. 3 and 4.

The maximum distance between the two quarks as seen from Fig. 3 is

$$d = 2L = 2p_0c/f = Mc^2/f \,. \tag{12}$$

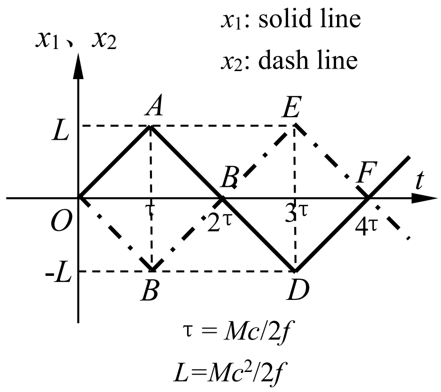


Fig. 3

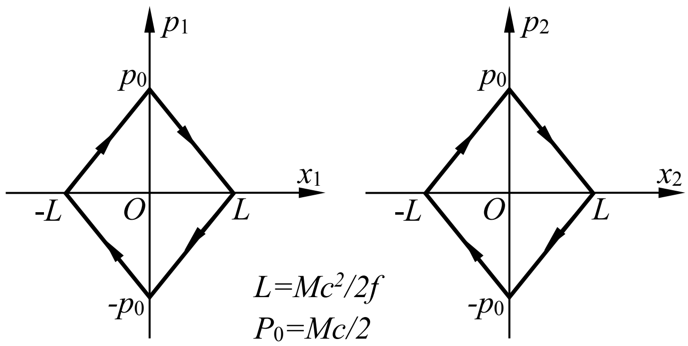


Fig. 4a Quark1
Fig. 4b Quark2

3) The reference frame S moves with a constant velocity V=0.6c relative to the Lab frame S'' in the x' axis direction, and the origins of the two frames are coincident at the beginning ($t = t' = 0$). The Lorentz transformation between these two frames is given by:

$$\begin{aligned} x' &= \gamma(x + \beta ct), \\ t' &= \gamma(t + \beta x/c), \end{aligned} \tag{13}$$

where $\beta = V/c$, and $\gamma = 1/\sqrt{1-\beta^2}$. With $V = 0.6c$, we have $\beta = 3/5$, and $\gamma = 5/4$. Since the Lorenta transformation is linear, a straight line in the (x, t) diagram transforms into a straight line the (x', t') diagram, thus we need only to calculate the coordinates of the turning points in the frame S'.

For quark 1, the coordinates of the turning points in the frames S and S' are as follows:

Frame S		Frame S'	
x_1	t_1	$x_1' = \gamma(x_1 + \beta c t_1)$	$t_1' = \gamma(t_1 + \beta x_1 / e)$
		$= \frac{5}{4}x_1 + \frac{3}{4}ct_1$	$= \frac{5}{4}t_1 + \frac{3}{4}x_1 / c$
0	0	0	0
L	τ	$\gamma(1+\beta)L = 2L$	$\gamma(1+\beta)\tau = 2\tau$
0	2τ	$2\gamma\beta L = \frac{3}{2}L$	$2\gamma\tau = \frac{5}{2}\tau$
$-L$	3τ	$\gamma(3\beta - 1)L = L$	$\gamma(3-\beta)\tau = 3\tau$
0	4τ	$4\gamma\beta L = 3L$	$4\gamma\tau = 5\tau$

where $L = p_0 c / f = Mc^2 / 2f$, $\tau = p_0 / f = Mc / 2f$.

For quark 2, we have

Frame S		Frame S'	
x_2	t_2	$x_2' = \gamma(x_2 + \beta c t_2)$	$t_2' = \gamma(t_2 + \beta x_2 / c)$
		$= \frac{5}{4}x_2 + \frac{3}{4}ct_2$	$= \frac{5}{4}t_2 + \frac{3}{4}x_2 / c$
0	0	0	0
$-L$	τ	$-\gamma(1-\beta)L = -\frac{1}{2}L$	$\gamma(1-\beta)\tau = \frac{1}{2}\tau$
0	2τ	$2\gamma\beta L = \frac{3}{2}L$	$2\gamma\tau = \frac{5}{2}\tau$
L	3τ	$\gamma(3\beta + 1)L = \frac{7}{2}L$	$\gamma(3+\beta)\tau = \frac{9}{2}\tau$
0	4τ	$4\gamma\beta L = 3L$	$4\gamma\tau = 5\tau$

With the above results, the (x', t') diagrams of the two quarks are shown in Fig. 5.

The equations of the straight lines *OA* and *OB* are:

$$x_1'(t') = ct'; \qquad 0 \le t' \le \gamma(1+\beta)\tau = 2\tau; \qquad \text{(14a)}$$

$$x_2'(t') = -ct'; \qquad 0 \le t' \le \gamma(1-\beta)\tau = \frac{1}{2}\tau \qquad \text{(14b)}$$

The distance between the two quarks attains its maximum d' when $t' = \frac{1}{2}\tau$, thus we have maximum distance

$$d' = 2c\gamma(1-\beta)\tau = 2\gamma(1-\beta)L = \frac{Mc^2}{2f}. \tag{15}$$

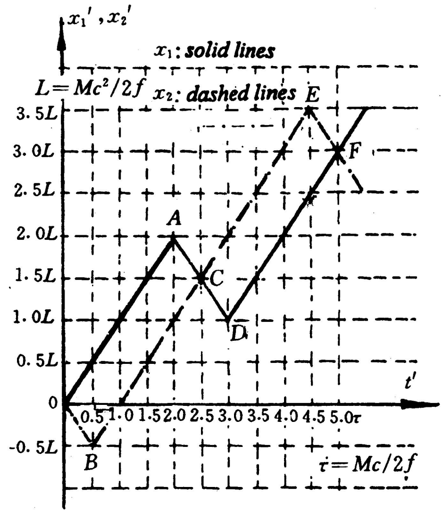


Fig. 5

4) It is given the meson moves with velocity V=0.6 crelative to the Lab frame, its energy measured in the Lab frame is

$$E' = \frac{Mc^2}{\sqrt{1-\beta^2}} = \frac{1}{0.8} \times 140 = 175\,\text{MeV}.$$

Grading Scheme

Part 1 2 points, distributed as follows:

0.4 point for the shape of $x(t)$ in Fig. 1;

0.3 point for 4 equal intervals in Fig. 1;

(0.3 for correct derivation of the formula only)

0.1 each for the coordinates of the turning points A and C, 0.4 point in total;

0.4 point for the shape of $p(x)$ in fig. 2; (0.2 for correct derivation only)

0.1 each for specification of p_0, $L = p_0 c / f$, $-p_0$, $-L$ and arrows, 0.5 point in total.

(0.05 each for correct calculations of coordinate of turning points only).

Part 2 4 points, distributed as follows:

0.6 each for the shape of $x_1(t)$ and $x_2(t)$, 1.2 points in total;

0.1 each for the coordinates of the turning points A, B, D and E in Fig. 3, 0.8 point in total;

0.3 each for the shape of $p_1(x_1)$ and $p_2(x_2)$, 0.6 point in total;

0.1 each for $p_0 = Mc/2$, $L = Mc^2/2f$, $-p_0$, $-L$ and arrows in Fig. 4a and Fig. 4b, 1 point in total;

0.4 point for $d = Mc^2/f$

Part 3 3 point, distributed as follows:

0.8 each for the shape of $x_1'(t')$ and $x_2'(t')$, 1.6 points in total;

0.1 each for the coordinates of the turning points A, B, D and E in Fig. 5, 0.8 point in total; (0.05 each for correct calculations of coordinate of turning points only).

0.6 point for $d' = Mc^2/2f$.

Part 4 1 point (0.5 point for the calculation formula; 0.5 point for the numerical value and units)

Theoretical Problem 2

SUPERCONDUCTING MAGNET

Super conducting magnets are widely used in laboratories. The most common form of super conducting magnets is a solenoid made of super conducting wire. The wonderful thing about a superconducting magnet is that it produces high magnetic fields without any energy dissipation due to Joule heating, since the electrical resistance of the superconducting wire becomes zero when the magnet is immersed in liquid helium at a temperature of 4.2 K. Usually, the magnet is provided with a specially designed superconducting switch, as shown in Fig. 1. The resistance r of the switch can be controlled: either r=0 in the superconducting state, or $r = r_n$ in the normal state. When the persistent mode, with a current circulating through the magnet and superconducting switch indefinitely. The persistent mode allows a steady magnetic field to be maintained for long periods with the external source cut off.

The details of the superconducting switch are not given in Fig. 1. It is usually a small length of superconducting wire wrapped with a heater wire and suitably thermally insulated from the liquid helium bath. On being heated, the temperature of the superconducting wire increases and it reverts to the resistive normal state. The typical value of r_n is a few ohms. Here we assume it to be 5 Ω. The inductance of a superconducting magnet depends on its size; assume it be 10 H for the magnet in Fig. 1. The total current I can be changed by adjusting the resistance R.

This problem will be graded by the plots only!

The arrows denote the positive direction of I, I_1 and I_2.

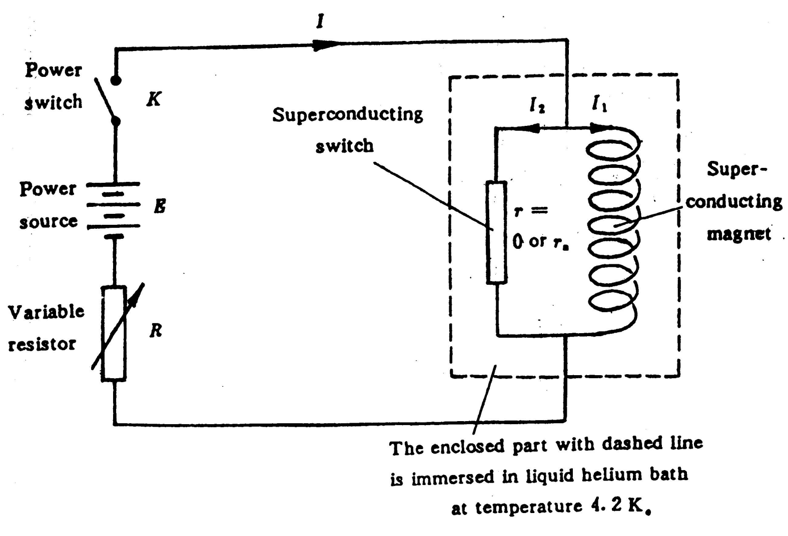


Fig. 1

1) If the total current I and the resistance r of the superconducting switch are controlled

to vary with time in the way shown in Figs, 2a and 2b respectively, and assuming the currents I_1 and I_2 flowing through the magnet and the switch respectively are equal at the beginning (Fig. 2c and Fig. 2d), how do they vary with time from t_1 to t_4? Plot your answer in Fig. 2c and Fig. 2d

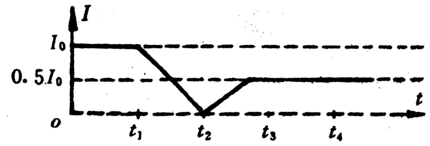
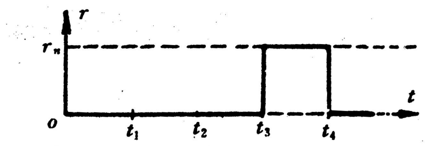
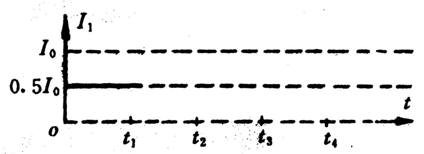


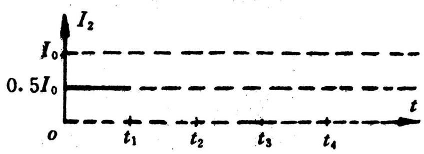
Fig.2a



2b



2c



2d

2) Suppose the power switch K is turned on at time t=0 when r=0, I_1=0 and R=7.5 Ω , and the total current I is 0.5A. With K kept closed, the resistance r of the superconducting switch is varied in he way shown in Fig. 3b. Plot the corresponding time dependences of I, I_1 and I_2 in Figs. 3a, 3c and 3d respectively.

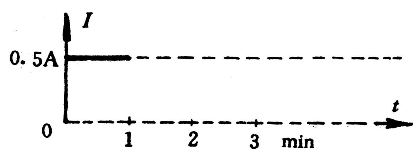
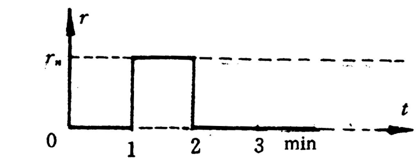
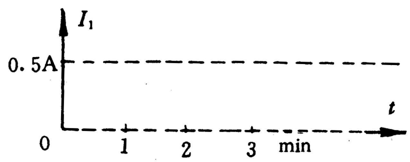


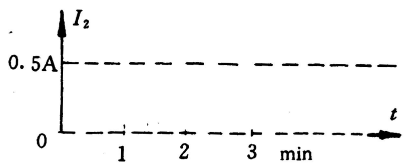
Fig. 3a



3b



3c



3d

3) Only small currents, less than 0.5A, are allowed to flow through the

superconducting switch when it is in the normal state, with larger currents the switch will be burnt out. Suppose the superconducting magnet is operated in a persistent mode, i. e. I=0, and I_1=i_1(e. g. 20A), I_2=-i_1, as shown in Fig. 4, from t=0 to t=3min. If the experiment is to be stopped by reducting the current through the magnet to zero, how would you do it? This has to be done in several operation steps. Plot the corresponding changes of I, r, I_1 and I_2 in Fig. 4

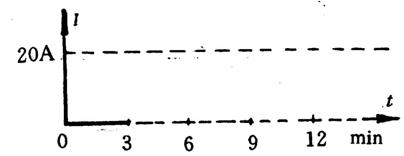
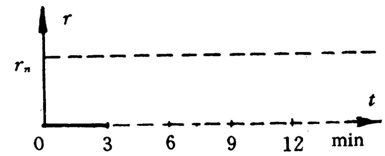
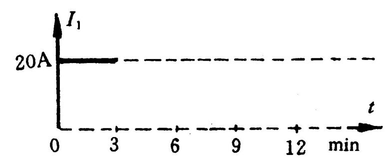


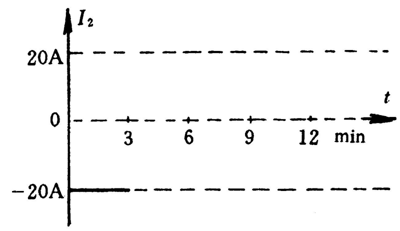
Fig. 4a



4b



4c



4d

4) Suppose the magnet is operated in a persistent mode with a persistent current of 20A (t=0 to t=3min. See Fig. 5). How would you change it to a persistent mode with a current of 30a? plot your answer in Fig. 5.

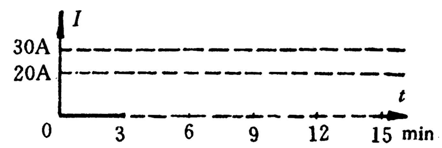
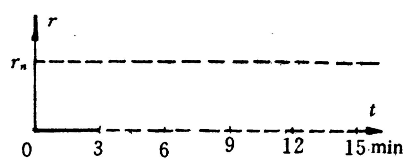
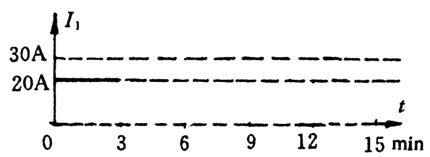


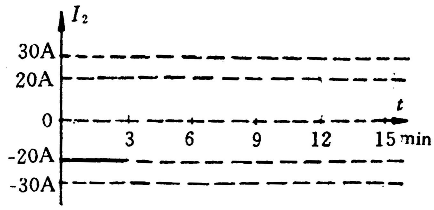
Fig. 5a



5b



5c



5d

Theoretical Problem 2—Solution

1) For t=t_1 to t_3

Since $r=0$, the voltage across the magnet $V_M = LdI_1/dt$ =0, therefore,

$$I_1 = I_1(t_1) = \frac{1}{2}I_0;$$

$$I_2 = I - I_1 = I - \frac{1}{2}I_0.$$

For t=t_3 to t_4

Since I_2=0 at t=t_3, and I is kept at $\frac{1}{2}I_0$ after

$t=t_3$, $V_M = I_2 r_n = 0$, therefore, I_1 and I_2 will not change.

$$I_1 = \frac{1}{2}I_0;$$

$$I_2 = 0$$

These results are shown in Fig. 6.

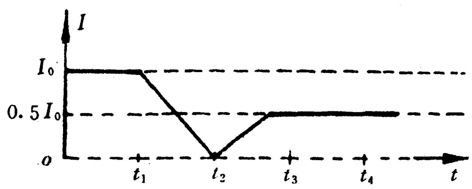
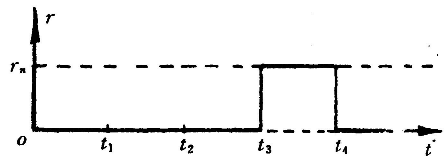
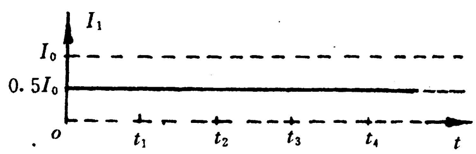


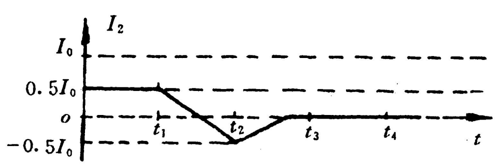
Fig. 6a



6b



6c



6d

2) For $t=0$ to $t=1\,\text{min}$:

Since $r=0$, $V_M = LdI_1/dt = 0$

$$I_1 = I_1(0) = 0$$

$$I_2 = I - I_1 = 0.5\,\text{A}.$$

At $t=1$ min, r suddenly jumps from O to r_n, I will drop from E/R to $E/(R+r_n)$ instantaneously, because I_1 can not change abruptly due to the inductance of the magnet coil. For E/R=0.5A, $R=7.5\Omega$ and $R_n=5\Omega$. I will drop to 0.3A.

For $t=1$ min to 2 min:

I, I_1 and I_2 gradually approach their steady values:

$$I = \frac{E}{R} = 0.5\,\text{A},$$

$$I_1 = I = 0.5\,\text{A}$$

$$I_2 = 0.$$

The time constant

$$\tau = \frac{L(R+r_n)}{Rr_n}.$$

When $L=10\,\text{H}$, $R=7.5\Omega$ and $r_n=5\Omega$, $\tau = 3\,\text{sec}$.

For $t=2\,\text{min}$ to 3 min:

Since $r=0$, I_1 and I_2 will not change, that is

$$I_1 = 0.5\,\text{A and } I_2 = 0$$

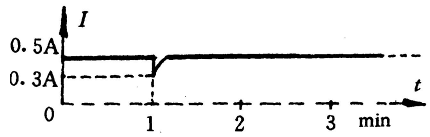
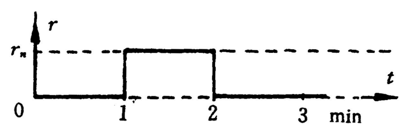
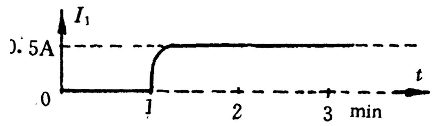


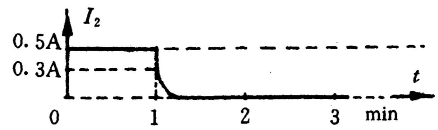
Fig. 7a



7b



7c



7d

3) The operation steps are:

First step

Turn on power switch *K*, and increase the total current *I* to 20 A, i. e. equal to I_1. Since the superconducting switch is in the state $r = 0$, so that $V_M = L \quad dI_1/dt = 0$, that is, I_1 can not change and I_2 increases by 20A, in other words, I_2 changes from -20 A to zero.

Second step

Switch r from 0 to r_n.

Third step

Gradually reduce *I* to zero while keeping $I_2 < 0.5$ A: since $I_2 = V_M / r_n$ and $V_m = L\, dI_1/dt$, when $L = 10$ H, $r_n = 5\Omega$, the requirement $I_2 < 0.5$ A corresponds to $dI_1/dt < 0.25$ A/sec, that is, a drop of <15A in 1 min. In Fig. 8 $dI/dt \sim 0.1$A/sec and dI_1/dt is around this value too, so the requirement has been fulfilled.

Final step

Switch r to zero when $V_M = 0$ and turn off the power switch *K*. These results are shown in Fig. 8.

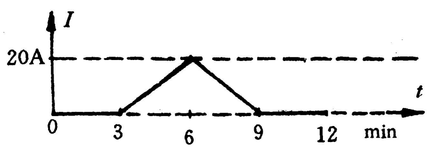
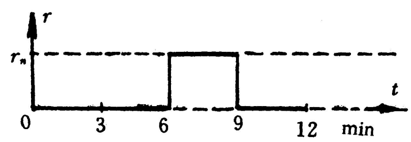
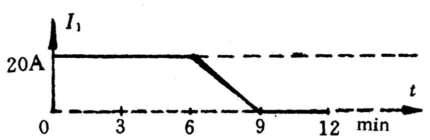


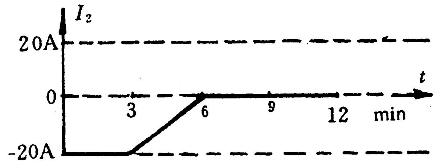
Fig. 8a



8b



8c



8d

4) **First step** and **second step** are the same as that in part 3, resulting in $I_2 = 0$.

Third step Increase *I* by 10 A to 30 A with a rate subject to the requirement $I_2 < 0.5$ A.

Fourth step Switch r to zero when $V_M = 0$.

Fifth step Reduce *I* to zero, $I_1 = 30$ A will not change because V_M is zero. $I_2 = I - I_1$ will change to -30 A. The current flowing through the magnet is thus closed by the superconducting switch.

Final step Turn off the power switch *K*. The magnet is operating in the persistent mode.

These results are shown in Fig. 9.

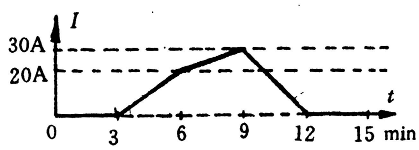
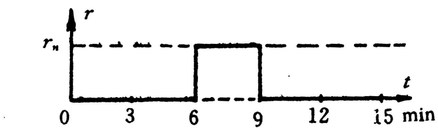
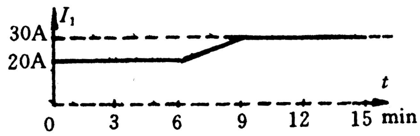


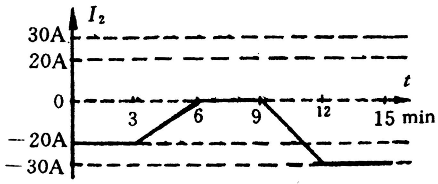
Fig. 9a



9b



9c



9d

Grading Scheme

Part 1, 2 points:

0.5 point for each of I_1, I_2 from $t = t_1$ to t_3 and I_1, I_2 from $t = t_3$ to t_4.

Part 2, 3 points:

0.3 point for each of I_1, I_2 from $t = 0$ to 1 min, I, I_1, I_2 at $t = 1$ min, and I_0, I_1, I_2 from $t=1$ to 2 min;

0.2 point for each of I, I_1, and I_2 from $t=2$ to 3 min.

Part 3, 2 points:

0.25 point for each section in Fig. 8 from $t=3$ to 9 min, 8 sections in total.

Part 4, 3 points:

0.25 point for each section in Fig. 9 from $t=3$ to 12 min, 12 sections in total.

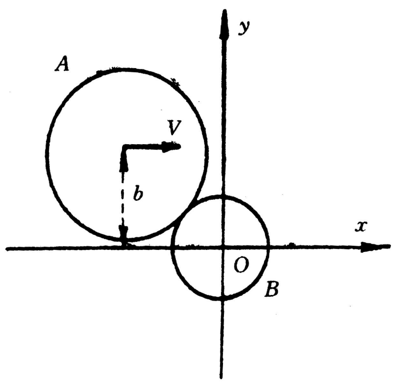
Theoretical Problem 3

COLLISION OF DISCS WITH SURFACE FRICTION

A homogeneous disc A of mass m and radius R_A moves translationally on a smooth horizontal *x*-*y* plane in the *x* direction with a velocity *V* (see the figure on the next page). The center of the disk is at a distance *b* from the x-axis. It collides with a stationary homogeneous disc B whose center is initially located at the origin of the coordinate system. The disc B has the same mass and the same thickness as A, but its radius is R_B. It is assumed that the velocities of the discs at their point of contact, in the direction perpendicular to the line joining their centers, are equal after the collision. It is also assumed that the magnitudes of the relative velocities of the discs along the line joining their centers are the same before and after the collision.

1) For such a collision determine the *X* and *Y* components of the velocities of the two discs after the collision, i. e. V'_{AX}, V'_{AY}, V'_{BX} and V'_{BY} in terms of m, R_A, R_B, V and b.

2) Determine the kinetic energies E'_A for disc A and E'_B for disc B after the collision in terms of m, R_A, R_B, V and b.



Theoretical Problem 3—Solution

1) When disc A collides with disc B, let *n* be the unit vector along the normal to the surfaces at the point of contact and *t* be the tangential unit vector as shown in the figure. Let φ be the angle between *n* and the *x* axis. Then we have

$$b = (R_A + R_B)\sin\varphi$$

The momentum components of *A* and *B* along *n* and *t* before collision are:

$$mV_{An} = mV\cos\varphi,\ mV_{Bn} = 0,$$

$$mV_{At} = mV\sin\varphi, mV_{Bt} = 0\,.$$

Denote the corresponding momentum components of A and B after collision by mV'_{An}, mV'_{Bn}, mV'_{At}, and mV'_{Bt}. Let ω_A and ω_B be the angular velocities of A and B about the axes through their centers after collision, and I_A and I_B be their corresponding moments of intertia. Then,

$$I_A = \frac{1}{2}mR_A^2, \qquad\qquad I_B = \frac{1}{2}mR_B^2$$

The conservation of momentum gives

$$mV\cos\varphi = mV'_{An} + mV'_{Bn}, \tag{1}$$

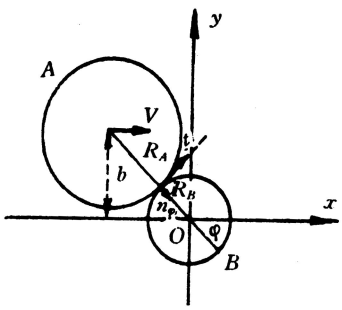
$$mV\sin\varphi = mV'_{At} + mV'_{tn}, \tag{2}$$

The conservation of angular momentum about the axis through O gives

$$mVb = mV'_{At}(R_A + R_B) + I_A\omega_A + I_B\omega_B \tag{3}$$

The impulse of the friction force exerted on B during collision will cause a momentum change of mV'_{At} along t and produces an angular momentum $I_B\omega_B$ simultaneously. They are related by.

$$mV'_{Bt}R_b = I_B\omega_B \tag{4}$$



During the collision at the point of contact A and B acquires the same tangential velocities, so we have

$$V'_{At} - \omega_A R_A = V'_{Bt} - \omega_B R_B \tag{5}$$

It is given that the magnitudes of the relative velocities along the normal direction of the two discs before and after collision are equal, i. e.

$$V\cos\varphi = V'_{Bn} - V'_{An}\,. \tag{6}$$

From Eqs. 1 and 6 we get

$$V'_{An} = 0 ,$$

$$V'_{Bn} = V \cos\varphi .$$

From Eqs. 2 to 5, we get

$$V'_{At} = \frac{5}{6} V \sin\varphi ,$$

$$V'_{Bt} = \frac{1}{6} V \sin\varphi ,$$

$$\omega_A = \frac{V \sin\varphi}{3R_A} ,$$

$$\omega_B = \frac{V \sin\varphi}{3R_B} .$$

The x and y components of the velocities after collision are:

$$V'_{Ax} = V'_{An} \cos\varphi + V'_{At} \sin\varphi = \frac{5Vb^2}{6(R_A + R_B)^2} , \tag{7}$$

$$V'_{Ay} = -V'_{An} \sin\varphi + V'_{At} \cos\varphi = \frac{5Vb\sqrt{(R_A + R_B)^2 - b^2}}{6(R_A + R_B)^2} , \tag{8}$$

$$V'_{Bx} = V'_{Bn} \cos\varphi + V'_{Bt} \sin\varphi = \left[1 - \frac{5b^2}{6(R_A + R_B)^2} \right] , \tag{9}$$

$$V'_{By} = -V'_{Bn} \sin\varphi + V'_{Bt} \cos\varphi = -\frac{5Vb\sqrt{(R_A + R_B)^2 - b^2}}{6(R_A + R_B)^2} , \tag{10}$$

2) After the collision, the kinetic energy of disc A is

$$E'_A = \frac{1}{2} m (V'^2_{Ax} + V'^2_{Ay}) + \frac{1}{2} I_A \omega_A^2 = \frac{3mV^2b^2}{8(R_A + R_B)^2} \tag{11}$$

while the kinetic energy of disc B is

$$E'_B = \frac{1}{2} m (V'^2_{Bx} + V'^2_{By}) + \frac{1}{2} I_B \omega_B^2 = \frac{1}{2} mV^2 \left[1 - \frac{11b^2}{12(R_A + R_B)^2} \right] \tag{12}$$

Grading Scheme

1. After the collision, the velocity components of discs A and B are shown in Eq. 7, 8, 9 and 10 of the solution respectively. The total points of this part is 8. 0. If the result in which all four velocity components are correct has not been obtained, the point is marked according to the following rules.

0.8 point for each correct velocity component;

0.8 point for the correct description of that the magnitudes of the relative velocities of the discs along the line joining their centers are the same before and after the collision.

0.8 point for the correct description of the conservation for angular momentum;

0.8 point for the correct description of the equal tangential velocity at the touching point;

0.8 point for the correct description of the relation between the impulse and the moment of the impulse.

2. After the collision, the kinetic energies of disc A and disc B are shown in Eqs. 11 and 12 of the solution respectively.

1.0 point for the correct kinetic energies of disc A;

1.0 point for the correct kinetic energies of disc B;

The total points of this part is 2.0

XXV INTERNATIONAL PHYSICS OLYMPIAD

BEIJING, PEOPLE'S REPUBLIC OF CHINA

PRACTICAL COMPETITION

July 15, 1994

Time available: 2.5 hours

READ THIS FIRST!

INSTRUCTIONS:

1. Use only the ball pen provided.
2. Your graphs should be drawn on the answer sheets attached to the problem.
3. Write your solution on the marked side of the paper only.
4. The draft papers are provided for doing numerical calculations and draft drawings.
5. Write at the top of every page:
    - The number of the problem
    - The number of the page of your report in each problem
    - The total number of pages in your report to the problem
    - Your name and code number

EXPERIMENTAL PROBLEM 1

Determination of light reflectivity of a transparent dielectric surface.

Experimental Apparatus

1. He-Ne Laser(~1.5mW).The light from this laser is not linearly polarized.

2. Two polarizers (**P**$_1$, **P**$_2$) with degree scale disk (Fig. 1), one (**P**$_1$) has been mounted in front of the laser output window as a polarizer, and another one can be fixed in a proper place of the drawing board by push-pins when it is necessary.

3. Two light intensity detectors (D_1, D_2) which consisted of a photocell and a microammeter (Fig. 2).

4. Glass beam splitter(B).

5. Transparent dielectric plate, whose reflectivity and refractive index are to be determined.

6. Sample table mounted on a semicircular degree scale plate with a coaxial swivel arm(Fig. 3).

7. Several push-.pins for fixing the sample table on the drawing board and as its rotation axis.

8. Slit aperture and viewing screen for adjusting the laser beam in the horizontal direction and for alignment of optical elements.

9. Lute for adhere of optical elements in a fixed place.

10. Wooden drawing board.

11. Plotting papers

Experiment Requirement

1. Determine the reflectivity of the p-component as a function of the incident angle (the electric field component, parallel to the plane of incidence is called the *p*-component).
   (a) Specify the transmission axis of the polarizer (A) by the position of the marked line on the degree scale disk in the *p*-componet measurement(the transmission axis is the direction of vibration of the electric field vector of the transmitted light).
   (b) Choose any one of the light intensity detector and set its micro-ammeter at the range of "×5". Verify the linear relation ship between the light intensity and the micro-ammeter reading. Draw the optical schematic diagram. Show your measured data and calculated results(including the calculation formula)in the farm of a table. Plot the linear relationship curve.

(c) Determine the reflectivity of the p-component as a function of the incident angle. Draw the optical schematic diagram. Show your measured data and calculated reflectivity(including the calculation formula)in the form of a table. Plot the reflectivity as a function of the incident angle.

2. Determine the refractive index of the sample as accurate as possible.

Explanation and Suggestion

1. Laser radiation avoid direct eye exposure.

2. Since the output power of the laser beam may fluctuate from time to time, the fluctuation of light output has to be monitored during the performance of the experiment and a correction of the experimental results has to be made.

3. The laser should be lighting all the time, even when you finish your experiment and leave the examination hall, the laser should be keeping in work.

4. The reflected light is totally plane polarized at an incident angle θ_B while $\mathrm{tg}\,\theta_B = n$ (refractive index).

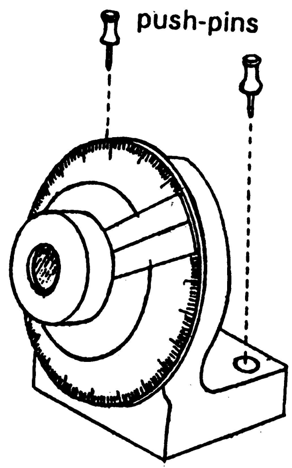


Fig. 1 polarizers with degree scale disk

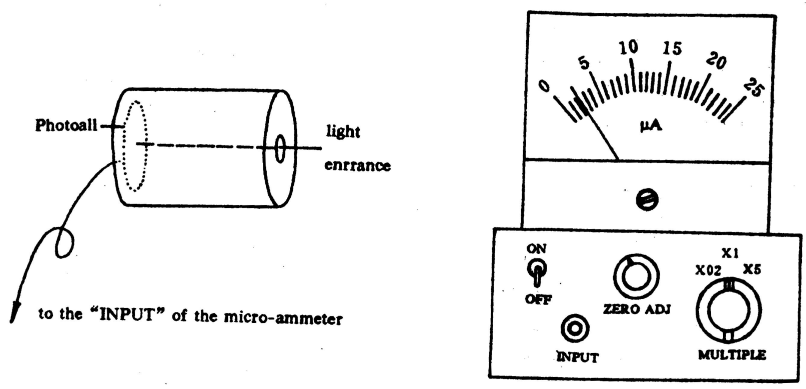


Fig. 2 Light intensity detector

(1) Insert the plug of photocell into the “INPUT” socket of microammeter

(2) Switching on the microammeter.

(3) Blocd off the light entrance hole in front of the photocell and adjust the scale reading of micro ammeter to “0”.

(4) Set the “Multiple” knob to a proper range.

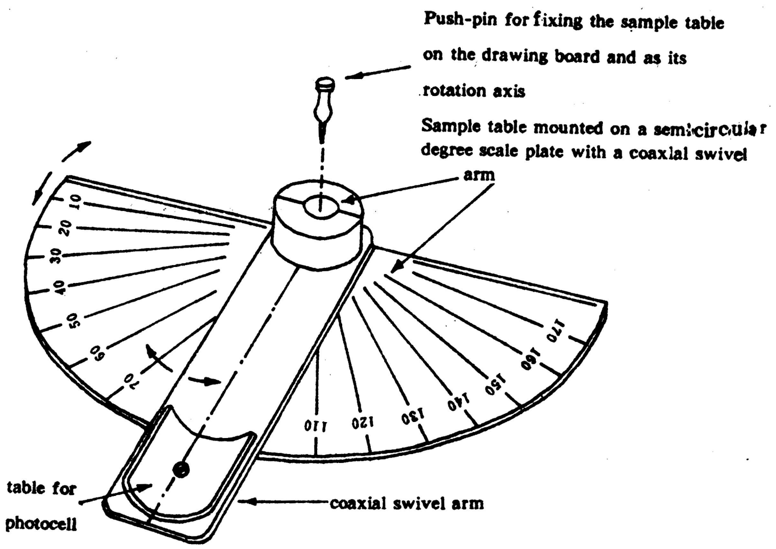
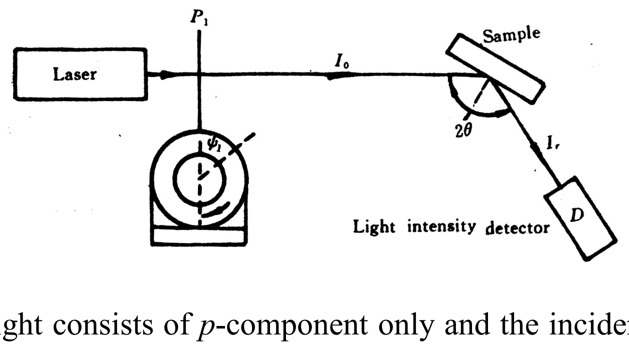


Fig.3 Sample table mounted on a semicircular degree scale plate

Experimental Problem 1——Solution

1. (a) Determine the transmission axis of the polarizer and the Brewster angle θ_B of the sample by using the fact that the rerlectivity of the *p*-component $R_p = 0$ at the Brewster angle.

Change the orientation of the transmission axis of P_1, specified by the position of the marked line on the degree scale disk (ψ) and the incident angle (θ_i) successively until the related intensity $I_r = 0$.



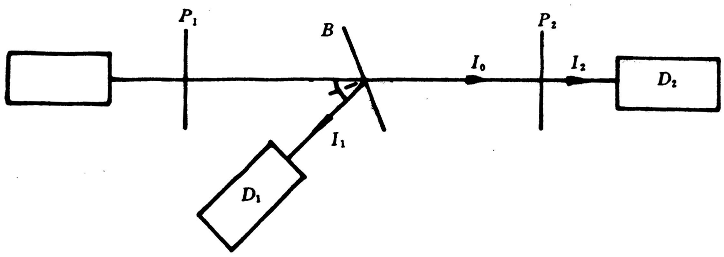
Now the incident light consists of *p*-component only and the incident angle is θ_B, the corresponding values ψ_1 and θ_B are shown below:

ψ_1	140.0°	322.0°	141.0°	322.5°
θ	56.4°	56.4°	56.2°	56.2°

$\psi_1 = 140.5° \pm 0.5°$ or 322.3° ±0.1°

The Brewster angle θ_B is 56.3° ±0.1°

1. (b) Verification of the linear relationship between the light intensity and the microammenter reading.



The intensity the transmitted light passing through two polarized P_1 and P_2 obeys Malus' law

$$I(\theta) = I_0 \cos^2 \theta$$

where I_0 is the intensity of the light polarized by p_1 and incident, I is the intensity of the transmitted light, and θ is the angle between the transmission axes of P_1 and p_2. Thus we can obtain light with various intensities for the verification by using two polarizers.

The experimental arrangement is shown in the figure.

The light intensity detector D_1 serves to monitor the intensity fluctuation of the incident beam (the ratio of I_1 to I_2 remain unchanged), and D_2 measures I_2. Let $i_1(\theta)$ and $i_2(\theta)$ be the readings of D_1 and D_2 respectively, and $\psi_2(\theta)$ be the reading of the marked line position. $i_2 = 0$ when $\theta = 90$ ° , the corresponding ψ_2 is $\psi_2(90$°), and the value of θ corresponding to ψ_2 is

$$\theta = |\psi_2 - \psi_2(90°) \pm 90°|$$

Data and results;

$$\psi_2(90°) = 4°$$

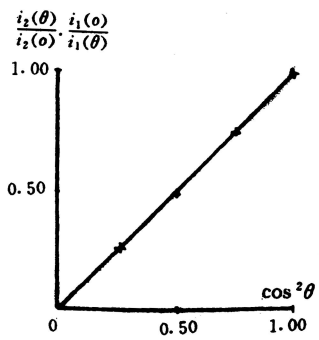
ψ_2	94.0°	64.0°	49.0°	34.0°	4.0°
θ	0.0°	30.0°	45.0°	60.0°	90.0°
$i_1(\theta)\mu A$	6.3×1	5.7×1	5.7×1	5.7×1	5.7×1
$i_2(\theta)\mu A$	18.7×5	12.7×5	8.2×5	4.0×5	0.0×5

From the above data we can obtain the values of $I(\theta)/I_2(\theta)$ from the formula

$$\frac{I(\theta)}{I_0} = \frac{i_2(\theta)}{i_1(\theta)} \cdot \frac{i_1(0)}{i_2(0)}$$

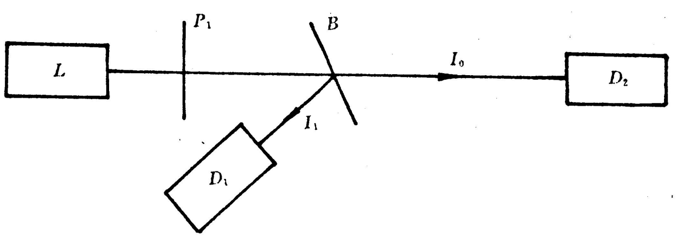
and compare them with $\cos^2\theta$ for examining the linear relationship. The results obtained are:

θ	0.0°	30.0°	45.0°	60.0°	90.0°
$\cos^2\theta$	1.00	0.75	0.50	0.25	0.00
$I(\theta)/I_0$	1.00	0.75	0.49	0.24	0.00

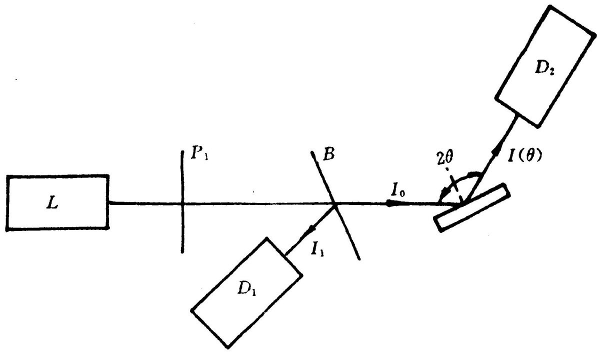


1. (c) Reflectivity measurement

The experimental arrangement shown below is used to determine the ratio of I_0 to I_1 which is proportional to the ratio of the reading (i_{20}) of D_2 to the corresponding reading (i_{10}) of D_1.



Then used the experimental arrangement shown below to measure the relativity R_p of the sample at various incident angle (θ) while the incident light consists of p-component only. Let $i_1(\theta)$ and $i_2(\theta)$ be the readings of D_1 and D_2 respectively.



Then the reflectivity is

$$R_p(\theta) = \frac{I(\theta)}{I_0} = \frac{i_2(\theta)}{i_1(\theta)} \cdot \frac{i_{10}}{i_{20}}$$

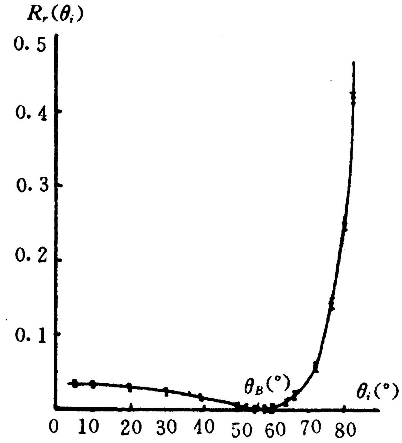
Data and results:

$$\psi_1 = 140.5°$$
$$i_{20} = 19.8 \times 5 \mu A$$
$$i_{10} = 13.3 \mu A$$

θ(°)	$i_2(\theta)$	$i_1(\mu A)$	$R_p(\theta)$
5	15.1×0.2	11.1	0.037
10	14.9×0.2	11.2	0.036
20	13.3×0.2	11.1	0.032
30	11.4×0.2	12.2	0.025
40	7.8×0.2	14.7	0.014
50	2.3×0.2	16.9	0.0037
53	0.7×0.2	11.3	0.0017
55	0.3×0.2	11.3	0.00059
56.3（dark）	～0	11.5	～0
58	0.3×0.2	11.5	0.0007
60	1.1×0.2	13.5	0.0024
64	6.5×0.2	16.7	0.011
66	7.8×0.2	11.8	0.018
68	16.3×0.2	15.0	0.029

72	5.3×0.1	11.7	0.061
76	13.1×1	14.0	0.13
80	4.4×5	11.7	0.25
84	9.1×5	14.5	0.42

The curve of reflectivity of p-component as a function of incident in plexiglass



2. The Brewster angle θ_B can be found from the above date as

$$\theta_B = 56.3° \pm 0.2°$$

The index of refraction can be calculated as

$$n = \tan\theta_B = 1.50 \pm 0.01$$

The sources of errors are:

1. Detector sensitivity is low.
2. The incident light does not consist of p-component only.
3. The degree scales are not uniform.

EXPERIMENTAL PROBLEM 1: Grading Scheme(10 points)

Part 1. Reflectivity of the p-component. 7 points, distributed as follows.

a. Determination of the transmission axis of the polarizer (A) in p-component measurement, 1 point.

(Error less than ±2°, 1.0point;
error less than ±3°, 0.7point;
error less than ±4°, 0.3point;
error less than ±5°, 0.1 point.)

b. Verification of the linearity of the light intensity detector(2 points). Draws the optical schematic diagram correctly, 1.0 point; (Without the correction of the fluctuation of the light intensity, 0.4 point only);

Uses $I/I_0 \sim \cos^2\theta$ figure to show the "linearity", 0.5 point;

Tabulate the measured data(with 5 points at least)correctly, 0.5 point.

c. Determination of the reflectivity of the p-component of the light as a function of incident angle, 4 points, distributed as follows.

Draws the optical schematic diagram correctly and tabulate the measured data perfectly, 2.0 points;

Plot the reflectivity as the function of incident angle with indication of errors, 2 points.

Part 2. Determination of the refractive index of sample, 3 point.

Brewster angle of sample, 1 point;

(Error less than ±1°, 1.0point;
error less than ±2°, 0.5point;
error less than ±3°, 0.2point;
error larger than ±3°, 0 point.)

The refractive index of sample, 0.5 point.

Discussion and determination of errors, 1.5 points.

EXPERIMENTAL PROBLEM 2

Black Box

Given a black box with two similar terminals. There are no more than three passive elements inside the black box. Find the values of elements in the equivalent circuit between the terminals. This box is not allowed to be opened.

Experimental Apparatus

1. Double channel oscilloscope with a panel illustration, showing the name and function of each knob
2. Audio frequency signal generator with a panel illustration, showing the name and function of each knob
3. Resistance box with a fixed value of l00 ohm(< ±0.5%)
4. Several connecting wires
5. For the coaxial cables, the wire in black color at the terminal is grounded.
6. Log-log paper, semi-log paper, and millimeter paper are provided for use if necessary

Note: The knobs, which were not shown on the panel illustration of the "signal generator" and "oscilloscope", have been set to the correct positions. It should not be touched by the student.

Experimental Requirements

1. Draw the circuit diagram in your experiment.

2. Show your measured data and the calculated results in the form of tables. Plot the experimental curves with the obtained results on the coordinate charts provided(indicate the title of the diagram and the titles and scale units of the coordinate axes)
3. Given the equivalent circuit of the black box and the names of the elements with their values in the equivalent circuit(write down the calculation formulas).

Instructions

1. Do your experiment in the frequency range between 100 Hz and 50kHz.
2. The output voltage of the signal generator should be less than 1.0V (peak-to-peak). Set the "Out Attenuation" switch to "20" db position and it should not be changed.
3. On connecting the wires, be careful to manage the wiring so as to minimize the 50Hz interference from the electric mains.

Instruction for Using XD2 Type Frequency Generator

1. Set the "Out Attenuation" to "20" db position and it should not be changed.
2. Set the "Damping Switch" to "Fast" position.
3. The indication of the voltmeter of the signal generator is the relative value, but not the true value of the output.
4. Neglect the error of the frequency readings.

Note: For XD22 Type Audio Frequency generator, there is no "Damping Switch", and the "output" switch should be set to the sine "~" position.

Instruction for Using SS-5702 Type Oscilloscope

1. Keep the "V mode" switch in "Dual" position.
2. The "Volts/div" (black) and the "variable control" (red) vary the gain of the vertical amplifier, and when the "variable control" (red) is ill the fully clockwise position, the black setting are calibrated.
3. The "Times/div" (Black) varies the horizontal sweep rate from 0.5 μ s/div to 0.2s/div, and they are calibrated when the "variable control" (red) is in the fully clockwise CAL position.
4. The "Trigging Source" (Trigging sweep signal) is used to select the trigging signal channel and the" level" control is used to adjust the amplitude of the trigging signal.
5. Measuring accuracy: ±4%.

Instruction for Using "Resistance Box"

The resistance of the "Resistance Box" has been set to a value of 100ohm, and it should not be changed.

Experimental problem 2...... Solution

1. The circuit diagram is shown in Fig. 1

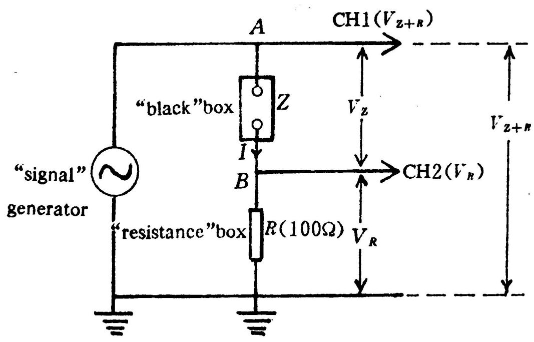


Fig. 1

We have the relation:

$$I = \frac{V_R}{R};$$

$$Z + R = \frac{V_{Z+R}}{I} = \frac{V_{Z+R}}{V_R} R$$

2. Measure the values of V_{Z+R} and V_R at various frequencies (f), the measured data and calculated value of Z+R are shown in table l. "The Z+R-f curve is plotted in Fig. 2

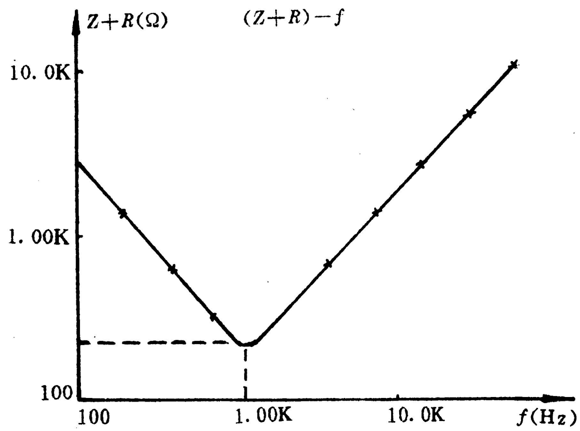


Table l. The magnitude of impedance verus frequency

$f(\times 10^3\text{ Hz})$	$U_{Z+R}(V_{pp})$	U_R mV$_{pp}$	$Z+R(\times 10^3\,\Omega)$
0.100	0.600	22.0	2.73
0.200	0.600	45.0	1.33
0.400	0.600	94.0	0.638
0.700	0.300	92.0	0.326
0.900	0.300	121	0.248

1.00	0.300	136	0.220
1.10	0.300	140	0.214
1.16	0.300	141	0.213
1.25	0.300	140	0.214
1.50	0.300	120	0.250
2.00	0.300	88.0	0.341
4.00	0.300	78.0	0.769
8.00	0.600	38.0	1.58
15.0	0.600	20.0	3.00
30.0	0.600	10.0	6.00
50.0	0.600	6.0	10.0

From table 1 and Fig. 2, we got the conclusions:

(1) Current resonance (minimum of *Z*) occurs at $f_0 \cong 1.16\times10^3$ Hz.

(2) $f \langle\langle f_0$, $Z \propto f$, $\Delta\varphi \approx -\pi/2$. The impedance of the "black box" at low frequency is dominated by a inductance.

(3) $f \rangle\rangle f_0$, $Z \propto f$, $\Delta\varphi \approx \pi/2$. The impedance of the "black box" at high frequency is dominated by a inductance.

(4) Equivalent circuit of the "black box"; *r*, *L* and *C* connected in series shown in Fig. 3.

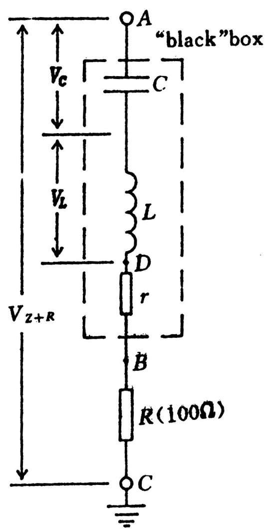


Fig. 3

3. Determination of the values of r, L and C.

(a) r

At resonance frequency f_0

$$V_C = -V_L$$

Then

$$Z + R = \frac{V_{Z+R}}{I} = \frac{V_{Z+R}}{V_R} R = r + R$$

From table 1, $r + R = 213\Omega$, it is given $R = 100\Omega$, so the equivalent resistance r in Fig. 3 is equal $113\,\Omega$.

(b) C

At low frequency, $z_L \approx 0$ in Fig. 3. So the circuit could be considered as a series RC circuit.

From phasor diagram, Fig. 4,

$$\frac{1}{\omega C} = Z_C = \frac{V_C}{I} = \frac{\sqrt{V_{Z+R}^2 - V_{R+r}^2}}{I}$$

Since $V_{R+r}^2 / V_{Z+R}^2 \approx 6 \times 10^{-3}$ at $f = 100$ Hz, V_{R+r}^2 can beneglected with respect to V_{Z+R}^2, so

$$\frac{1}{\omega C} \approx \frac{V_{Z+R}}{I} \approx Z + R = 2.73 \times 10^3\,\Omega$$

$$C \approx \frac{1}{\omega(Z+R)} = 0.58\mu f\ .$$

$$C \cong 0.58\mu f\ .$$

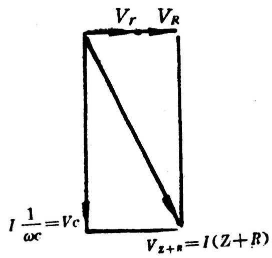


Fig. 4

(c) L

At high frequency, $Z_L \approx 0$ in Fig. 3. So the circuit could be considered as a series RL circuit.

From phasor diagram, Fig. 5,

$$|V_L| = \sqrt{V_{Z+R}^2 - V_{r+R}^2}\ ,$$

Since $V_{r+R}^2 / V_{Z+R}^2 \approx 4.5 \times 10^{-4}$ at $f = 50$ kHz, V_{r+R}^2 can be neglected with respect to V_{Z+R}^2, so

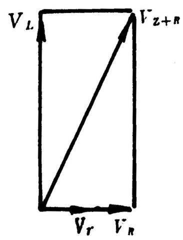


Fig. 5

$$\omega L = Z_L = \frac{V_L}{I} = \frac{|V_{Z+R}|}{I} \approx Z + R = 10^4\,\Omega \tag{3}$$

$$L = \frac{Z + R}{\omega} = 31.8\,\text{mH}.$$

Error estimation:

It is given, precision of the resistance box reading $\Delta R / R \approx 0.5\%$

precision of the voltmeter reading $\Delta V / V \approx 4\%$

(1) Resistance r : at resonance frequency f_0

$$r + R = \frac{V_{Z+R}}{V_R} R$$

$$\frac{\Delta(r+R)}{r+R} = \frac{\Delta V_{Z+R}}{V_{Z+R}} + \frac{\Delta V_R}{V_R} + \frac{\Delta R}{R} \approx 4\% + 4\% + 0.5\% = 8.5\%$$

$$\Delta r = 16\Omega$$

(2) Capacitance C: (Neglect the error of the frequency reading)

$$\frac{1}{\omega C} \cong Z_C = \frac{V_{Z+R}}{V_R} R$$

$$\frac{\Delta C}{C} = \frac{\Delta V_{Z+R}}{V_{Z+R}} + \frac{\Delta V_R}{V_R} + \frac{\Delta R}{R} \approx 8.8\%$$

The approximation $V_C \approx V_{Z+R}$ will introduce apercentageerror 0.3%

(3) Inductance L: Similar to the results of capacitance C, but the percentage error introduced by the approximation $V_L \approx V_{Z+R}$ is much small (0.003%) and thus negligible.

$$\frac{\Delta L}{L} \approx 8.5\%.$$

Experimental Problem 2: Grading Scheme (10 points maximum)

1. Measuring circuit is correct as shown in Fig.(a)

……2.0point

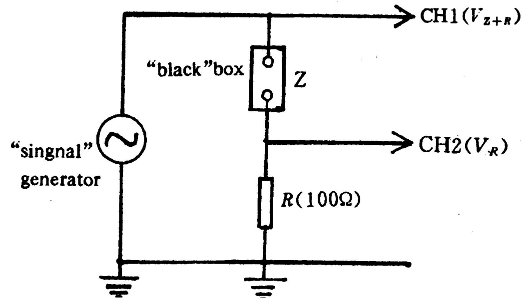


Fig. a

2. Correct data table and figure to show the characteristic of the black box

······2.0 points

3. The equivalent circuit of the black box, and the names of the elements with their values in the equivalent circuit are correct

total 6.0 points

(a) R, L and C are connected in series

······1.5 point

(L and C are connected in series

······1.0 point)

(b) Correct value (error less than 15%) for each element

······0.5 point (×3)

(error between 15% and 30% 0.3)

(error between 30% and 50% 0.1)

(c) Correct calculation formula for each element

······0.5 point (×3)

(d) Error estimate is reasonable for each element

······0.5 points (×3)

Experimental Question 1

Terminal velocity in a viscous liquid

An object falling in a liquid will eventually reach a constant velocity, called the *terminal velocity*. The aim of this experiment is to measure the terminal velocities of objects falling through glycerine.

For a sphere of radius r falling at velocity v through a viscous liquid, the viscous force F is given by $F = 6\pi\eta rv$. Here η is a property of the liquid called the *viscosity*. In this experiment you will measure the terminal velocity of metal cylinders (because cylinders are easier to make than spheres). The diameter of each cylinder is equal to its length, and we will assume the viscous force on such a cylinder is similar to the viscous force on a sphere of the same diameter, $2r$:

$$F_{cyl} = 6\pi\kappa\eta r^m v \tag{1}$$

where $\kappa = 1$, $m = 1$ for a sphere.

Preliminary

Calculation of terminal velocity (2 marks)

If ρ is the density of the culinder and ρ' is the density of the liquid, show that the terminal velocity v_T of the cylinder is given by

$$v_T = Cr^{3-m}(\rho - \rho') \tag{2}$$

where C is a constant and derive a expression for C.

Experiment

Use the equipment available to determine the numerical value of the exponent m (10 marks) and the density of glycerine (8 marks).

Notes

- For consistency, try to ensure that the cylinders fall in the same orientation, with the axis of the cylinder horizontal.
- The tolerances on the diameter and the length of the cylinders are 0.05 mm (you need not measure them yourself).
- There is a brass sieve inside the container that you should use to retrieve the metal cylinders. Important: make sure the sieve is in place before dropping objects into the glycerine, otherwise you will not be able to retrieve them for repeat measurements.
- When glycerine absorbs water from the atmosphere, it becomes less viscous. Ensure that the cylinder of glycerine is covered with the plastic film provided when not in use.
- Do not mix cylinders of different size and different material after the experiment.

Material	**Density ($\mathbf{kgm^{-3}}$)**
Aluminium	2.70×10^3
Titanium	4.54×10^3
Stainless steel	7.87×10^3
Copper	8.96×10^3

Solution to Experimental Question 1

Preliminary: Calculation of Terminal Velocity

When the cylinder is moving at its terminal velocity, the resultant of the three forces acting on the cylinder, gravity, viscous drag and buoyant force, is zero.

$$V\rho g - 6\pi\kappa\eta r^m v_T - V\rho' g = 0$$

where $V = 2\pi r^3$ is the volume of a cylinder (whose height is $2r$).

This gives

$$v_r = Cr^{3-m}(\rho - \rho')$$

where

$$C = \frac{g}{3\kappa\eta}$$

Experiment

Determination of the exponent m

Aluminium cylinders of different diameters are dropped into the glycerine. Fall times between specified marks on the measuring cylinder containing the glycerine are recorded for each cylinder. A preliminary experiment should establish that the cylinders have reached their terminal velocity before detailed results are obtained. The measurements are repeated several times for each cylinder and an average fall time is calculated. Table 1 shows a typical set of data. To find the value of m a graph of log(fall time) as a function of log(diameter) is plotted as in figure 1. The slope of the resulting straight line graph is $3 - m$ from which a value of m can be determined. A reasonable value for m is 1.33 with an uncertainty of order ± 0.1. The uncertainty is estimated by the deviation from the line of best fit through the data points obtained by drawing other possible lines.

Determination of the density of glycerine

Cylinders with the same geometry but different densities are dropped into the glycerine and timed as in the first part of the experiment. Table 2 shows a typical set of results. From equation (2) a linear plot of $1/t$ as a function of density should yield a straight line with an intercept on the density axis corresponding to the density of glycerine. Figure 2 shows a typical plot. Alternatively the terminal velocities could be calculated and plotted against density which would again lead to the same intercept on the density axis. The uncertainty in the measurement can be estimated by drawing other possible straight lines through the data points and noting the change in the value of the intercept.

Diameter (mm)	Individual readings (s)						Mean (s)
10	1.44	1.56	1.44	1.37	1.44	1.41	1.44
4	6.22	6.06	6.16	6.13	6.13	6.22	6.15
8	1.80	1.82	1.78	1.84	1.82	1.81	1.82
5	4.06	4.34	4.09	4.12	4.25	4.13	4.13

Table 1: Sample data set

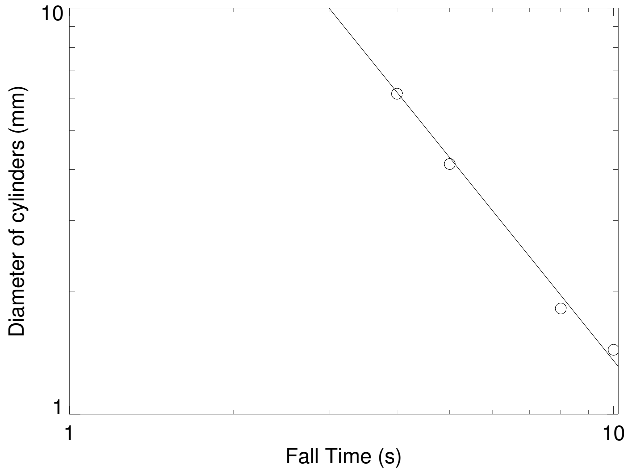


Figure 1: Sample plot

$$\text{Slope } = -\frac{58.2}{66.2} \div \frac{48.5}{93} = -1.67 \qquad \therefore m = 3 - 1.67 = 1.33$$

Material	Individual readings (s)						Mean (s)
Ti	3.00	2.91	2.97	2.91	2.84	2.75	2.91
Cu	1.25	1.25	1.28	1.25	1.22	1.22	1.25
S.Steel	1.31	1.32	1.38	1.44	1.31	1.34	1.33
Al	6.03	6.09	6.09	6.16	6.06	6.06	6.08

Table 2: Sample data set

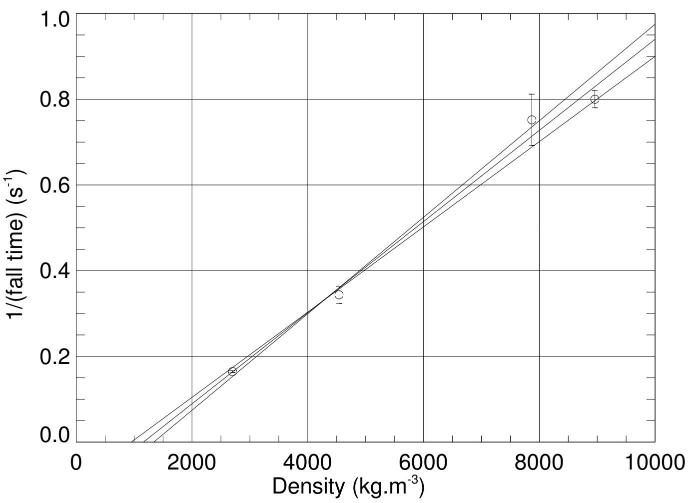


Figure 2: Sample plot

$$\rho' = (1.1 \pm 0.2) \times 10^3 \text{ kg.m}^{-3}$$

Detailed mark allocation

Section I	
Reasonable range of data points with a scatter of ~ 0.1 s	[2]
Check that the cylinders have reached their terminal velocity	
Visual check, or check referred to	[1]
Specific data presented	[1]
Labelled log-log graph	[2]
Data points for all samples, with a reasonable scatter about a straight line on the log-log graph	[1]
Calculation of $(3 - m)$ from graph	[1]
including estimate of error in determining m	[1]
Reasonable value of m, ~ 1.33	[1]
Subtotal	[10]
Section 2	
Reasonable range of data points	[1]
Check that the cylinders have reached their terminal velocity	[1]
Labelled graph of $(\text{falltime})^{-1}$ vs. density of cylinder	[1]
Data points for all samples, with a reasonable scatter about a straight line on the $(\text{falltime})^{-1}$ vs. density of cylinder graph	[1]
Calculation of the density of glycerine (ρ') from this graph	[1]
Estimate of uncertainty in ρ'	[1]
Reasonable value of ρ'. "Correct" value is 1.260 kg.m^{-3}	[1]
Subtotal	[8]
TOTAL	20

Experimental Question 2

Diffraction and Scattering of Laser Light

The aim of this experiment is to demonstrate and quantify to some extent the reflection, diffraction, and scattering of light, using visible radiation from a Laser Diode source. A metal ruler is employed as a diffraction grating, and a perspex tank, containing water and diluted milk, is used to determine reflection and scattering phenomena.

Section 1 (6 marks)

Place the 150 mm length metal ruler provided so that it is nearly normal to the incident laser beam, and so that the laserr beam illuminates several rulings on it. Observe a number of "spots" of light on the white paper screen provided, caused by the phenomenon of diffraction.

Draw the overall geometry you have employed and measure the position and separation of these spots with the screen at a distance of approximately 1.5 metres from the ruler.

Using the relation

$$N\lambda = h \sin\beta$$

where N is the order of diffraction
λ is the radiation wavelength
h is the grating period
β is the angle of diffraction

and the information obtained from your measurements, determine the wavelength of the laser radiation.

Section 2 (4 marks)

Now insert the empty perspex tank provided into the space between the laser and the white paper screen. Set the tank at approximately normal incidence to the laser beam.

(i) Observe a reduction in the emergent beam intensity, and estimate the percentage value of this reduction. Some calibrated transmission discs are provided to assist with this estimation. Remember that the human eye has a logarithmic response.

This intensity reduction is caused primarily by reflection losses at the aid/perspex boundaries, of which there are four in this case. THe reflection coefficient for normal incidence at each boundary, R, which is the ratio of the reflectied to incident intensities, is given by

$$R = \{(n_1 - n_2)/(n_1 + n_2)\}^2$$

where n_1 and n_2 are the refractive indices before and after the boundary. The corresponding transmission coefficient, assuming zero absorption in the perspex, is fiven by

$$T = 1 - R \quad .$$

(ii) Assuming a refractive index of 1.59 for the perspex and neglecting the effect of multiple reflections and cogerence, calculate the intensity transmission coefficient of the empty perspex tank. Compare this result with the estimate you made in Part (i) of this Section.

Section 3 (1 mark)

Without moving the perspex tank, repeat the observations and calculations in Section 2 with the 50 mL of water provided in a beaker now added to the tank. Assume the refractive index of water to be 1.33.

Section 4 (10 marks)

(i) Add 0.5 mL (12 drops) of milk (the scattering material) to the 50 mL of water in the perspex tank, and stir well. Measure as accurately as possible the total angle through which the laser light is scattered, and the diameter of the emerging light patch at the exit face of the tank, noting that these quantities are related. Also estimate the reduction in transmitted intensity, as in earlier sections.

(ii) Add a further 0.5 mL of milk to the tank, and repeat the measurements requested in part (i).

(iii) Repeat the process in part (ii) until very little or no transmitted laser light can be observed.

(iv) Determine the relationship between scattering angle and milk concentration in the tank.

(v) Use your results, and the relationship

$$I = I_0 e^{-\mu z} = T_{milk} \times I_0$$

where

I_0	is the input intensity
I	is the emerging intensity
z	is the distance in the tank
μ	is the attenuation coefficient and equals a constant times the concentration of the scatterer
T_{milk}	is the transmission coefficient for the milk

to obtain an estimate for the value of μ for a scatterer concentration of 10%.

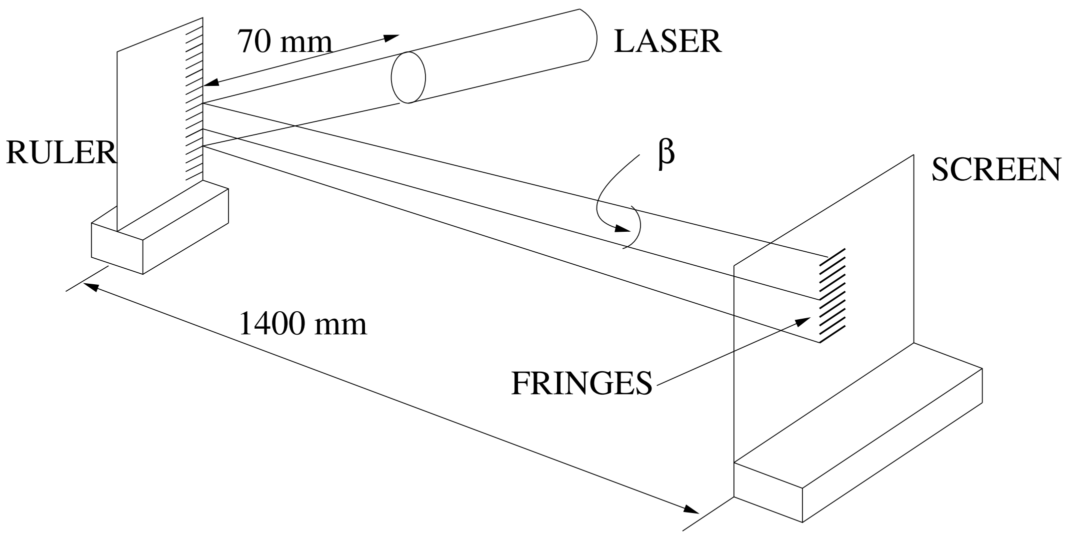
Solution to Experimental Question 2

Section 1

i. A typical geometric layout is as shown below.

(a) Maximum distance from ruler to screen is advised to increase the spread of the diffraction pattern.

(b) Note that the grating (ruler) lines are horizontal, so that diffraction is in the vertical direction.



ii. Vis a vis the diffraction phenomenon, $\beta = \left(\frac{y}{1400 \text{ mm}}\right)$

The angle β is measured using either a protractor (not recommended) or by measuring the value of the fringe separation on the screen, y, for a given order N.

If the separation between 20 orders is measured, then $N = \pm 10$ ($N = 0$ is central zero order).

The values of y should be tabulated for $N = 10$. If students choose other orders, this is also acceptable.

N	± 10	± 10	± 10	± 10	± 10	± 10	± 10	± 10	± 10	± 10
$2y$ mm	39.0	38.5	39.5	41.0	37.5	38.0	39.0	38.0	37.0	37.5
y mm	19.5	19.25	19.75	20.5	18.75	19.0	19.5	19.0	18.5	18.75

Mean Value $= (19.25 \pm 1.25)$ mm

i.e. Mean "spot" distance $= 19.25$ mm for order $N = 10$.

From observation of the ruler itself, the grating period, $h = (0.50 \pm 0.02)$ mm.

Thus in the relation

$$\begin{aligned}
N\lambda &= \pm h \sin\beta \\
N &= 10 \\
h &= 0.5 \text{ mm} \\
\sin\beta \simeq \beta &= \frac{y}{1400 \text{ mm}} = 0.01375 \\
10\lambda &= 0.006875 \text{ mm} \\
\lambda &= 0.0006875 \text{ mm}
\end{aligned}$$

Since β is small, $\dfrac{\delta\lambda}{\lambda} \simeq \dfrac{\delta h}{h} + \dfrac{\delta y}{y} \simeq 10\%$

i.e. measured $\lambda = (690 \pm 70)$ nm

The accepted value is 680 nm so that the departure from accepted value equals 1.5%.

Section 2

This section tests the student's ability to make semi-quantitative measurements and the use of judgement in making observations.

i. Using the $T = 50\%$ transmission disc, students should note that the transmission through the tank is greater than this value. Using a linear approximation, 75% could well be estimated. Using the hint about the eye's logarithmic response, the transmission through the tank could be estimated to be as high as 85%.

   Any figure for transmission between 75% and 85% is acceptable.

ii. Calculation of the transmission through the tank, using

$$T = 1 - R = 1 - \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2$$

   for each of the four surfaces of the tank, and assuming $n = 1.59$ for the perspex, results in a total transmission

$$T_{\text{total}} = 80.80\%$$

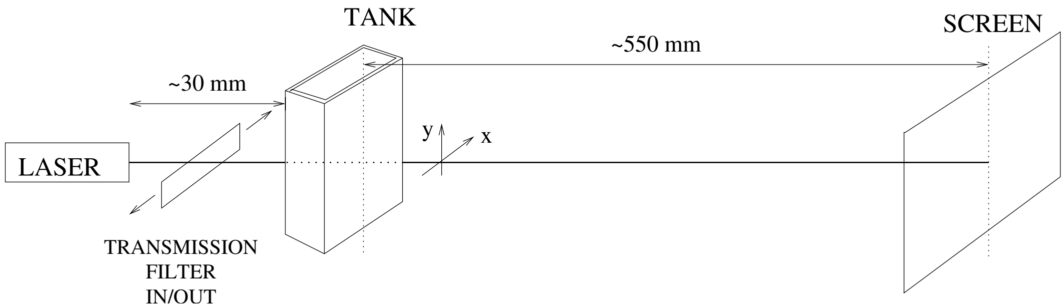
Section 3

With water in the tank, surfaces 2 and 3 become perspex/water interfaces instead of perspex/air interfacs, as in (ii).

The resultant value is

$$T_{\text{total}} = 88.5\%$$

Section 4

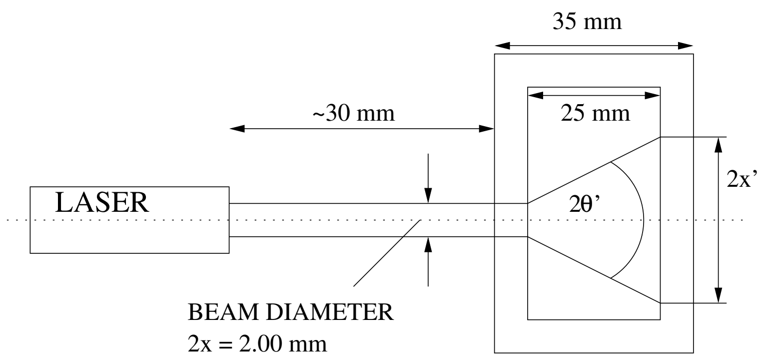


Possible configuration for section 4 (and sections 2 and 3)

With pure water in the tank only, we see from Section 3 that the transmission T is

$$T_{\text{Water}} \simeq 88\%$$

The aim here is to determine the beam divergence (scatter) and transmission as a function of milk concentration. Looking down on the tank, one sees



i. The entrance beam diameter is 2.00 mm. The following is an example of the calculations expected: With 0.5 mL milk added to the 50 mL water, we find

$$\text{Scatterer concentration} = \frac{0.5}{50} = 1\% = 0.01$$

Scattering angle

$$2x' = 2.2 \text{ mm} \quad ; \quad 2\theta' = \frac{2x'}{30} = 0.073$$

Transmission estimated with the assistance of the neutral density filters

$$T_{\text{total}} = 0.7 \quad .$$

Hence

$$T_{\text{milk}} = \frac{0.7}{0.88} = 0.79$$

Note that

$$T_{\text{milk}} = \frac{T_{\text{total}}}{T_{\text{water}}} \quad \text{and} \quad T_{\text{water}} = 0.88 \tag{1}$$

If students miss the relationship (1), deduct one mark.

ii. & iii. One thus obtains the following table of results. $2\theta'$ can be determined as shown above, OR by looking down onto the tank and using the protractor to measure the value of $2\theta'$. It is important to note that even in the presence of scattering, there is still a direct beam being transmitted. It is much stronger than the scattered radiation intensity, and some skill will be required in measuring the scattering angle $2\theta'$ using either method. Making the correct observations requires observational judgement on the part of the student.

Typical results are as follows:

Milk volume (mL)	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
% Concentration	0	1	2	3	4	5	6	7	8
$2x'$	2.00	2.2	6.2	9.4	12	Protractor			
$2\theta'$ (Degrees)	~ 0	4	12	18	23	28	36	41	48
T_{milk}	1.0	0.79	0.45	0.22	0.15	0.12	0.08	0.06	0.05

iii. From the graphed results in Figure 1, one obtains an approximately linear relationship between milk concentration, C, and scattering angle, $2\theta'$ ($= \phi$) of the form

$$\phi = 6C \quad .$$

iv. Assuming the given relation

$$I = I_0 e^{-\mu z} = T_{\text{milk}} I_0$$

where z is the distance into the tank containing milk/water.

We have

$$T_{\text{milk}} = e^{-\mu z}$$

Thus

$$\ln T_{\text{milk}} = -\mu z \quad \text{, and} \quad \mu = \text{constant} \times C$$

Hence $\ln T_{\text{milk}} = -\alpha z C$.

Since z is a constant in this experiment, a plot of $\ln T_{\text{milk}}$ as a function of C should yield a straight line. Typical data for such a plot are as follows:

% Concentration	0	1	2	3	4	5	6	7	8
T_{milk}	1.0	0.79	0.45	0.22	0.15	0.12	0.08	0.06	0.05
$\ln T_{\text{milk}}$	0	-0.24	-0.8	-1.51	-1.90	-2.12	-2.53	-2.81	-3.00

An approximately linear relationship is obtained, as shown in Figure 2, between $\ln T_{\text{milk}}$ and C, the concentration viz.

$$\ln T_{\text{milk}} \simeq -0.4C = -\mu z$$

Thus we can write

$$T_{\text{milk}} = e^{-0.4C} = e^{-\mu z}$$

For the tank used, $z = 25$ mm and thus

$$0.4C = 25\mu \quad \text{or} \quad \mu = 0.016C \quad \text{whence} \quad \alpha = 0.016 \text{ mm}^{-1}\%^{-1}$$

By extrapolation of the graph of $\ln T_{\text{milk}}$ versus concentration C, one finds that for a scatterer concentration of 10%

$$\mu = 0.160 \text{ mm}^{-1} \quad .$$

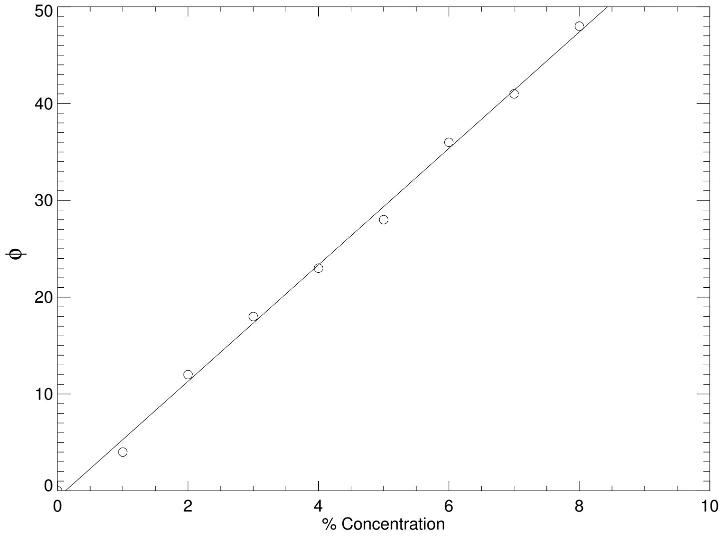


Figure 1: Sample plot

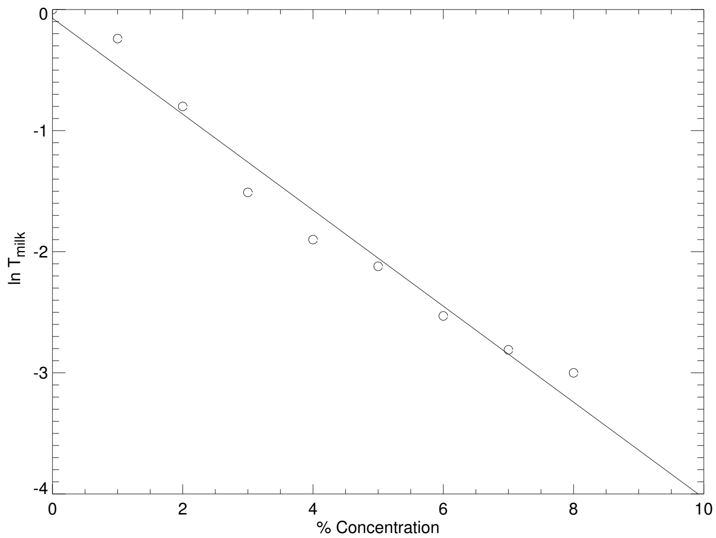


Figure 2: Sample plot

Detailed Mark Allocation

Section 1	
A clear diagram illustrating geometry used with appropriate allocations	[1]
Optimal geometry used - as per model solution (laser close to ruler)	[1]
Multiple measurements made to ascertain errors involved	[1]
Correctly tabulated results	[1]
Sources of error including suggestion of ruler variation (suggested by non-ideal diffraction pattern)	[1]
Calculation of uncertainty	[1]
Final result	[2]
Allocated as per:	
$\pm 10\%$ (612, 748 nm)	[2]
$\pm 20\%$ (544, 816 nm)	[1]
$\pm$ anything worse	[0]
Section 2	
For evidence of practical determination of transmission rather than simply "back calculating". Practical range $70 - 90\%$	[1]
For correct calculation of transmission (no more than 3 significant figures stated)	[1]
Section 3	
Correct calculation with no more than 3 significant figures stated and an indication that the measurement was performed	[1]
Section 4	
Illustrative diagram including viewing geometry used, i.e. horizontal/vertical	[1]
For recognizing the difference between scattered light and the straight-through beam	[1]
For taking the T_{water} into account when calculating T_{milk}	[1]
Correctly calculated and tabulated results of T_{milk} with results within 20% of model solution	[1]
Using a graphical technique for determining the relationship between scatter angle and milk concentration	[1]
Using a graphical technique to extrapolate T_{milk} to 10% concentration	[1]
Final result for μ	[2]
Allocated as $\pm 40\%$ [2], $\pm 60\%$ [1], anything worse [0]	
A reasonable attempt to consider uncertainties	[1]
TOTAL	20

OSLO 1996
27th INTERNATIONAL
PHYSICS OLYMPIAD

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OSLO, NORWAY

THEORETICAL COMPETITION
JULY 2 1996

Time available: 5 hours

READ THIS FIRST :

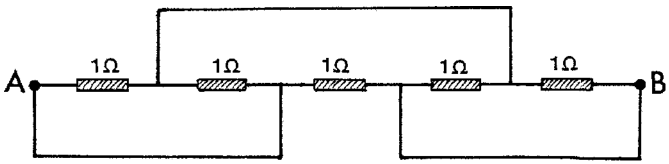
1. Use only the pen provided
2. Use only the marked side of the paper
3. Each problem should be answered on separate sheets
4. In your answers please use primarily equations and numbers, and as little *text* as possible
5. Write at the top of *every* sheet in your report:
   - Your candidate number (IPhO identification number)
   - The problem number and section identification, e.g. 2/a
   - Number each sheet consecutively
6. Write on the front page the total number of sheets in your report

This set of problems consists of 7 pages.

PROBLEM 1

(The five parts of this problem are unrelated)

a) Five 1Ω resistances are connected as shown in the figure. The resistance in the conducting wires (fully drawn lines) is negligible.



Determine the resulting resistance R between A and B. *(1 point)*

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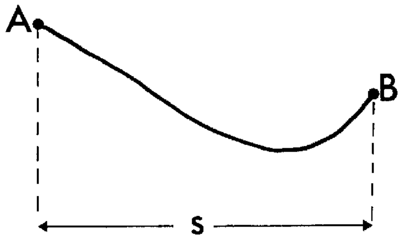
b)



A skier starts from rest at point A and slides down the hill, without turning or braking. The friction coefficient is μ. When he stops at point B, his horizontal displacement is s. What is the height difference h between points A and B? (The velocity of the skier is small so that the additional pressure on the snow due to the curvature can be neglected. Neglect also the friction of air and the dependence of μ on the velocity of the skier.) *(1.5 points)*

---

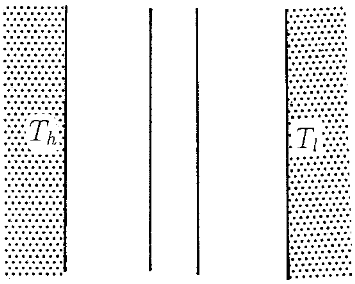
c) A thermally insulated piece of metal is heated under atmospheric pressure by an electric current so that it receives electric energy at a constant power P. This leads to an increase of the absolute temperature T of the metal with time t as follows:

$$T(t) = T_0\left[1 + a(t - t_0)\right]^{1/4}.$$

Here a, t_0 and T_0 are constants. Determine the heat capacity $C_p(T)$ of the metal (temperature dependent in the temperature range of the experiment). *(2 points)*

d) A black plane surface at a constant high temperature T_h is parallel to another black plane surface at a constant lower temperature T_l. Between the plates is vacuum.

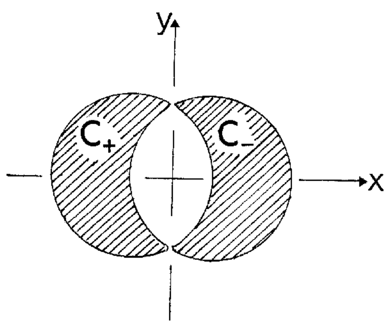
In order to reduce the heat flow due to radiation, a heat shield consisting of two thin black plates, thermally isolated from each other, is placed between the warm and the cold surfaces and parallel to these. After some time stationary conditions are obtained.



By what factor ξ is the stationary heat flow reduced due to the presence of the heat shield? Neglect end effects due to the finite size of the surfaces. *(1.5 points)*

---

e) Two straight and very long nonmagnetic conductors C_+ and C_-, insulated from each other, carry a current I in the positive and the negative z direction, respectively. The cross sections of the conductors (hatched in the figure) are limited by circles of diameter D in the x-y plane, with a distance $D/2$ between the centres. Thereby the resulting cross sections each have an area $(\frac{1}{12}\pi + \frac{1}{8}\sqrt{3})D^2$. The current in each conductor is uniformly distributed over the cross section.

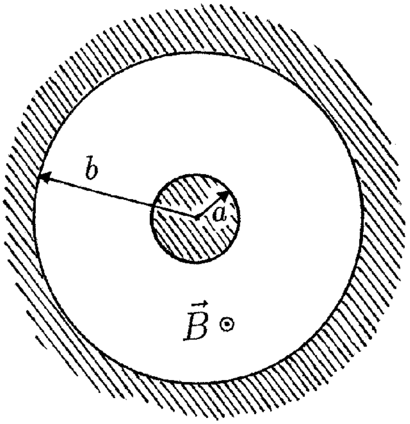


Determine the magnetic field $B(x,y)$ in the space between the conductors. *(4 points)*

PROBLEM 2

The space between a pair of coaxial cylindrical conductors is evacuated. The radius of the inner cylinder is a, and the inner radius of the outer cylinder is b, as shown in the figure below. The outer cylinder, called the anode, may be given a positive potential V relative to the inner cylinder. A static homogeneous magnetic field $\vec{B}$ parallel to the cylinder axis, directed out of the plane of the figure, is also present. Induced charges in the conductors are neglected.

We study the dynamics of electrons with rest mass m and charge $-e$. The electrons are released at the surface of the inner cylinder.



a) First the potential V is turned on, but $\vec{B} = 0$. An electron is set free with negligible velocity at the surface of the inner cylinder. Determine its speed v when it hits the anode. Give the answer both when a non-relativistic treatment is sufficient, and when it is not. *(1 point)*

For the remaining parts of this problem a non-relativistic treatment suffices.

b) Now $V = 0$, but the homogeneous magnetic field $\vec{B}$ is present. An electron starts out with an initial velocity $\vec{v}_0$ in the radial direction. For magnetic fields larger than a critical value B_c, the electron will not reach the anode. Make a sketch of the trajectory of the electron when B is slightly more than B_c. Determine B_c. *(2 points)*

From now on *both* the potential V *and* the homogeneous magnetic field $\vec{B}$ are present.

c) The magnetic field will give the electron a non-zero angular momentum L with respect to the cylinder axis. Write down an equation for the rate of change *dL/dt* of the angular momentum. Show that this equation implies that

$$L - keBr^2$$

is constant during the motion, where k is a definite pure number. Here r is the distance from the cylinder axis. Determine the value of k. *(3 points)*

d) Consider an electron, released from the inner cylinder with negligible velocity, that does not reach the anode, but has a maximal distance from the cylinder axis equal to r_m. Determine the speed v at the point where the radial distance is maximal, in terms of r_m. *(1 point)*

e) We are interested in using the magnetic field to regulate the electron current to the anode. For B larger than a critical magnetic field B_c, an electron, released with negligible velocity, will not reach the anode. Determine B_c.
(1 point)

f) If the electrons are set free by heating the inner cylinder an electron will in general have an initial nonzero velocity at the surface of the inner cylinder. The component of the initial velocity parallel to $\vec{B}$ is v_B, the components orthogonal to $\vec{B}$ are v_r (in the radial direction) and v_φ (in the azimuthal direction, i.e. orthogonal to the radial direction).

Determine for this situation the critical magnetic field B_c for reaching the anode. *(2 points)*

PROBLEM 3

In this problem we consider some gross features of the magnitude of mid-ocean tides on earth. We simplify the problem by making the following assumptions:

(i) The earth and the moon are considered to be an isolated system,
(ii) the distance between the moon and the earth is assumed to be constant,
(iii) the earth is assumed to be completely covered by an ocean,
(iv) the dynamic effects of the rotation of the earth around its axis are neglected, and
(v) the gravitational attraction of the earth can be determined as if all mass were concentrated at the centre of the earth.

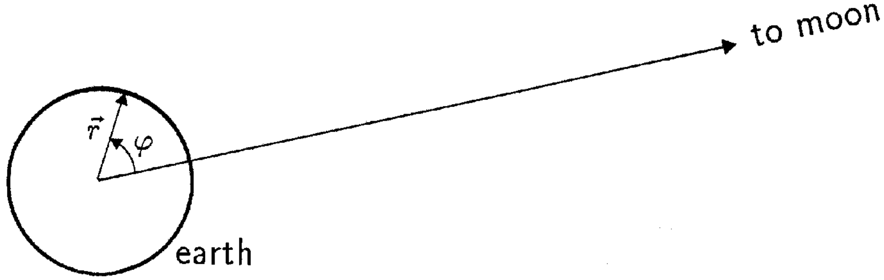
The following data are given:

Mass of the earth: $M = 5.98 \cdot 10^{24}$ kg
Mass of the moon: $M_m = 7.3 \cdot 10^{22}$ kg
Radius of the earth: $R = 6.37 \cdot 10^{6}$ m
Distance between centre of the earth and centre of the moon:
$L = 3.84 \cdot 10^{8}$ m
The gravitational constant: $G = 6.67 \cdot 10^{-11}$ m^3 kg^{-1} s^{-2}.

a) The moon and the earth rotate with angular velocity ω about their common centre of mass, C. How far is C from the centre of the earth? (Denote this distance by l.)

Determine the numerical value of ω. *(2 points)*

We now use a frame of reference that is co-rotating with the moon and the center of the earth around C. In this frame of reference the shape of the liquid surface of the earth is static.



In the plane P through C and orthogonal to the axis of rotation the position of a point mass on the liquid surface of the earth can be described by polar coordinates r, φ as shown in the figure. Here r is the distance from the centre of the earth.

We will study the shape

$$r(\varphi) = R + h(\varphi)$$

of the liquid surface of the earth in the plane P.

b) Consider a mass point (mass m) on the liquid surface of the earth (in the plane P). In our frame of reference it is acted upon by a centrifugal force and by gravitational forces from the moon and the earth. Write down an expression for the potential energy corresponding to these three forces.

Note: Any force $F(r)$, radially directed with respect to some origin, is the negative derivative of a spherically symmetric potential energy $V(r)$:
$F(r) = -V'(r)$. *(3 points)*

c) Find, in terms of the given quantities M, M_m, etc, the approximate form $h(\varphi)$ of the tidal bulge. What is the difference in meters between high tide and low tide in this model?

You may use the approximate expression

$$\frac{1}{\sqrt{1 + a^2 - 2a\cos\theta}} \approx 1 + a\cos\theta + \tfrac{1}{2}a^2(3\cos^2\theta - 1),$$

valid for a much less than unity.

In this analysis make simplifying approximations whenever they are reasonable. *(5 points)*

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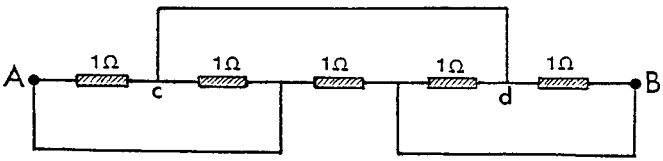
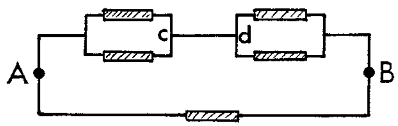
THEORETICAL COMPETITION
JULY 2 1996

Solution Problem 1

a) The system of resistances can be redrawn as shown in the figure:





The equivalent drawing of the circuit shows that the resistance between point c and point A is 0.5Ω, and the same between point d and point B. The resistance between points A and B thus consists of two connections in parallel: the direct 1Ω connection and a connection consisting of two 0.5Ω resistances in series, in other words two parallel 1Ω connections. This yields

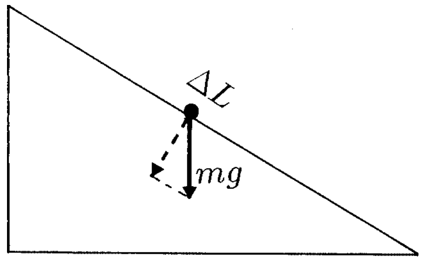
$$R = \underline{\underline{0.5\ \Omega}}\ .$$

b) For a sufficiently short horizontal displacement Δs the path can be considered straight. If the corresponding length of the path element is ΔL, the friction force is given by

$$\mu\, mg \frac{\Delta s}{\Delta L}$$

and the work done by the friction force equals force times displacement:

$$\mu\ mg \frac{\Delta s}{\Delta L} \cdot \Delta L = \mu\ mg\Delta s.$$



Adding up, we find that along the whole path the total work done by friction forces i $\mu\ \ mg\, s$. By energy conservation this must equal the decrease $mg\, h$ in potential energy of the skier. Hence

$$h = \underline{\underline{\mu s}}.$$

---

c) Let the temperature increase in a small time interval dt be dT. During this time interval the metal receives an energy $P\, dt$.

The heat capacity is the ratio between the energy supplied and the temperature increase:

$$C_p = \frac{Pdt}{dT} = \frac{P}{dT/dt}.$$

The experimental results correspond to

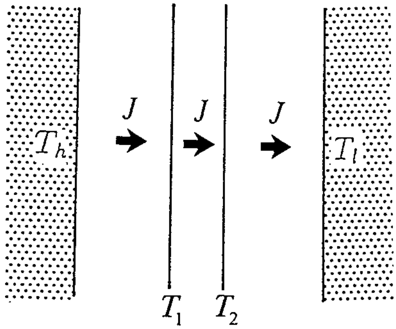
$$\frac{dT}{dt} = \frac{T_0}{4} a[1 + a(t - t_0)]^{-3/4} = T_0 \frac{a}{4}\left(\frac{T_0}{T}\right)^3.$$

Hence

$$C_p = \frac{P}{dT/dt} = \underline{\frac{4P}{aT_0^{\ 4}} T^3}.$$

(*Comment*: At low, but not extremely low, temperatures heat capacities of metals follow such a T^3 law.)

d)



Under stationary conditions the net heat flow is the same everywhere:

$$J = \sigma(T_h^4 - T_1^4)$$
$$J = \sigma(T_1^4 - T_2^4)$$
$$J = \sigma(T_2^4 - T_l^4)$$

Adding these three equations we get

$$3J = \sigma(T_h^4 - T_l^4) = J_0,$$

where J_0 is the heat flow in the absence of the heat shield. Thus $\xi = J/J_0$ takes the value

$$\xi = \underline{\underline{1/3}}.$$

---

e) The magnetic field can be determined as the superposition of the fields of two *cylindrical* conductors, since the effects of the currents in the area of intersection cancel. Each of the cylindrical conductors must carry a larger current I', determined so that the fraction I of it is carried by the actual cross section (the moon-shaped area). The ratio between the currents I and I' equals the ratio between the cross section areas:

$$\frac{I}{I'} = \frac{(\frac{\pi}{12} + \frac{\sqrt{3}}{8})D^2}{\frac{\pi}{4} D^2} = \frac{2\pi + 3\sqrt{3}}{6\pi}.$$

Inside one cylindrical conductor carrying a current I' Ampère's law yields at a distance r from the axis an azimuthal field

$$B_\phi = \frac{\mu_0}{2\pi r} \frac{I' \pi r^2}{\frac{\pi}{4} D^2} = \frac{2\mu_0 I' r}{\pi D^2}.$$

The cartesian components of this are

$$B_x = -B_\phi \frac{y}{r} = -\frac{2\mu_0 I'y}{\pi D^2}; \qquad B_y = B_\phi \frac{x}{r} = \frac{2\mu_0 I'x}{\pi D^2}.$$

For the superposed fields, the currents are $\pm I'$ and the corresponding cylinder axes are located at $x = \mp D/4$.

The two x-components add up to zero, while the y-components yield

$$B_y = \frac{2\mu_0}{\pi D^2}[I'(x + D/4) - I'(x - D/4)] = \frac{\mu_0 I'}{\pi D} = \underline{\underline{\frac{6\mu_0 I}{(2\pi + 3\sqrt{3})D}}},$$

i.e., a *constant* field. The direction is along the positive y-axis.

Solution Problem 2

a) The potential energy gain eV is converted into kinetic energy. Thus

$$\tfrac{1}{2}mv^2 = eV \qquad \text{(non-relativistically)}$$

$$\frac{mc^2}{\sqrt{1 - v^2/c^2}} - mc^2 = eV \qquad \text{(relativistically).}$$

Hence

$$v = \begin{cases} \sqrt{2eV/m} & \text{(non - relativistically)} \\ c\sqrt{1 - \left(\dfrac{mc^2}{mc^2 + eV}\right)^2} & \text{(relativistically).} \end{cases} \qquad (1)$$

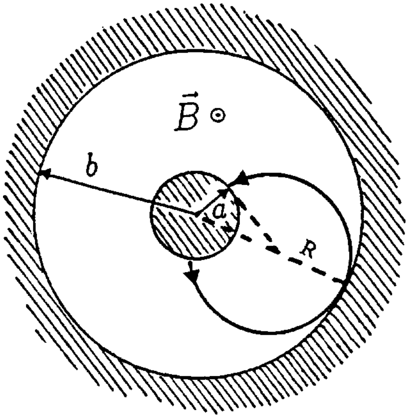
b) When $V = 0$ the electron moves in a homogeneous static magnetic field. The magnetic Lorentz force acts orthogonal to the velocity and the electron will move in a circle. The initial velocity is tangential to the circle.

The radius R of the orbit (the "cyclotron radius") is determined by equating the centripetal force and the Lorentz force:

$$eBv_0 = \frac{mv_0^2}{R} \; ,$$

i.e.

$$B = \frac{mv_0}{eR} \; . \tag{2}$$



From the figure we see that in the critical case the radius R of the circle satisfies

$$\sqrt{a^2 + R^2} = b - R$$

By squaring we obtain

$$a^2 + R^2 = b^2 - 2bR + R^2,$$

i.e.

$$R = (b^2 - a^2) / 2b \; .$$

Insertion of this value for the radius into the expression (2) gives the critical field

$$B_c = \frac{mv_0}{eR} = \underline{\underline{\frac{2bmv_0}{(b^2 - a^2)e}}}.$$

c) The change in angular momentum with time is produced by a torque. Here the azimuthal component F_ϕ of the Lorentz force $\vec{F} = (-e)\vec{B}\times\vec{v}$ provides a torque $F_\phi r$. It is only the radial component $v_r = dr/dt$ of the velocity that provides an azimuthal Lorentz force. Hence

$$\underline{\underline{\frac{dL}{dt} = eBr\frac{dr}{dt}}},$$

which can be rewritten as

$$\frac{d}{dt}(L - \frac{eBr^2}{2}) = 0.$$

Hence

$$C = \underline{\underline{L - \tfrac{1}{2} eBr^2}} \qquad (3)$$

is constant during the motion. The dimensionless number k in the problem text is thus $k = \underline{1/2}$.

d) We evaluate the constant C, equation (3), at the surface of the inner cylinder and at the maximal distance r_{m} :

$$0 - \tfrac{1}{2} eBa^2 = mvr_m - \tfrac{1}{2} eBr_m^2$$

which gives

$$\underline{v = \frac{eB(r_m^2 - a^2)}{2mr_m}}. \qquad (4)$$

Alternative solution: One may first determine the electric potential $V(r)$ as function of the radial distance. In cylindrical geometry the field falls off inversely proportional to r, which requires a logarithmic potential, $V(s) = c_1 \ln r + c_2$. When the two constants are determined to yield $V(a) = 0$ and $V(b) = V$ we have

$$V(r) = V \frac{\ln(r / a)}{\ln(b / a)}.$$

The gain in potential energy, $sV(r_m)$, is converted into kinetic energy:

$$\frac{1}{2} mv^2 = eV \frac{\ln(r_m / a)}{\ln(b / a)}.$$

Thus

$$v = \sqrt{\frac{2eV}{m} \frac{\ln(r_m / a)}{\ln(b / a)}}. \qquad (5)$$

(4) and (5) seem to be different answers. This is only apparent since r_m is not an independent parameter, but determined by B and V so that the two answers are identical.

e) For the critical magnetic field the maximal distance r_m equals b, the radius of the outer cylinder, and the speed at the turning point is then

$$v = \frac{eB(b^2 - a^2)}{2mb}.$$

Since the Lorentz force does no work, the corresponding kinetic energy $\frac{1}{2}mv^2$ equals eV (question a):

$$v = \sqrt{2eV/m}$$

The last two equations are consistent when

$$\frac{eB(b^2 - a^2)}{2mb} = \sqrt{2eV/m}.$$

The critical magnetic field for current cut-off is therefore

$$\underline{\underline{B_c = \frac{2b}{b^2 - a^2}\sqrt{\frac{2mV}{e}}.}}$$

f) The Lorentz force has no component parallel to the magnetic field, and consequently the velocity component v_B is constant under the motion. The corresponding displacement parallel to the cylinder axis has no relevance for the question of reaching the anode.

Let v denote the final azimuthal speed of an electron that barely reaches the anode. Conservation of energy implies that

$$\tfrac{1}{2}m(v_B^2 + v_\phi^2 + v_r^2) + eV = \tfrac{1}{2}m(v_B^2 + v^2),$$

giving

$$v = \sqrt{v_r^2 + v_\phi^2 + 2eV/m}. \tag{6}$$

Evaluating the constant C in (3) at both cylinder surfaces for the critical situation we have

$$mv_\phi a - \tfrac{1}{2}eB_c a^2 = mvb - \tfrac{1}{2}eB_c b^2.$$

Insertion of the value (6) for the velocity v yields the critical field

$$B_c = \frac{2m(vb - v_\phi a)}{e(b^2 - a^2)} = \frac{2mb}{e(b^2 - a^2)}\left[\sqrt{v_r^2 + v_\phi^2 + 2eV/m} - v_\phi a/b\right].$$

Solution Problem 3

a) With the centre of the earth as origin, let the centre of mass C be located at $\vec{l}$. The distance l is determined by

$$M\,l = M_m\,(L - l),$$

which gives

$$l = \frac{M_m}{M + M_m}L = \underline{4.63 \cdot 10^6\,\text{m},} \tag{1}$$

less than R, and thus inside the earth.

The centrifugal force must balance the gravitational attraction between the moon and the earth:

$$M\omega^2 l = G\frac{MM_m}{L^2},$$

which gives

$$\omega = \sqrt{\frac{GM_m}{L^2 l}} = \underline{\underline{\sqrt{\frac{G(M + M_m)}{L^3}}}} = \underline{\underline{2.67 \cdot 10^{-6}\,\text{s}^{-1}}}. \tag{2}$$

(This corresponds to a period $2\pi/\omega = 27.2$ days.) We have used (1) to eliminate l.

b) The potential energy of the mass point m consists of three contributions:

(1) Potential energy because of rotation (in the rotating frame of reference, see the problem text),

$$-\frac{1}{2}m\omega^2 r_1^2,$$

where $\vec{r}_1$ is the distance from C. This corresponds to the centrifugal force $m\omega^2 r_1$, directed outwards from C.

(2) Gravitational attraction to the earth,

$$-G\frac{mM}{r}.$$

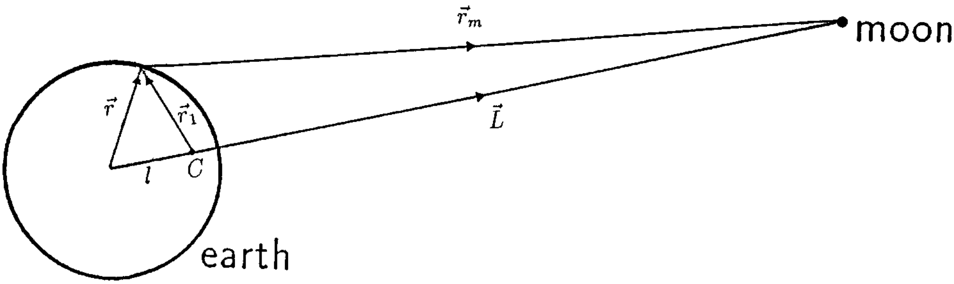
(3) Gravitational attraction to the moon,

$$- G\frac{mM_m}{|\vec{r}_m|},$$

where $\vec{r}_m$ is the distance from the moon.

Describing the position of m by polar coordinates r; ϕ in the plane orthogonal to the axis of rotation (see figure), we have

$$\vec{r}_1^{\,2} = (\vec{r} - \vec{l}\,)^2 = r^2 - 2rl\cos\phi + l^2 .$$



Adding the three potential energy contributions, we obtain

$$V(\vec{r}) = -\frac{1}{2}m\omega^2(r^2 - 2rl\cos\phi + l^2) - G\frac{mM}{r} - G\frac{mM_m}{|\vec{r}_m|}. \tag{3}$$

Here l is given by (1) and

$$|\vec{r}_m| = \sqrt{(\vec{L} - \vec{r})^2} = \sqrt{L^2 - 2\vec{L}\vec{r} + r^2} = L\sqrt{1 + (r/L)^2 - 2(r/L)\cos\phi} .$$

c) Since the ratio $r/L = a$ is very small, we may use the expansion

$$\frac{1}{\sqrt{1 + a^2 - 2a\cos\phi}} = 1 + a\cos\phi + a^2\,\tfrac{1}{2}(3\cos^2\phi - 1).$$

Insertion into the expression (3) for the potential energy gives

$$V(r,\phi)/m = -\tfrac{1}{2}\omega^2 r^2 - \frac{GM}{r} - \frac{GM_m r^2}{2L^3}(3\cos^2\phi - 1), \tag{4}$$

apart from a constant. We have used that

$$m\omega^2 rl\cos\phi - GmM_m\frac{r}{L^2}\cos\phi = 0,$$

when the value of ω_2, equation (2), is inserted.

The form of the liquid surface is such that a mass point has the same energy *V everywhere on the surface*. (This is equivalent to requiring no net force tangential to the surface.) Putting

$$r = R + h,$$

where the tide h is much smaller than R, we have approximately

$$\frac{1}{r} = \frac{1}{R+h} = \frac{1}{R} \cdot \frac{1}{1+(h/R)} \cong \frac{1}{R}(1-\frac{h}{R}) = \frac{1}{R} - \frac{h}{R^2},$$

as well as

$$r^2 = R^2 + 2Rh + h^2 \cong R^2 + 2Rh.$$

Inserting this, and the value (2) of ω into (4), we have

$$V(r,\phi)/m = -\frac{G(M+M_m)R}{L^3}h + \frac{GM}{R^2}h - \frac{GM_m r^2}{2L^3}(3\cos^2\phi - 1), \quad (5)$$

again apart from a constant.

The magnitude of the first term on the right-hand side of (5) is a factor

$$\frac{(M+M_m)}{M}\left(\frac{R}{L}\right)^3 \cong 10^{-5}$$

smaller than the second term, thus negligible. If the remaining two terms in equation (5) compensate each other, i.e.,

$$h = \frac{M_m r^2 R^2}{2ML^3}(3\cos^2\phi - 1),$$

then the mass point m has the same energy everywhere on the surface. Here r^2 can safely be approximated by R^2, giving the tidal bulge

$$\underline{\underline{h = \frac{M_m R^4}{2ML^3}(3\cos^2\phi - 1).}}$$

The largest value $h_{\max} = M_m R^4 / ML^3$ occurs for $\phi = 0$ or π, in the direction of the moon or in the opposite direction, while the smallest value

28th International Physics Olympiad
Sudbury, Canada

THEORETICAL COMPETITION

Thursday, July 17th, 1997

Time Available: 5 hours

Read This First:

1. Use only the pen provided.
2. Use only the front side of the answer sheets and paper.
3. In your answers please use *as little text as possible*; express yourself primarily in equations, numbers and figures. **Summarize your results on the answer sheet**.
4. Please indicate on the first page the total number of pages you used.
5. At the end of the exam please put your answer sheets, pages and graphs in order.

This set of problems consists of 11 pages.

Examination prepared at: University of British Columbia
Department of Physics and Astronomy
Committee Chair: Chris Waltham

Hosted by: Laurentian University

Theory Question No.1

Scaling

(a) A small mass hangs on the end of a massless ideal spring and oscillates up and down at its natural frequency f. If the spring is cut in half and the mass reattached at the end, what is the new frequency f'? (1.5 *marks*)

(b) The radius of a hydrogen atom in its ground state is a_0 = 0.0529 nm (the "Bohr radius"). What is the radius a' of a "muonic-hydrogen" atom in which the electron is replaced by an identically charged muon, with mass 207 times that of the electron? Assume the proton mass is much larger than that of the muon and electron. (2 *marks*)

(c) The mean temperature of the earth is T = 287 K. What would the new mean temperature T' be if the mean distance between the earth and the sun was reduced by 1%?
(2 *marks*)

(d) On a given day, the air is dry and has a density $\rho = 1.2500$ kg/m^3. The next day the humidity has increased and the air is 2% by mass water vapour. The pressure and temperature are the same as the day before. What is the air density ρ' now? (2 *marks*)

Mean molecular weight of dry air: 28.8 (g/mol)
Molecular weight of water: 18 (g/mol)

Assume ideal-gas behaviour.

(e) A type of helicopter can hover if the mechanical power output of its engine is P. If another helicopter is made which is an exact ½-scale replica (in all linear dimensions) of the first, what mechanical power P' is required for it to hover? (*2.5 marks*)

Theory Question 1: Answer Sheet **STUDENT CODE:**___________________

(a) Frequency f':

(b) Radius a':

(c) Temperature T':

(d) Density ρ':

(e) Power P':

Theory Question No.2

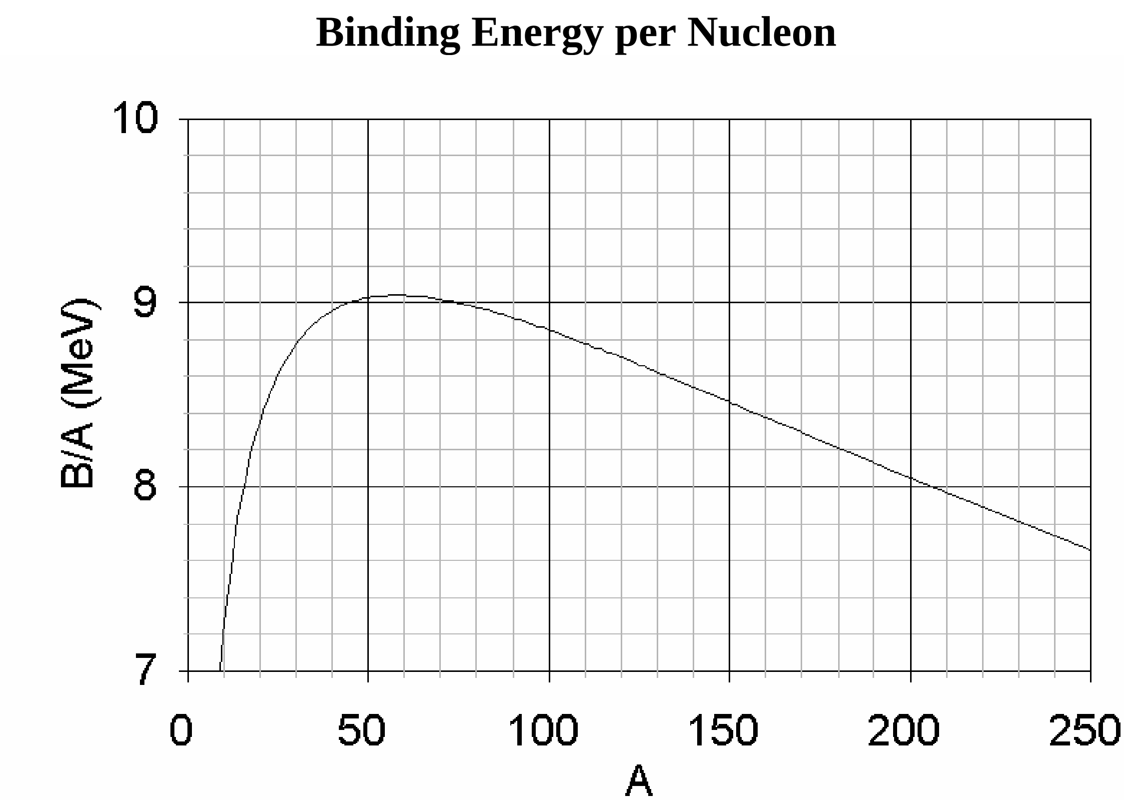
Nuclear Masses and Stability

All energies in this question are expressed in MeV - millions of electron volts.
One MeV = 1.6×10^{-13} J, but it is not necessary to know this to solve the problem.

The mass M of an atomic nucleus with Z protons and N neutrons (i.e. the mass number $A = N + Z$) is the sum of masses of the free constituent nucleons (protons and neutrons) minus the binding energy B/c^2.

$$Mc^2 = Zm_pc^2 + Nm_nc^2 - B$$

The graph shown below plots the maximum value of B/A for a given value of A, vs. A. The greater the value of B/A, in general, the more stable is the nucleus.



(a) Above a certain mass number A_α , nuclei have binding energies which are always small enough to allow the emission of alpha-particles (*A=4*). Use a linear approximation to this curve above *A* = 100 to estimate A_α. (3 *marks*)

For this model, assume the following:

- Both initial and final nuclei are represented on this curve.
- The total binding energy of the alpha-particle is given by B_4 = 25.0 MeV (this cannot be read off the graph!).

(b) The binding energy of an atomic nucleus with *Z* protons and *N* neutrons (*A*=*N*+*Z*) is given by a semi-empirical formula:

$$B = a_v A - a_s A^{2/3} - a_c Z^2 A^{-1/3} - a_a \frac{(N-Z)^2}{A} - \delta$$

The value of δ is given by:

$+ a_p A^{-3/4}$ for odd-N/odd-Z nuclei

0 for even-N/odd-Z or odd-N/even-Z nuclei

$- a_p A^{-3/4}$ for even-N/even-Z nuclei

The values of the coefficients are:

a_v = 15.8 MeV; a_s = 16.8 MeV; a_c = 0.72 MeV; a_a = 23.5 MeV; a_p = 33.5 MeV.

(i) Derive an expression for the proton number Z_{max} of the nucleus with the largest binding energy for a given mass number *A* . Ignore the δ-term for this part only. (2 *marks*)

(ii) What is the value of *Z* for the *A* = 200 nucleus with the largest *B/A*? Include the effect of the δ-term. (2 *marks*)

(iii) Consider the three nuclei with A = 128 listed in the table on the answer sheet. Determine which ones are energetically stable and which ones have sufficient energy to decay by the processes listed below. Determine Z_{max} as defined in part (i) and fill out the table on your answer sheet.

In filling out the table, please:

- Mark processes which are energetically allowed thus: √
- Mark processes which are NOT energetically allowed thus: 0
- Consider only transitions between these three nuclei.

Decay processes:

(1) β^-- decay; emission from the nucleus of an electron
(2) β^+- decay; emission from the nucleus of a positron
(3) $\beta^-\beta^-$ - decay; emission from the nucleus of two electrons simultaneously
(4) Electron capture; capture of an *atomic* electron by the nucleus.

The rest mass energy of an electron (and positron) is $m_ec^2 = 0.51$ MeV; that of a proton is $m_pc^2 = 938.27$ MeV; that of a neutron is $m_nc^2 = 939.57$ MeV.

(3 *marks*)

Question 2: Answer Sheet **STUDENT CODE:** ___________________

(a) Numerical value for A_α :

(b) (i) Expression for Z_{max} :

(b) (ii) Numerical value of Z :

(b) (iii)

Nucleus/Process	β^- - decay	β^+ - decay	Electron-capture	$\beta^-\beta^-$ - decay
${}^{128}_{53}\mathrm{I}$				
${}^{128}_{54}\mathrm{Xe}$				
${}^{128}_{55}\mathrm{Cs}$				

Notation : ${}^{A}_{Z}X$

X = Chemical Symbol

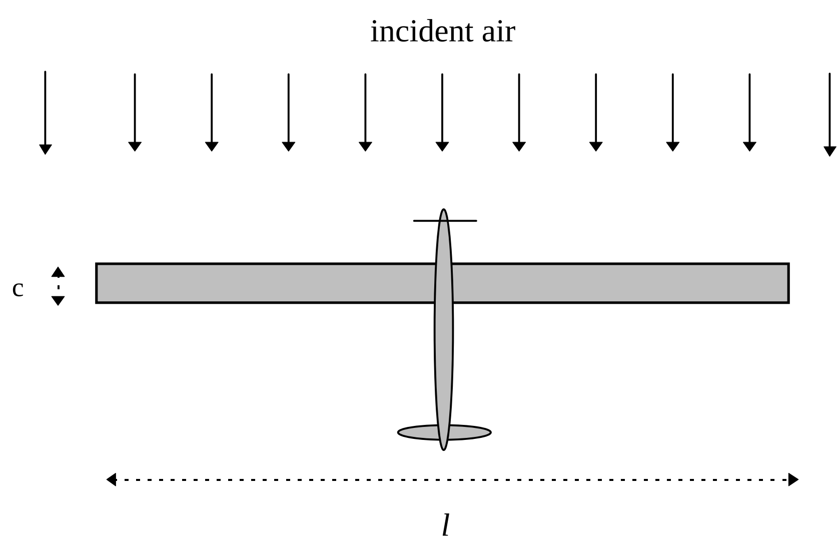
Theory Question No.3

Solar-Powered Aircraft

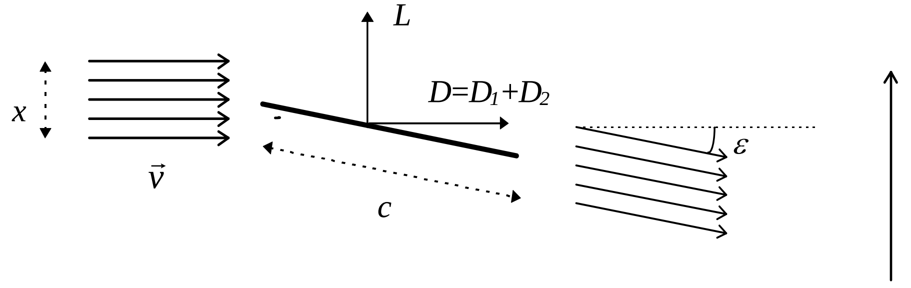
We wish to design an aircraft which will stay aloft using solar power alone. The most efficient type of layout is one with a wing whose top surface is completely covered in solar cells. The cells supply electrical power with which the motor drives the propeller.

Consider a wing of rectangular plan-form with span l, chord (width) c; the wing area is $S = cl$, and the wing aspect ratio $A = l / c$. We can get an approximate idea of the wing's performance by considering a slice of air of height x and length l being deflected downward at a small angle ε with only a very small change in speed. Control surfaces can be used to select an optimal value of ε for flight. This simple model corresponds closely to reality if $x = \pi l /4$, and we can assume this to be the case. The total mass of the aircraft is M and it flies horizontally with velocity $\vec{v}$ relative to the surrounding air. In the following calculations consider only the air flow around the wing.

Top view of aircraft (in its own frame of reference):



Side view of wing (in a frame of reference moving with the aircraft):



incident air wing section air leaving wing vertical(up)

Ignore the modification of the airflow due to the propeller.

(a) Consider the change in momentum of the air moving past the wing, with *no* change in speed while it does so. Derive expressions for the vertical lift force L and the horizontal drag force D_1 on the wing in terms of wing dimensions, v, ε, and the air density ρ. Assume the direction of air flow is always parallel to the plane of the side-view diagram. (*3 marks*)

(b) There is an additional horizontal drag force D_2 caused by the friction of air flowing over the surface of the wing. The air slows slightly, with a change of speed Δv (<< 1% of v) given by:

$$\frac{\Delta v}{v} = \frac{f}{A}$$

The value of f is independent of ε.

Find an expression (in terms of M, f, A, S, ρ and g- the acceleration due to gravity) for the flight speed v_0 corresponding to a minimum power being needed to maintain this aircraft in flight at constant altitude and velocity. Neglect terms of order ($\varepsilon^2 f$) or higher. (*3 marks*)

You may find the following small angle approximation useful:

$$1 - \cos\varepsilon \approx \frac{\sin^2\varepsilon}{2}$$

(c) On the answer sheet, sketch a graph of power P versus flight speed v . Show the separate contributions to the power needed from the two sources of drag. Find an expression (in terms of M, f, A, S, ρ and g) for the minimum power, P_{min} . (*2 marks*)

(d) If the solar cells can supply sufficient energy so that the electric motors and propellers generate mechanical power of I = 10 watts per square metre of wing area, calculate the maximum wing loading Mg/S (N/m^2) for this power and flight speed v_0 (m/s). Assume ρ = 1.25 kg/m^3, f = 0.004, A = 10. (*2 marks*)

Question 3: Answer Sheet **STUDENT CODE:**______________________

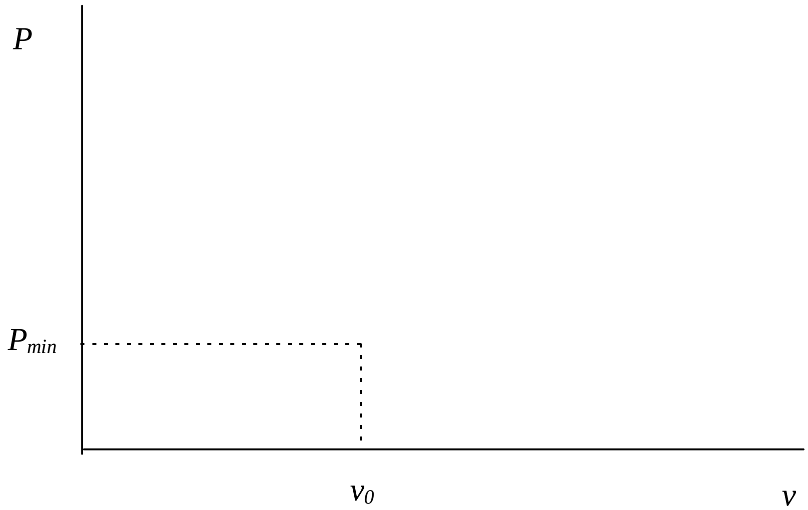
(a) Expression for L :

(a) Expression for D_1 :

(b) Expression for D_2 :

(b) Expression for v_0 :

(c)



(c) Expression for P_{min} :

(d) Maximum value of Mg/S :

(d) Numerical value of v_0 :

29^{th} International Physics Olympiad
Reykjavík, Iceland

Theoretical competition

Saturday, July 4^{th}, 1998

9 a.m. – 2 p.m.

Read this first:

1. Use only the pen provided.
2. Use only the front side of the answer sheets.
3. Use *as little text as possible* in your answers; express yourself primarily with equations, numbers and figures. **Summarize your results on the answer sheets.**
4. For anything but your answers and your graphs use the blank answer sheets. This applies e.g. when you are asked to *show that* … and also for all calculations you want to be considered for evaluation.
5. You may often be able to solve later parts of a problem without having solved the previous ones. In such cases you may take the result of a previous part as given, in the form stated in the problem text.
6. Please indicate on all sheets your team name, student number, number of page and total number of pages. On the blank answer sheets also indicate the problem number.
7. At the end of the exam please put your answer sheets in order. You may leave on your table material which you do not wish to be evaluated.

This set of problems consists of 11 pages in total.

Examination prepared at:
University of Iceland, Department of Physics, in collaboration with physicists from the National Energy Authority.

1 Rolling of a hexagonal prism[1]

1.1 Problem text

Consider a long, solid, rigid, regular hexagonal prism like a common type of pencil (Figure 1.1). The mass of the prism is M and it is uniformly distributed. The length of each side of the cross-sectional hexagon is a. The moment of inertia I of the hexagonal prism about its central axis is

$$I = \frac{5}{12} M a^2 \tag{1.1}$$

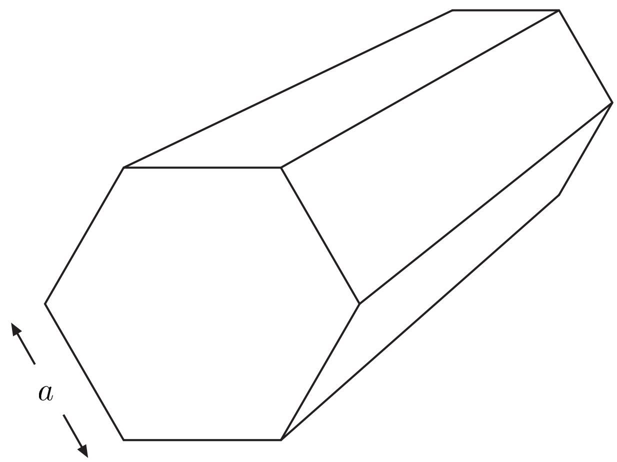


Figure 1.1: *A solid prism with the cross section of a regular hexagon.*

The moment of inertia I' about an edge of the prism is

$$I' = \frac{17}{12} M a^2 \tag{1.2}$$

a) (3.5 points) The prism is initially at rest with its axis horizontal on an inclined plane which makes a small angle θ with the horizontal (Figure 1.2). Assume that the surfaces of the prism are slightly concave so that the prism only touches the plane at its edges. The effect of this concavity on the moment of inertia can be ignored. The prism is now displaced from rest and starts an uneven rolling down the plane. Assume that friction prevents any sliding and that the prism does not lose contact with the plane. The angular velocity just before a given edge hits the plane is ω_i while ω_f is the angular velocity immediately after the impact.

Show that we may write

$$\omega_f = s\omega_i \tag{1.3}$$

and write the value of the coefficient s on the answer sheet.

[1] Authors: Leó Kristjánsson and Thorsteinn Vilhjálmsson

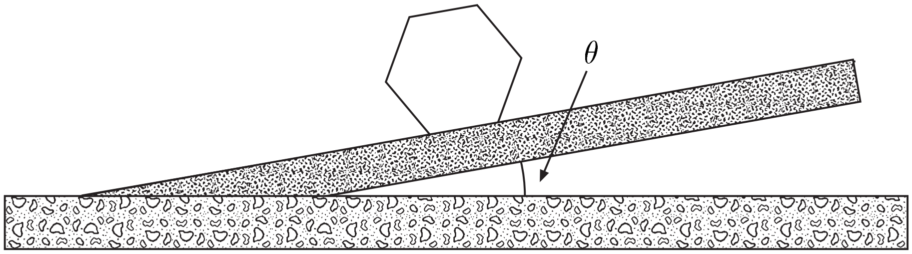


Figure 1.2: *A hexagonal prism lying on an inclined plane.*

b) (1 point) The kinetic energy of the prism just before and after impact is similarly K_i and K_f.

Show that we may write

$$K_f = rK_i \tag{1.4}$$

and write the value of the coefficient r on the answer sheet.

c) (1.5 points) For the next impact to occur K_i must exceed a minimum value $K_{i,min}$ which may be written in the form

$$K_{i,min} = \delta Mga \tag{1.5}$$

where $g = 9.81\ \mathrm{m/s^2}$ is the acceleration of gravity.

Find the coefficient δ in terms of the slope angle θ and the coefficient r. Write your answer on the answer sheet. (Use the algebraic symbol r, not its value).

d) (2 points) If the condition of part (c) is satisfied, the kinetic energy K_i will approach a fixed value $K_{i,0}$ as the prism rolls down the incline.

Given that the limit exists, show that $K_{i,0}$ may be written as:

$$K_{i,0} = \kappa Mga \tag{1.6}$$

and write the coefficient κ in terms of θ and r on the answer sheet.

e) (2 points) Calculate, to within 0.1°, the minimum slope angle θ_0, for which the uneven rolling, once started, will continue indefinitely. Write your numerical answer on the answer sheet.

1.2 Solution

a)
Solution Method 1

At the impact the prism starts rotating about a new axis, i.e. the edge which just hit the plane. The force from the plane has no torque about this axis, so that the angular momentum about the edge is conserved during the brief interval of impact. The linear momentum of the prism as a whole has the same direction as the velocity of the center of mass ($\vec{P} = M\ \vec{v}_C$ where the subscript C refers to the center of mass), and this direction is easy to follow when we know the axis of rotation at a given time. Just before impact $\vec{P}$ is directed 30° downwards relative to the plane, but will after impact point 30° upwards from the plane, see Figure 1.3.

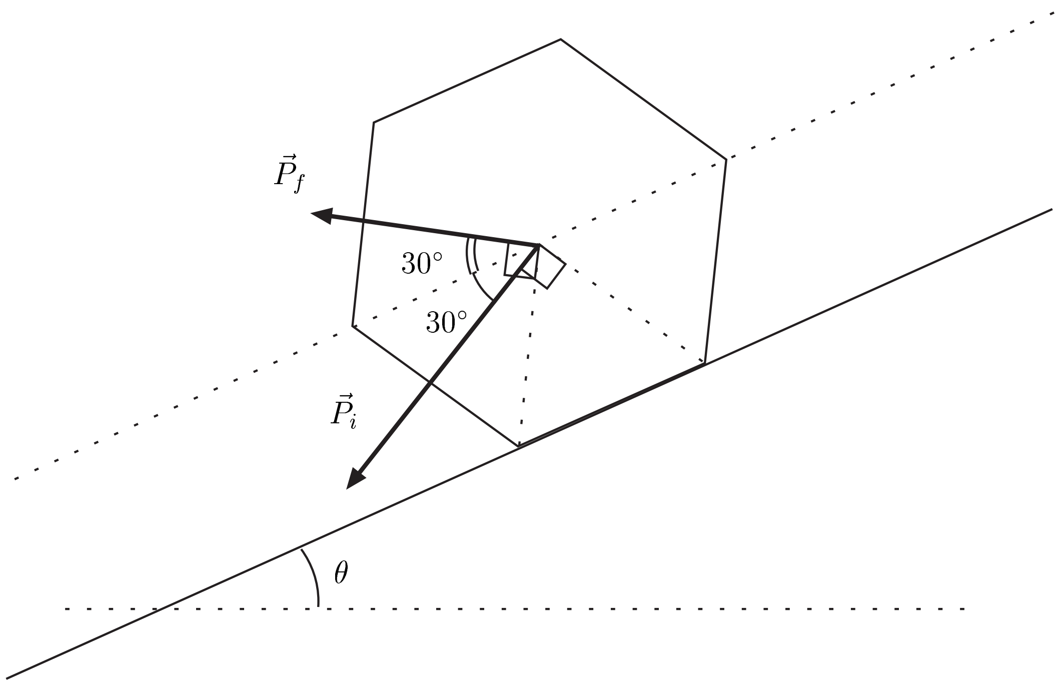


Figure 1.3: *The linear momentum of the prism as a whole, before and after impact.*

To find the angular momentum about the edge of impact just before the impact we use the equation relating angular momentum $\vec{L}$ about an arbitrary axis to the angular momentum $\vec{L}_C$ about an axis through the center of mass parallel to the first one:

$$\vec{L} = \vec{L}_C + M\ \vec{r}_C \times \vec{v}_C \tag{1.7}$$

where the subscript C refers to the center of mass. Here, this is applied to an axis at the point of impact so that $\vec{r}_C$ is the vector from that point to the center of mass (Figure 1.3). The vectors on the right hand side of equation (1.7) both have the same direction. Hence we get for the quantities just before impact[2]

$$\begin{aligned} |\vec{r}_C \times \vec{v}_{Ci}| &= r_C\ v_{Ci} \sin 30^\circ = a^2\ \omega_i\ /\ 2 && (1.8) \\ L_i = I\ \omega_i + \frac{1}{2}\ M\ a^2\ \omega_i &= \left(\frac{5}{12} + \frac{1}{2}\right)\ M\ a^2\ \omega_i = \frac{11}{12}\ M\ a^2\ \omega_i && (1.9) \end{aligned}$$

On the other hand, angular momentum about the edge just after impact is, from equation (1.2):[3]

[2]This may also be done by using Steiner's theorem twice, going from the previous axis of impact to the center of mass and from there to the new axis of impact.

[3]Alternatively:

$$L_f = I'\omega_f = \frac{17}{12} M a^2 \omega_f \tag{1.10}$$

where the subscript f always refers to the situation just after impact. We may notice that the difference comes about because of the different directions of $\vec{v}_{Ci}$ and $\vec{v}_{Cf}$. Now, when we state the conservation of angular momentum, $L_i = L_f$, we obtain a relation between the angular velocities as follows:

$$\omega_f = \frac{11/12}{17/12}\, \omega_i = \frac{11}{17}\, \omega_i \tag{1.11}$$

We thus get:

$$\boldsymbol{s = 11/17} \tag{1.12}$$

We may note that s is independent of a, ω_i, and θ.

Solution Method 2

On impact the prism receives an impulse $\vec{P}$ [N · s] from the plane at the edge where the impact occurs. There is no reaction at the edge which is leaving the plane. The impulse has a component $P_{\|}$ parallel to the inclined plane (positive upwards along the incline in Figure 1.3 and a component $P_{\perp}$ perpendicular to the plane (positive upwards from the plane in the same figure).

We can set up three equations with the three unknowns $P_{\|}$, $P_{\perp}$ and the ratio $s = \frac{\omega_f}{\omega_i}$. The quantity $P_{\|}$ is the change in the parallel component of the linear momentum of the prism and $P_{\perp}$ is the corresponding change in perpendicular linear momentum. Thus:

$$\begin{aligned} P_{\|} &= M\left(\omega_i - \omega_f\right) a \cdot \frac{\sqrt{3}}{2} &(1.13)\\ P_{\perp} &= M\left(\omega_i + \omega_f\right) a \cdot \frac{1}{2}. &(1.14) \end{aligned}$$

We finally have:

$$P_{\perp} a \frac{1}{2} - P_{\|} a \frac{\sqrt{3}}{2} = I\left(\omega_i - \omega_f\right) \tag{1.15}$$

since the right hand side is the change in angular momentum about the center of mass. Equations (1.13), (1.14) and (1.15) can now be solved for the ratio $s = \frac{\omega_f}{\omega_i}$ giving, of course, the same result as before.

$$\begin{aligned} L_f &= I\,\omega_f + M\,\left|\vec{r}_C \times \vec{v}_{Cf}\right| = I\omega_f + Ma^2\,\omega_f \sin 90^\circ \\ &= \left(\frac{5}{12} + 1\right) Ma^2\,\omega_f = \frac{17}{12}\,Ma^2\,\omega_f \end{aligned}$$

b)

The linear speed of the center of mass just before impact is $a\omega_i$ and just after impact it is $a\omega_f$. We know that we can always write the kinetic energy of a rotating rigid body as a sum of „internal“ and „external“ kinetic energy:

$$K_{tot} = \frac{1}{2}\ I\ \omega^2 + \frac{1}{2}\ M\ v_C^2 \tag{1.16}$$

From this we see that in our case the kinetic energy K_{tot} is proportional to ω^2 both before and after impact so that we get

$$K_f = r\ K_i = \left(\frac{11}{17}\right)^2 K_i = \frac{121}{289} K_i \tag{1.17}$$

so

$$\boldsymbol{r = 121/289} \approx 0.419 \tag{1.18}$$

c)

The kinetic energy K_f after the impact must be sufficient to lift the center of mass to its highest position, straight above the point of contact. The angle through which $\vec{r}_C$ moves for this is

$$x = \frac{\alpha}{2} - \theta \tag{1.19}$$

where $\alpha = 60°$ is the top angle of the triangles meeting at the center of the polygon.[4] The energy for this lifting of the center of mass is

$$E_0 = Mga(1 - \cos x) = Mga\,(1 - \cos(30° - \theta)) \tag{1.20}$$

and we get the condition

$$K_f = rK_i > E_0 = Mga\,(1 - \cos(30° - \theta)) \tag{1.21}$$

thus

$$\boldsymbol{\delta = \frac{1}{r}\,(1 - \cos(30° - \theta))} \tag{1.22}$$

(Note that $\cos(30° - \theta) = \frac{\sqrt{3}}{2}\cos\theta + \frac{1}{2}\sin\theta$).

d)

Let $K_{i,n}$ and $K_{f,n}$ be the kinetic energies just before and just after the nth impact. We have shown that we have the relation

[4] In the general case $\alpha = 2\pi/N$.

$$K_{f,n} = r\ K_{i,n} \tag{1.23}$$

where $r = \frac{121}{289}$ for a hexagonal prism. Between subsequent impacts the height of the center of mass of the prism decreases by $a \sin\theta$ and its kinetic energy increases for this reason by

$$\Delta = Mga \sin\theta \tag{1.24}$$

We therefore have

$$K_{i,n+1} = rK_{i,n} + \Delta. \tag{1.25}$$

One does not have to write out the complete expression $K_{i,n}$ as a function of $K_{i,1}$ and n to find the limit. This would actually be a proof that the limit exists (see below) but this is given in the problem text. Hence one can make $K_{i,n+1} \approx K_{i,n}$ arbitrarily accurate for sufficiently large n. The limit $K_{i,0}$ must thus satisfy the iterative formula, i.e.

$$K_{i,0} = rK_{i,0} + \Delta \tag{1.26}$$

yielding the solution

$$K_{i,0} = \frac{\Delta}{1-r}. \tag{1.27}$$

i.e.

$$\boldsymbol{\kappa = \frac{\sin\theta}{1-r}} \tag{1.28}$$

We can also solve the problem explicitly by writing out the full expressions:

$$\begin{aligned}
K_{i,2} &= r\ K_{i,1} + \Delta & (1.29)\\
K_{i,3} &= r\ K_{i,2} + \Delta = r^2 K_{i,1} + (1+r)\Delta & (1.30)\\
\ldots & & \\
K_{i,n} &= r^{n-1} K_{i,1} + (1 + r + \ldots + r^{n-2})\Delta & (1.31)\\
&= r^{n-1} K_{i,1} + \frac{1-r^{n-1}}{1-r}\Delta & (1.32)
\end{aligned}$$

In the limit of $n \to \infty$ we get

$$K_{i,n} \to K_{i,0} = \frac{\Delta}{1-r} \tag{1.33}$$

which is, of course, the same result as before.

If we calculate the change in kinetic energy through a whole cycle, i.e. from just before impact number n until just before impact $n+1$ we get

$$\begin{aligned}
\Delta K_{i,n} = K_{i,n+1} - K_{i,n} &= (r-1) r^{n-1} K_{i,1} + r^{n-1}\Delta & (1.34)\\
&= r^{n-1}\left(\Delta - (1-r)K_{i,1}\right) & (1.35)
\end{aligned}$$

This is positive if the initial value $K_{i,1} < K_{i,0}$ so that $K_{i,n}$ will then increase up to the limit value $K_{i,0}$. If, on the other hand, $K_{i,1} > K_{i,0}$, the kinetic energy $K_{i,n}$ just before impact will decrease down to the limit $K_{i,0}$.

All of this may remind you of motion with friction which increases with speed. Mathematically speaking, the main difference is that we here are dealing with difference equations instead of differential equations.

e)

For indefinite continuation the limit value of K_i in part (d) must be larger than the minimum value for continuation found in part (c):

$$\frac{1}{1-r}\Delta = \frac{1}{1-r} Mga \sin\theta > Mga\left(1 - \cos(30^\circ - \theta)\right)/r \tag{1.36}$$

We put $A = \frac{r}{1-r} = \frac{121}{168}$:

$$A\sin\theta > 1 - \cos 30^\circ \cos\theta - \sin 30^\circ \sin\theta \tag{1.37}$$
$$(A + 1/2)\sin\theta + \sqrt{3}/2\cos\theta > 1 \tag{1.38}$$

To solve this we define[5]

$$u = \arccos\left(\frac{A + 1/2}{\sqrt{(A+1/2)^2 + 3/4}}\right) \approx 35.36^\circ \tag{1.39}$$

and obtain

$$\cos u \sin\theta + \sin u \cos\theta > 1/\sqrt{(A+1/2)^2 + 3/4} \tag{1.40}$$
$$\sin(u + \theta) > 1/\sqrt{(A+1/2)^2 + 3/4} \tag{1.41}$$
$$\theta > \arcsin\{1/\sqrt{(A+1/2)^2 + 3/4}\} - u \approx 41.94^\circ - 35.36^\circ = 6.58^\circ \tag{1.42}$$

That is

$$\boldsymbol{\theta_0 \approx 6.58^\circ} \tag{1.43}$$

If $\theta > \theta_0$ *and* the kinetic energy before the first impact is sufficient according to part (c), we will, under the assumptions made, get an indefinite "rolling".

[5]You can of course solve any of the inequalities in a purely numerical way, e.g. by progressive guessing or by using the approximations $\sin\phi \approx \phi$ and $\cos\phi \approx 1 - \phi^2/2$.

1.3 Grading scheme

Part **2(a)**	
Answer: $s = \omega_f/\omega_i = 11/17$, equation (1.12)	**3.5**

Part **2(b)**	
Answer: $r = K_f/K_i = s^2 = 121/289$, equation (1.18)	**1.0**

Part **2(c)**	
Answer: $K_{i,min}$ by δ, equation (1.22)	**1.5**

Part **2(d)**	
Answer: Limit $K_{i,0}$ by $\kappa = \sin\theta/(1-r)$, equation (1.28)	**2.0**

Part **2(e)**	
Answer: Minimum angle $\theta_0 = 6.58°$, equation (1.43)	**2.0**

2 Water under an ice cap[6]

2.1 Problem text

An ice cap is a thick sheet of ice (up to a few km in thickness) resting on the ground below and extending horizontally over tens or hundreds of km. In this problem we consider the melting of ice and the behavior of water under a temperate ice cap, i.e. an ice cap at the melting point. We may assume that under such conditions the ice causes pressure variations as a viscous fluid, but deforms in a brittle fashion, principally by vertical movement. For the purposes of this problem the following information is given.

Density of water:	$\rho_w = 1.000 \cdot 10^3$ kg/m^3
Density of ice:	$\rho_i = 0.917 \cdot 10^3$ kg/m^3
Specific heat of ice:	$c_i = 2.1 \cdot 10^3$ J/(kg °C)
Specific latent heat of ice:	$L_i = 3.4 \cdot 10^5$ J/kg
Density of rock and magma:	$\rho_r = 2.9 \cdot 10^3$ kg/m^3
Specific heat of rock and magma:	$c_r = 700$ J/(kg °C)
Specific latent heat of rock and magma:	$L_r = 4.2 \cdot 10^5$ J/kg
Average outward heat flow through the surface of the earth:	$J_Q = 0.06$ W/m^2
Melting point of ice:	$T_0 = 0$°C, constant

a) (0.5 points) Consider a thick ice cap at a location of average heat flow from the interior of the earth. Using the data from the table, calculate the thickness d of the ice layer melted every year and write your answer in the designated box on the answer sheet.

b) (3.5 points) Consider now the upper surface of an ice cap. The ground below the ice cap has a slope angle α. The upper surface of the cap slopes by an angle β as shown in Figure 2.1. The vertical thickness of the ice at $x = 0$ is h_0. Hence the lower and upper surfaces of the ice cap can be described by the equations

$$y_1 = x \tan\alpha, \quad y_2 = h_0 + x \tan\beta \tag{2.1}$$

Derive an expression for the pressure p at the bottom of the ice cap as a function of the horizontal coordinate x and write it on the answer sheet.

Formulate mathematically a condition between β and α, so that water in a layer between the ice cap and the ground will flow in neither direction. Show that the condition is of the form $\tan\beta = s\tan\alpha$. Find the coefficient s and write the result in a symbolic form on the answer sheet.

The line $y_1 = 0.8\,x$ in Figure 2.2 shows the surface of the earth below an ice cap. The vertical thickness h_0 at $x = 0$ is 2 km. Assume that water at the bottom is in equilibrium.

On a graph answer sheet draw the line y_1 and add a line y_2 showing the upper surface of the ice. Indicate on the figure which line is which.

[6] Authors: Gudni Axelsson and Thorsteinn Vilhjálmsson

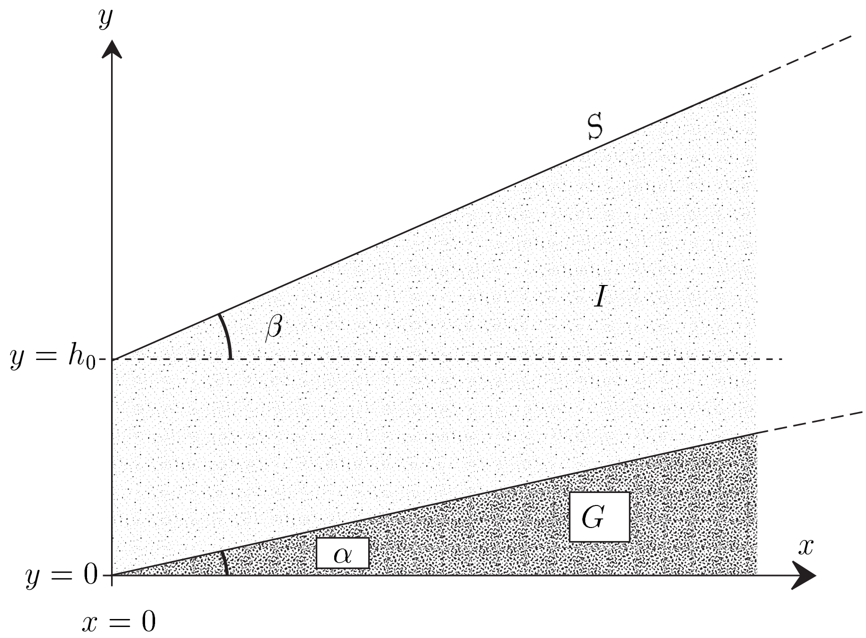


Figure 2.1: *Cross section of an ice cap with a plane surface resting on an inclined plane ground. S: surface, G: ground, I: ice cap.*

c) (1 point) Within a large ice sheet on horizontal ground and originally of constant thickness $D = 2.0$ km, a conical body of water of height $H = 1.0$ km and radius $r = 1.0$ km is formed rather suddenly by melting of the ice (Figure 2.3). We assume that the remaining ice adapts to this by vertical motion only.

Show analytically on a blank answer sheet and pictorially on a graph answer sheet, the shape of the surface of the ice cap after the water cone has formed and hydrostatic equilibrium has been reached.

d) (5 points) In its annual expedition an international group of scientists explores a temperate ice cap in Antarctica. The area is normally a wide plateau but this time they find a deep crater-like depression, formed like a top-down cone with a depth h of 100 m and a radius r of 500 m (Figure 2.4). The thickness of the ice in the area is 2000 m.

After a discussion the scientists conclude that most probably there was a minor volcanic eruption below the ice cap. A small amount of magma (molten rock) intruded at the bottom of the ice cap, solidified and cooled, melting a certain volume of ice. The scientists try as follows to estimate the volume of the intrusion and get an idea of what became of the melt water.

Assume that the ice only moved vertically. Also assume that the magma was completely molten and at 1200°C at the start. For simplicity, assume further that the intrusion had the form of a cone with a circular base vertically below the conical depression in the surface. The time for the rising of the magma was short relative to the time for the exchange of heat in the process. The heat flow is assumed to have been primarily vertical such that the volume melted from the ice at any time is bounded by a conical surface centered above the center of the magma intrusion.

Given these assumptions the melting of the ice takes place in two steps. At first the water is not in pressure equilibrium at the surface of the magma and hence flows away. The water flowing away can be assumed to have a temperature of 0°C. Subsequently,

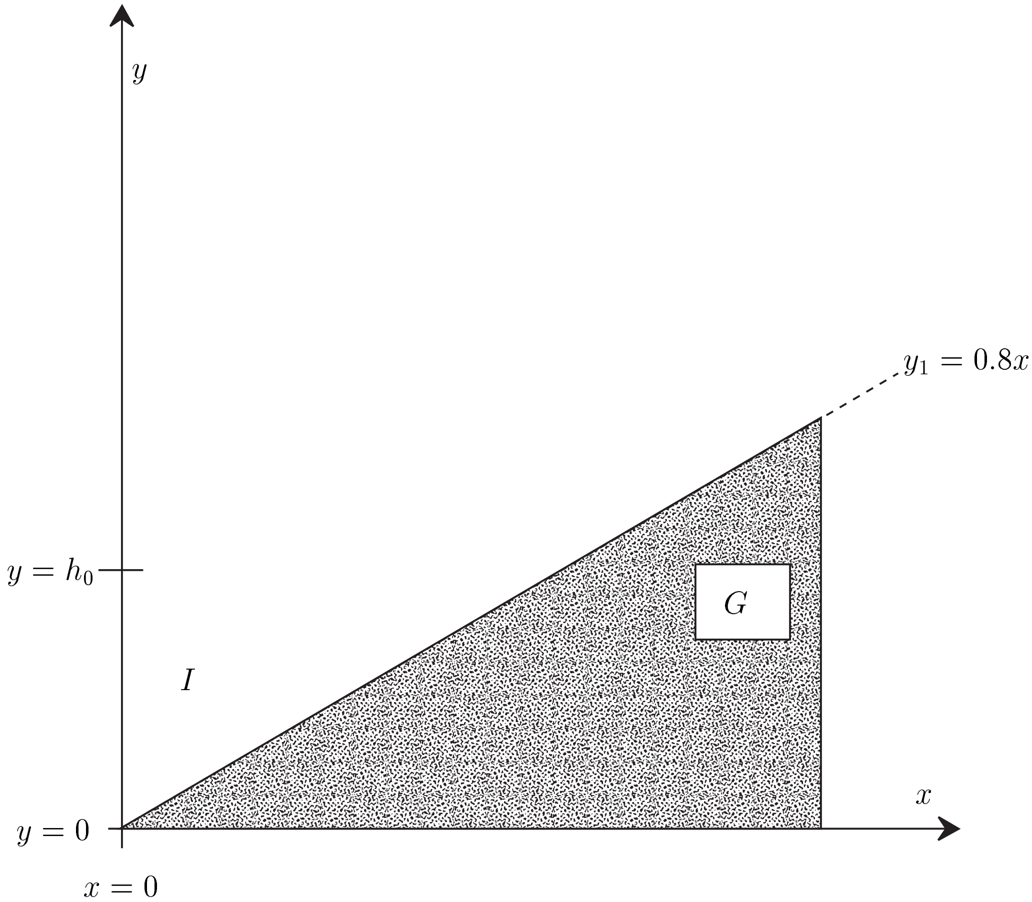


Figure 2.2: *Cross section of a temperate ice cap resting on an inclined ground with water at the bottom in equilibrium. G: ground, I: ice cap.*

hydrostatic equilibrium is reached and the water accumulates above the intrusion instead of flowing away.

When thermal equilibrium has been reached, you are asked to determine the following quantities. Write the answers on the answer sheet.

1. The height H of the top of the water cone formed under the ice cap, relative to the original bottom of the ice cap.
2. The height h_1 of the intrusion.
3. The total mass m_{tot} of the water produced and the mass m' of water that flows away.

Plot on a graph answer sheet, to scale, the shapes of the rock intrusion and of the body of water remaining. Use the coordinate system suggested in Figure 2.4.

2.2 Solution

a)

Based on the conservation of energy we have

$$J_Q \cdot 1 \text{ year} = L_i \rho_i d \tag{2.2}$$

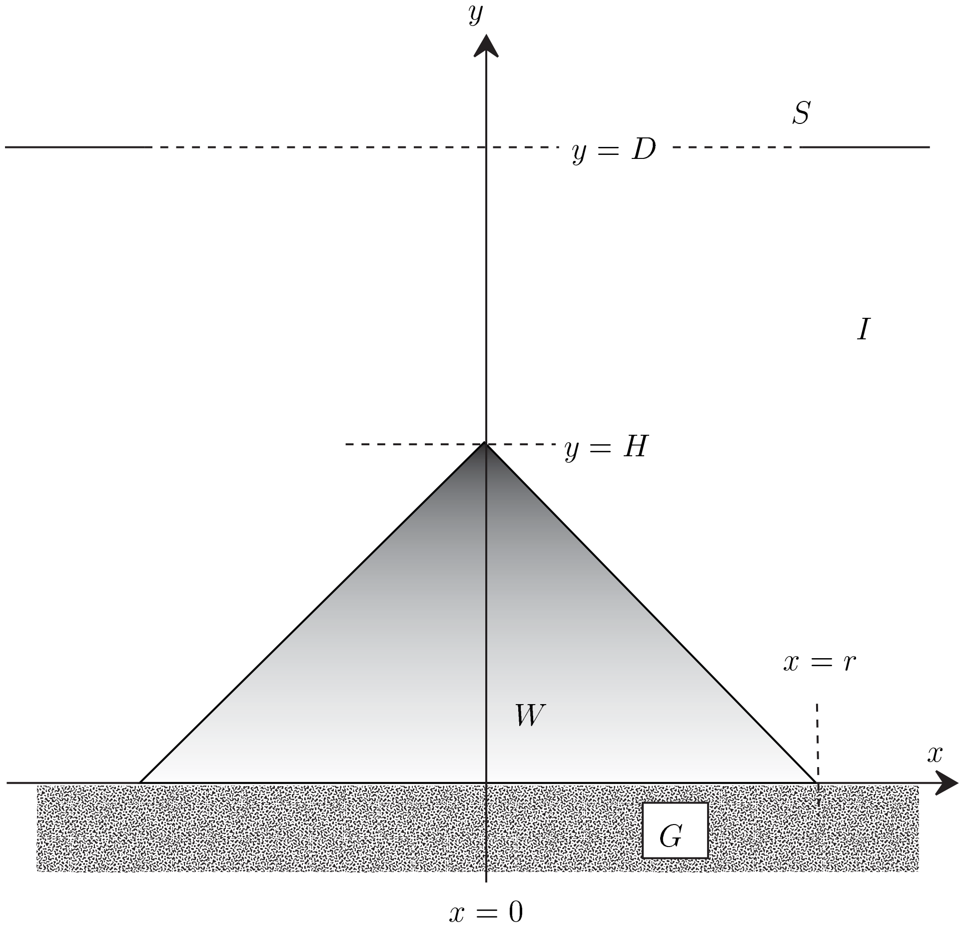


Figure 2.3: *A vertical section through the mid-plane of a water cone inside an ice cap. S: surface, W: water, G: ground, I: ice cap.*

$$\boldsymbol{d} = \frac{J_Q \cdot 1 \text{ year}}{L_i \ \rho_i} = \frac{0.06 \text{ J s}^{-1} \text{ m}^{-2} \ 365.25 \ \cdot 24 \ \cdot 60 \ \cdot 60 \text{ s}}{3.4 \cdot 10^5 \text{ J/kg } 917 \text{ kg/m}^3} = \mathbf{6.1 \cdot 10^{-3} m} \tag{2.3}$$

b)

Let p_a be the atmospheric pressure, taken to be constant. At a depth z inside the ice cap the pressure is given by:

$$p = \rho_i g z + p_a \tag{2.4}$$

Therefore, at the bottom of the ice cap, where $z = y_2 - y_1$:

$$\begin{aligned} \boldsymbol{p} &= \rho_i g(y_2 - y_1) + p_a & (2.5) \\ &= \boldsymbol{\rho_i g x (\tan\beta - \tan\alpha) + \rho_i g h_0 + p_a} & (2.6) \end{aligned}$$

For water not to move at the base of the ice cap the pressure must be hydrostatic (trivial, but can be seen from Bernoulli's equation), i.e.

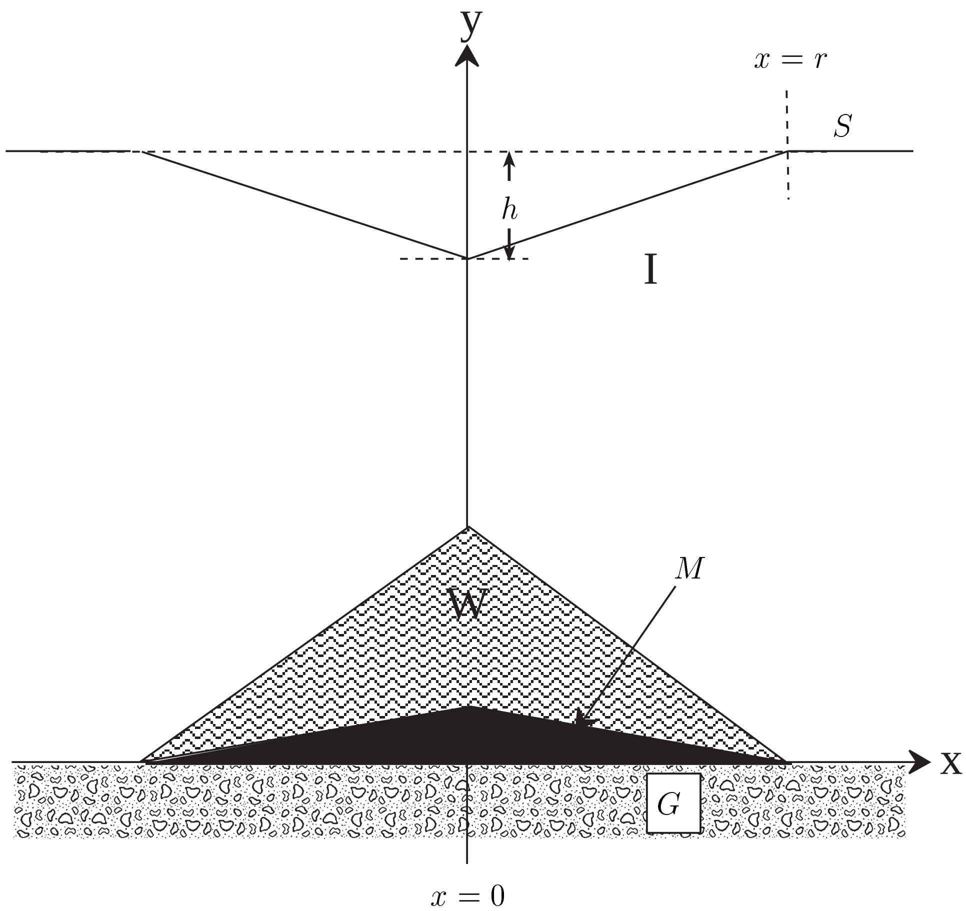


Figure 2.4: *A vertical and central cross section of a conical depression in a temperate ice cap. S: surface, G: ground, I: ice cap, M: rock/magma intrusion, W: water. Note that the figure is NOT drawn to scale.*

$$p = constant - \rho_w g y_1 \tag{2.7}$$
$$= constant - \rho_w g x \tan\alpha \tag{2.8}$$

Therefore

$$\rho_i g x(\tan\beta - \tan\alpha) = -\rho_w g x \tan\alpha \tag{2.9}$$

leading to

$$\boldsymbol{\tan\beta} = -\frac{\rho_w - \rho_i}{\rho_i}\ \tan\alpha = \boldsymbol{-\frac{\Delta\rho}{\rho_i}\ \tan\alpha} \approx -0.091\ \tan\alpha \tag{2.10}$$
$$\boldsymbol{s = -\Delta\rho/\rho_i = -0.091} \tag{2.11}$$
$$\tag{2.12}$$

where the minus-sign is significant.

This can also be seen in various ways by looking at a mass element of water at the bottom of the ice and demanding equilibrium. – We now proceed with the solution.

With $\tan\alpha = 0.8$, we get $\tan\beta = -0.073$ and

$$\boldsymbol{y_2 = 2\ \textbf{km} - 0.073\ x} \tag{2.13}$$

The students are supposed to draw this line on a graph.

c)

Since the ice adapts by vertical motion only we see that the conical depression at the surface will have the same radius of 1.0 km as the intrusion. According to (b) it will have a depth of

$$\begin{aligned} h &= |r\tan\beta| = \frac{\Delta\rho}{\rho_i} r\tan\alpha & (2.14)\\ &= \frac{\Delta\rho}{\rho_i} H & (2.15)\\ &= 0.091 \ \cdot 1 \text{ km} = 91 \text{ m.} & (2.16) \end{aligned}$$

The students are supposed to show this result as a graph.

d)

The volume of a circular cone is $V = \frac{1}{3}\pi r^2 h$. We assume that the height of the intrusion is h_1. We may say that it firstly melts an ice cone of its own volume $V_1 = \frac{1}{3}\pi r^2 h_1$. Pressure equilibrium has not yet been reached. Hence the water will flow away and the ice will keep contact with the face of the intrusion making the upper surface of the ice horizontal again. The intrusion then melts a volume equivalent to a cone of height $h_2 = \frac{\Delta\rho}{\rho_i} h_1$ whereupon pressure equilibrium has been reached (following part (c)). During this second phase the melted water will also flow away. Assuming that the intrusion still has not cooled down to 0°C the intrusion will further melt a volume equivalent to a cone of height h_3, its water accumulating in place, forming a cone of height $h_3' = \frac{\rho_i}{\rho_w}\ h_3$ relative to the top of the intrusion. The total height of the ice cone melted is

$$h_{tot} = h_1 + h_2 + h_3 \tag{2.17}$$

The depth of the depression at the surface will be given by

$$h = \frac{\Delta\rho}{\rho_i}\ (h_1 + h_3') \tag{2.18}$$

which is most easily seen by considering pressure equilibrium in the final situation (again following part (c)). Thus, the requested height of the top of the water cone is

$$\boldsymbol{H} = h_1 + h_3' = \frac{\rho_i}{\Delta\rho}\ h = \mathbf{1.1 \times 10^3 m} \tag{2.19}$$

The heat balance gives

$$\frac{1}{3}\ \pi\ r^2\ \{\rho_r h_1 (L_r\ + c_r\ \Delta T) - \rho_i\ L_i\ h_{tot}\} = 0 \tag{2.20}$$

where $\Delta T = 1200°\mathrm{C}$ is the change in temperature of the rock intrusion. Following equation (2.17) and using the facts that $h_2 = \frac{\Delta\rho}{\rho_i} h_1$ and $h_3 = \frac{\rho_w}{\rho_i} h_3'$ we obtain

$$h_{tot} = h_1 + \frac{\Delta\rho}{\rho_i} h_1 + \frac{\rho_w}{\rho_i} h_3' = \frac{\rho_w}{\rho_i} (h_1 + h_3') \tag{2.21}$$

Therefore (using equation (2.19))

$$h_{tot} = \frac{\rho_w}{\rho_i} (h_1 + h_3') = \frac{\rho_w}{\rho_i}\ H = \frac{\rho_w}{\Delta\rho}\ h = 1.20 \cdot 10^3 \mathrm{m} \tag{2.22}$$

This implies that the cone does not reach the surface of the ice cap. Inserting the result into the equation (2.20) we can solve for h_1:

$$\rho_r\ h_1\ (L_r +\ c_r\ \Delta T) = \frac{\rho_i\ \rho_w\ L_i\ h}{\Delta\rho} \tag{2.23}$$

$$\boldsymbol{h_1} = \frac{\rho_i\ \rho_w\ L_i\ h}{\Delta\rho\ \rho_r\ (L_r +\ c_r\ \Delta T)} \tag{2.24}$$

$$= \mathbf{103\ m} \tag{2.25}$$

The total mass of water formed is of course equal to the mass of the ice melted and is

$$\boldsymbol{m_{tot}} = \rho_i\ (1/3)\ \pi\ r^2\ h_{tot} = \mathbf{2.9 \cdot 10^{11}\ kg} \tag{2.26}$$

The mass of the water which flows away is

$$\boldsymbol{m'} = \frac{h_1 + h_2}{h_{tot}}\ m_{tot} = \frac{\rho_w\ h_1}{\rho_i\ h_{tot}}\ m_{tot} = \mathbf{2.7 \cdot 10^{10}\ kg} \tag{2.27}$$

The students are finally expected to plot the shapes of the rock intrusion and the water body.

2.3 Grading scheme

2(a)	
Answer: equation (2.3), $d = 6.1 \cdot 10^{-3}$ m	**0.5**

2(b)	
Answer i): equation (2.6): $p = \rho_i g x(\tan\beta - \tan\alpha) + \rho_i g h_0 + p_a$	**1.0**
Answer ii): equation (2.10): $s = -\frac{\rho_w - \rho_i}{\rho_i} = -\frac{\Delta\rho}{\rho_i}$	**2.0**
Answer iii): Graph based on equation (2.13)	**0.5**

2(c)	
Answer: Depth, radius and graph, $r = 1000$ m, $h = 91$ m	**1.0**

2(d)	
Answer i): Height of water cone as in (2.19): $H = 1.1 \cdot 10^3$ m	**2.0**
Answer ii): Height of intrusion as in (2.25): $h_1 = 103$ m	**1.0**
Answer iii): Total mass of melt water as in (2.26): $m_{tot} = 2.9 \cdot 10^{11}$ kg	**0.5**
Answer iv): Mass of water flowing away as in (2.27): $m' = 2.7 \cdot 10^{10}$ kg	**1.0**
Answer v): Graph	**0.5**

3 Faster than light?[7]

3.1 Problem text

In this problem we analyze and interpret measurements made in 1994 on radio wave emission from a compound source within our galaxy.

The receiver was tuned to a broad band of radio waves of wavelengths of several centimeters. Figure 3.1 shows a series of images recorded at different times. The contours indicate constant radiation strength in much the same way as altitude contours on a geographical map. In the figure the two maxima are interpreted as showing two objects moving away from a common center shown by crosses in the images. (The center, which is assumed to be fixed in space, is also a strong radiation emitter but mainly at other wavelengths). The measurements conducted on the various dates were made at the same time of day.

The scale of the figure is given by a line segment showing one arc second (as). (1 as = 1/3600 of a degree). The distance to the celestial body at the center of the figure, indicated by crosses, is estimated to be $R = 12.5$ kpc. A kiloparsec (kpc) equals $3.09 \cdot 10^{19}$ m. The speed of light is c $= 3.00 \cdot 10^{8}$ m/s. Error calculations are not required in the solution.

a) (2 points) We denote the angular positions of the two ejected radio emitters, relative to the common center, by $\theta_1(t)$ and $\theta_2(t)$, where the subscripts 1 and 2 refer to the left and right hand ones, respectively, and t is the time of observation. The angular speeds, as seen from the Earth, are ω_1 and ω_2. The corresponding apparent transverse linear speeds of the two sources are denoted by $v'_{1,\perp}$ and $v'_{2,\perp}$.

Using Figure 3.1, make a graph to find the numerical values of ω_1 and ω_2 in milli-arc-seconds per day (mas/d). Also determine the numerical values of $v'_{1,\perp}$ and $v'_{2,\perp}$, and write all answers on the answer sheet. (You may be puzzled by some of the results).

b) (3 points) In order to resolve the puzzle arising in part (a), consider a light-source moving with velocity $\vec{v}$ at an angle ϕ ($0 \leq \phi \leq \pi$) to the direction towards a distant observer O (Figure 3.2). The speed may be written as $v = \beta c$, where c is the speed of light. The distance to the source, as measured by the observer, is R. The angular speed of the source, as seen from the observer, is ω, and the apparent linear speed perpendicular to the line of sight is $v'_{\perp}$.

Find ω and $v'_{\perp}$ in terms of β, R and ϕ and write your answer on the answer sheet.

c) (1 point) We assume that the two ejected objects, described in the introduction and in part (a), are moving in opposite directions with equal speeds $v = \beta c$. Then the results of part (b) make it possible to calculate β and ϕ from the angular speeds ω_1 and ω_2 and the distance R. Here ϕ is the angle defined in part (b), for the left hand object, corresponding to subscript 1 in part (a).

Derive formulas for β and ϕ in terms of known quantities and determine their numerical values from the data in part (a). Write your answers in the designated fields on the answer sheet.

d) (2 points) In the one-body situation of part (b), find the condition for the apparent perpendicular speed $v'_{\perp}$ to be larger than the speed of light c.

[7] Authors: Einar Gudmundsson, Knútur Árnason and Thorsteinn Vilhjálmsson

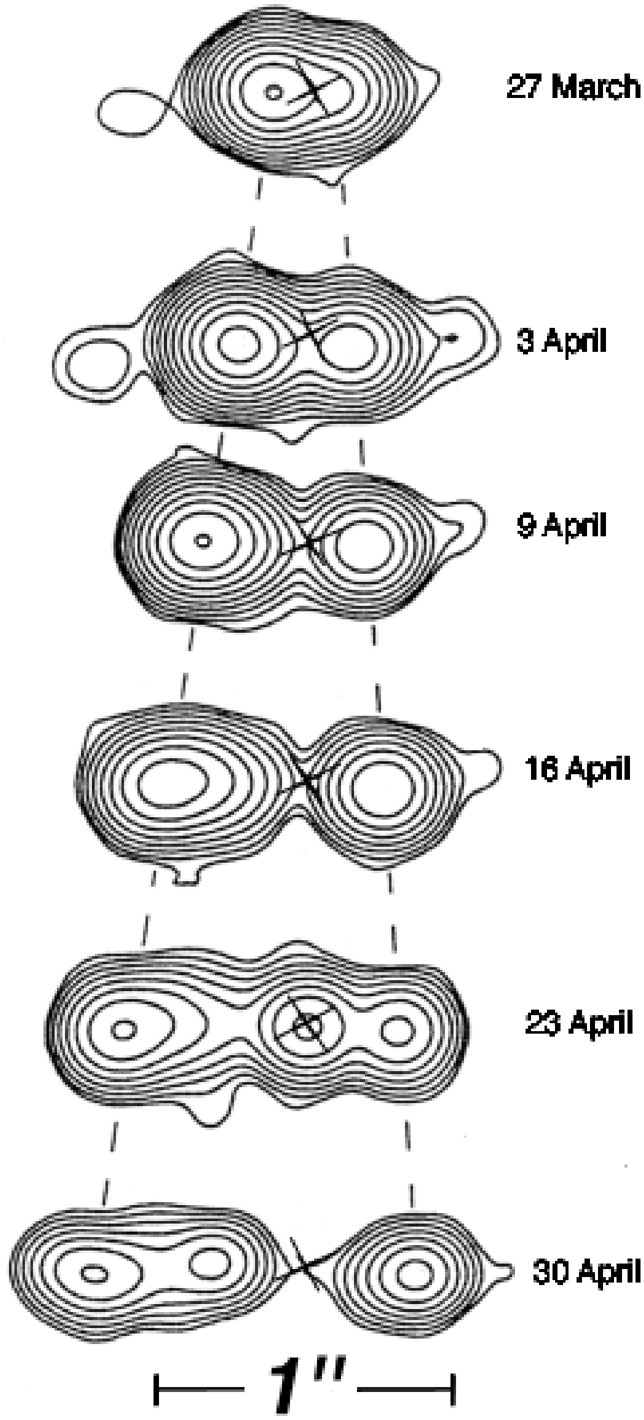


Figure 3.1: *Radio emission from a source in our galaxy.*

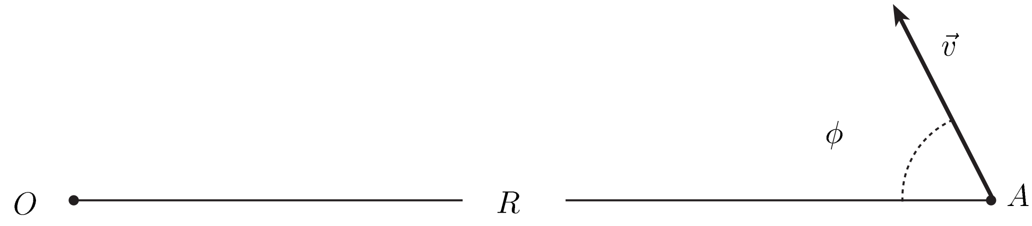


Figure 3.2: *The observer is at O and the original position of the light source is at A. The velocity vector is $\vec{v}$.*

Write the condition in the form $\beta > f(\phi)$ and provide an analytic expression for the function f on the answer sheet.

Draw on the graph answer sheet the physically relevant region of the (β, ϕ)-plane. Show by shading in which part of this region the condition $v'_\perp > c$ holds.

e) (1 point) Still in the one-body situation of part (b), find an expression for the maximum value $(v'_\perp)_{max}$ of the apparent perpendicular speed $v'_\perp$ for a given β and write it in the designated field on the answer sheet. Note that this speed increases without limit when $\beta \to 1$.

f) (1 point) The estimate for R given in the introduction is not very reliable. Scientists have therefore started speculating on a better and more direct method for determining R. One idea for this goes as follows. Assume that we can identify and measure the Doppler shifted wavelengths λ_1 and λ_2 of radiation from the two ejected objects, corresponding to the same known original wavelength λ_0 in the rest frames of the objects.

Starting from the equations for the relativistic Doppler shift, $\lambda = \lambda_0(1 - \beta\cos\phi)(1 - \beta^2)^{-1/2}$, and assuming, as before, that both objects have the same speed, v, show that the unknown $\beta = v/c$ can be expressed in terms of λ_0, λ_1, and λ_2 as

$$\beta = \sqrt{1 - \frac{\alpha\, \lambda_0^2}{(\lambda_1 + \lambda_2)^2}} \,. \tag{3.1}$$

Write the numerical value of the coefficient α in the designated field on the answer sheet.

You may note that this means that the suggested wavelength measurements will in practice provide a new estimate of the distance.

3.2 Solution

a) On Figure 3.1 we mark the centers of the sources as neatly as we can. Let $\theta_1(t)$ be the angular distance of the left center from the cross as a function of time and $\theta_2(t)$ the angular distance of the right center. We measure these quantities on the figure at the given times by a ruler and convert to arcseconds according to the given scale. This results in the following numerical data:

time [days]	θ_1 [as]	θ_2 [as]
0	0.139	0.076
7	0.253	0.139
13	0.354	0.190
20	0.468	0.253
27	0.601	0.316
34	0.709	0.367

The uncertainty in the readings by the ruler is estimated to be ± 0.5 mm, resulting in the uncertainty of $\pm$ 0.013 as in the θ values. We plot the data in Figure 3.3.

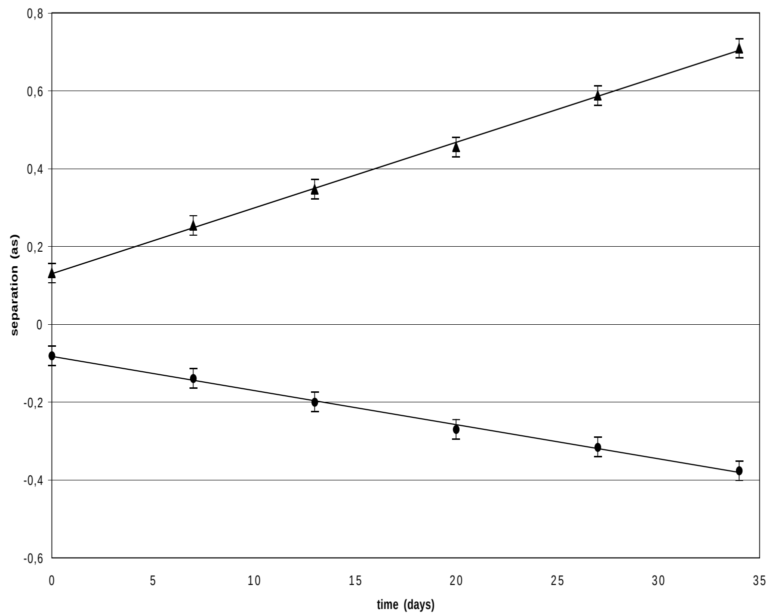


Figure 3.3: *The angular distances θ_1 and θ_2 (in as) as functions of the time in days.*

Fitting straight lines through the data results in:

$$\boldsymbol{\omega_1} = \mathrm{d}\theta_1/\mathrm{dt} = \mathbf{(17.0 \pm 1.0)\ mas/day} = 9.54 \cdot 10^{-13}\mathrm{rad/s} \tag{3.2}$$

$$\boldsymbol{\omega_2} = \mathrm{d}\theta_2/\mathrm{dt} = \mathbf{(8.7 \pm 1.0)\ mas/day} = 4.88 \cdot 10^{-13}\mathrm{rad/s} \tag{3.3}$$

$$\boldsymbol{v'_{1,\perp}} = \omega_1\ R = 9.54 \cdot 10^{-13} \cdot 12.5 \cdot 3.09 \cdot 10^{19} \tag{3.4}$$

$$= \mathbf{3.68 \cdot 10^8 m/s \approx (1.23 \pm 0.07)}\ \boldsymbol{c} \tag{3.5}$$

$$\boldsymbol{v'_{2,\perp}} = \mathbf{1.89 \cdot 10^8 m/s \approx (0.63 \pm 0.07)}\ \boldsymbol{c} \tag{3.6}$$

b) We consider the motion of the source during the time interval Δt from the point A to the point A', see Figure 3.4.

We then have

$$\vec{r}_{AA'} = \vec{r}_{A'} - \vec{r}_A = \vec{v} \cdot \Delta t\ . \tag{3.7}$$

Now let $\Delta t'$ denote the difference in arrival times at O of the signals from A and A'. Due to the different distances to A and A' and the finite speed of light, c, we have

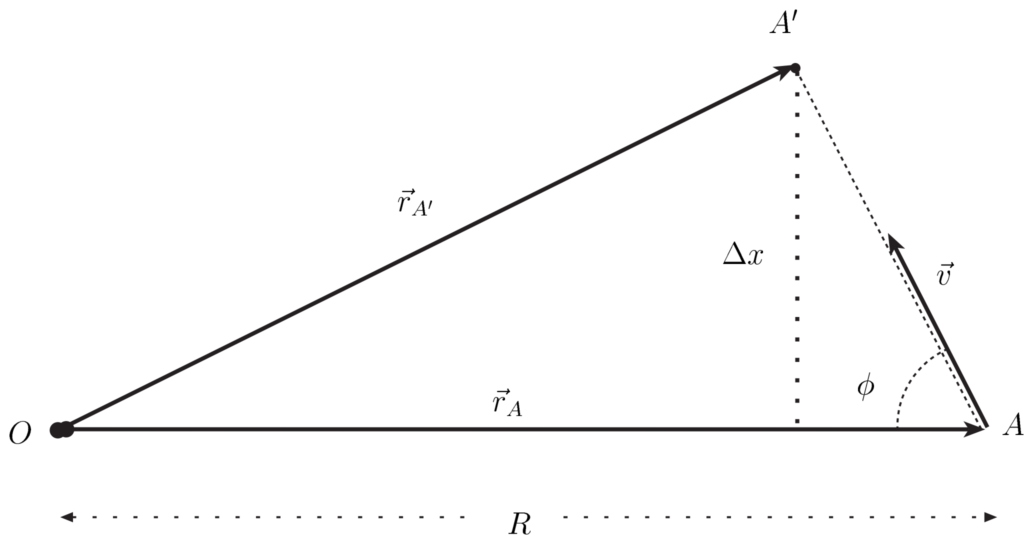


Figure 3.4: *The observer is at O and the original position of the source is at A. The velocity vector is $\vec{v}$.*

$$\Delta t' = \Delta t + (r_{A'} - r_A)/c \ . \tag{3.8}$$

For small Δt, such that $v\ \Delta t << r_A = R$, we have

$$r_{A'} - r_A \approx -v\ \Delta t \cos\phi \tag{3.9}$$

and hence

$$\Delta t' \approx \Delta t\ (1 - \beta\cos\phi)\ ;\ \beta = v/c \ . \tag{3.10}$$

This implies that an observer at O will find the apparent transverse speed of the source to be

$$\boldsymbol{v'_\perp} = \frac{\Delta x}{\Delta t'} = \frac{\Delta x}{\Delta t\ (1 - \beta\cos\phi)} = \boldsymbol{\frac{c\beta\sin\phi}{1 - \beta\cos\phi}} \tag{3.11}$$

where we have used that the real transverse speed in the reference frame of the observer is $v_\perp = \Delta x/\Delta t = c\beta\sin\phi$.

The angular speed observed at O is

$$\boldsymbol{\omega} = \frac{v'_\perp}{R} = \boldsymbol{\frac{c\beta\sin\phi}{R\ (1 - \beta\cos\phi)}} \tag{3.12}$$

c) Figure 3.5 shows the situation in this case. Note the relations given in the caption. Taking $\phi = \phi_1$ we have $\sin\phi_2 = \sin\phi$ and $\cos\phi_2 = -\cos\phi$. Equation (3.12) then gives:

$$\omega_1 = \frac{\beta\ c\ \sin\phi}{R\ (1 - \beta\cos\phi)} \tag{3.13}$$

$$\omega_2 = \frac{\beta\ c\ \sin\phi}{R\ (1 + \beta\cos\phi)} \ . \tag{3.14}$$

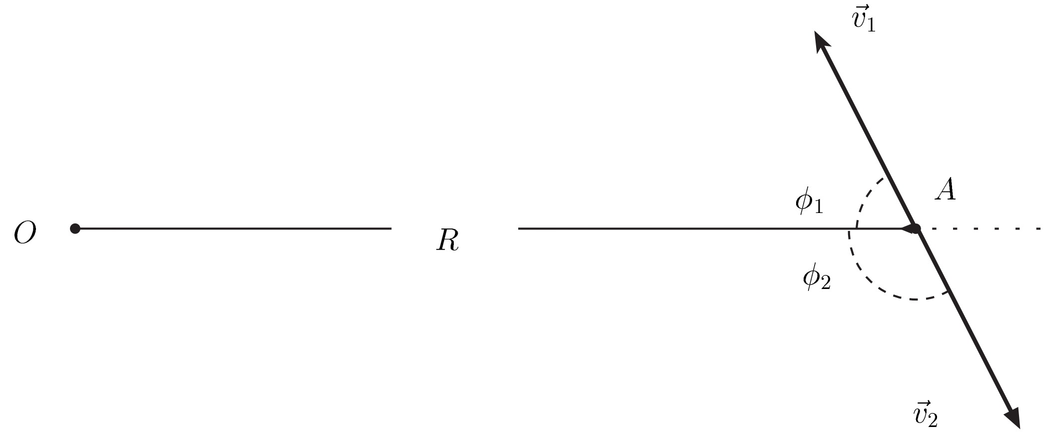


Figure 3.5: *If the two objects have equal speeds but opposite velocities we have* $v_1 = v_2 = v,\ \beta_1 = \beta_2 = \beta$ *and* $\phi_2 = \pi - \phi_1$.

The quantities ω_1, ω_2 and R are given, but β and ϕ are to be determined as stated in the problem text. Simple algebra gives:

$$
\begin{aligned}
(1 - \beta\ \cos\phi)\ \omega_1\ \omega_2 &= \beta\ c\ \sin\phi\ \omega_2/R && (3.15)\\
(1 + \beta\ \cos\phi)\ \omega_2\ \omega_1 &= \beta\ c\ \sin\phi\ \omega_1/R\ . && (3.16)
\end{aligned}
$$

Subtracting (3.15) from (3.16) gives:

$$2\ \beta\ \cos\phi\ \omega_2\ \omega_1 = \beta\ c\ \sin\phi\ (\omega_1 - \omega_2)/R \tag{3.17}$$

$$\tan\phi = \frac{2\ R\ \omega_2\ \omega_1}{c\ \ (\omega_1 - \omega_2)} \tag{3.18}$$

$$\boldsymbol{\phi = \mathbf{arctan}\left(\frac{2\ R\ \omega_2\ \omega_1}{c\ \ (\omega_1 - \omega_2)}\right)}\ . \tag{3.19}$$

Dividing (3.15) by (3.16) gives β in terms of $\cos\phi$ and the known quantities ω_1 and ω_2:

$$\omega_1\ (1 - \beta\ \cos\phi) = \omega_2\ (1 + \beta\ \cos\phi) \tag{3.20}$$

$$\boldsymbol{\beta} = \frac{\boldsymbol{\omega_1 - \omega_2}}{\boldsymbol{\cos\phi\ (\omega_1 + \omega_2)}}\ . \tag{3.21}$$

Inserting the values of ω_1 and ω_2 from part (a) and the given values of R and c we get:

$$
\begin{aligned}
\boldsymbol{\phi} &= \arctan(2.57) = \mathbf{1.20\ rad = 68.8^\circ \pm 2^\circ} && (3.22)\\
\boldsymbol{\beta} &= \mathbf{0.892 \pm 0.08} && (3.23)
\end{aligned}
$$

d) Equation (3.11) shows that the observer will find the apparent transverse speed to be larger than or equal to the speed of light if and only if:

$$\frac{\beta \sin \phi}{1 - \beta \cos \phi} \geq 1. \tag{3.24}$$

If $\beta < 1$ condition (3.24) is equivalent to:

$$\begin{aligned}
\beta \sin \phi &\geq 1 - \beta \cos \phi && (3.25)\\
\beta \left(\sin \phi + \cos \phi\right) &\geq 1 && (3.26)\\
\beta \sqrt{2} \left(\sin \phi \cos \frac{\pi}{4} + \cos \phi \sin \frac{\pi}{4}\right) &\geq 1 && (3.27)\\
\sin \left(\phi + \frac{\pi}{4}\right) &\geq \frac{1}{\beta \sqrt{2}} && (3.28)
\end{aligned}$$

and hence (3.24) is satisfied if:

$$\beta > \boldsymbol{f(\phi)} = \left(\sqrt{\mathbf{2}}\, \mathbf{sin}(\boldsymbol{\phi} + \boldsymbol{\pi}/\mathbf{4})\right)^{-\mathbf{1}}. \tag{3.29}$$

The physically relevant region in the (β, ϕ)-plane is:

$$(\boldsymbol{\beta}, \boldsymbol{\phi}) \in [\mathbf{0}, \mathbf{1}[\times[\mathbf{0}, \boldsymbol{\pi}]\,. \tag{3.30}$$

It is obvious that (3.24) can only be satisfied for $\phi \in [0, \pi/2]$ and (3.28) can only have a solution for ϕ if $\beta \geq 1/\sqrt{2}$.

We therefore take a closer look at the region

$$(\beta, \phi) \;\in\; [2^{-1/2}, 1[\; \times \; [0, \pi/2] \tag{3.31}$$

The mapping

$$(\beta, \phi) \mapsto \beta \sin \left(\phi + \frac{\pi}{4}\right) \tag{3.32}$$

is continuous in this region. It is therefor sufficient to look at the boundary of the region, defined by the equality sign in (3.28):

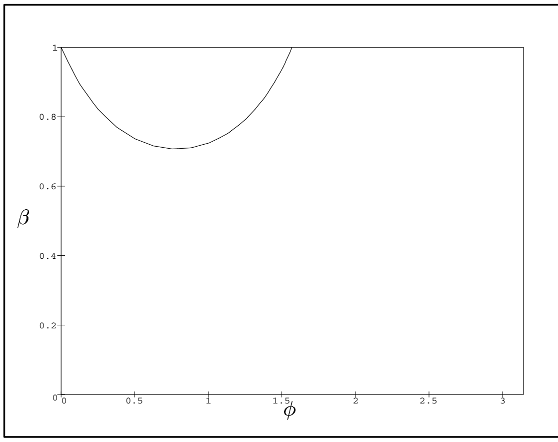
$$\beta \sin \left(\phi + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} \tag{3.33}$$

This defines β as a function of ϕ which is shown in Figure 3.6 as the curve bounding the shaded area where $v'_{\perp} > c$.

e) To find the extrema of $v'_{\perp}$ as a function of ϕ we differentiate (3.11) and get

$$\frac{d}{d\phi}\left(\frac{v'_{\perp}}{c}\right) = \frac{\beta(\cos \phi - \beta)}{(1 - \beta \cos \phi)^2}\,. \tag{3.34}$$

This is zero for $\phi = \phi_m$ where:



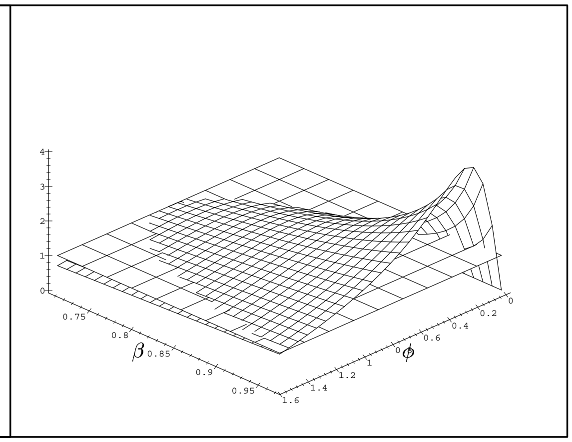


Figure 3.6: *The region between the horizontal line and the curve in the upper left hand corner shows where* $v'_\perp/c > 1$.

Figure 3.7: *The curved surface is* $v'_\perp/c$ *as a function of* β *and* ϕ. *The plane represents the constant function* $\beta = 1$.

$$\cos\phi_m = \beta\ ;\ \phi_m = \arccos\beta \in]0, \pi/2] \tag{3.35}$$

To see that this is indeed a maximum, we differentiate (3.34) again and get:

$$\frac{d^2}{d\phi^2}\left(\frac{v'_\perp}{c}\right) = -\beta\left(\frac{\sin\phi}{(1-\beta\cos\phi)^2} + 2\frac{\beta\sin\phi(\cos\phi-\beta)}{(1-\beta\cos\phi)^3}\right) \tag{3.36}$$

At the extremum

$$\frac{d^2}{d\phi^2}\left(\frac{v'_\perp}{c}\right)\Bigg|_{\phi_m} = -\frac{\beta\sin\phi_m}{(1-\beta^2)^2} < 0 \tag{3.37}$$

showing that ϕ_m corresponds to a maximum. From (3.11) and (3.35) the maximum apparent transverse speed is given:

$$\boldsymbol{(v'_\perp)_{max} = \frac{\beta c}{\sqrt{1-\beta^2}}} \tag{3.38}$$

From this and (3.35) we see that

$$(v'_\perp)_{max} \underset{\beta\to 1}{\longrightarrow} \infty\ ;\ \phi_m \underset{\beta\to 1}{\longrightarrow} 0\ . \tag{3.39}$$

Figure 3.7 shows $v'_\perp/c$ as a function of β and ϕ in the region $(\beta, \phi) \in [2^{-1/2}, 1[\times [0, \pi/2]$.

f) We have the equations for relativistic Doppler-shift:

$$\frac{\lambda_{1,2}}{\lambda_0} = \frac{1 \mp \beta \cos\phi}{\sqrt{1-\beta^2}} \tag{3.40}$$

We add them, define an auxiliary ratio ρ and solve for β.

$$\rho := \frac{\lambda_1 + \lambda_2}{2\ \lambda_0} = \frac{1}{\sqrt{1-\beta^2}} \tag{3.41}$$

$$\rho^2\ (1-\beta^2) = 1 \tag{3.42}$$

$$\beta = \sqrt{1 - 1/\rho^2} = \sqrt{1 - \frac{4\ \lambda_0^2}{(\lambda_1+\lambda_2)^2}} \tag{3.43}$$

giving

$$\boldsymbol{\alpha = 4} \tag{3.44}$$

Adding equation (3.43) to the set of equations (3.18) and (3.21) we have three equations which can be solved for the three unknowns β, ϕ and R. For instance, we may calculate β from (3.43), insert that into (3.21), and solve for ϕ. The distance R can then be obtained from (3.18). Thus the measurement of the Doppler-shifted wavelengths turns out to give an estimate of the distance to the source provided that ω_1 and ω_2 are known.

3.3 Grading scheme

Part 1(a)	
Answer i): equation (3.2), ω_1 in the range (16.5-17.5) mas/day	**0.8**
Answer ii): equation (3.3), ω_2 in the range (8.2-9.2) mas/day	**0.8**
Answer iii): equation (3.4), for $v'_{1,\perp}$ in the range (1.13-1.30)c	**0.2**
Answer iv): equation (3.6), for $v'_{2,\perp}$ in the range (0.56-0.70)c	**0.2**

Part 1(b)	
Answer i): $v'_\perp(\beta,\phi)$, equation (3.11)	**2.5**
Answer ii): $\omega(\beta,\phi)$, equation (3.12)	**0.5**

Part 1(c)	
Answer i): $\phi(\omega_1,\omega_2)$, equation (3.19)	**0.3**
Answer ii): $\beta(\omega_1,\omega_2)$, equation (3.21)	**0.3**
Answer iii): ϕ numerical in the range 67° - 71°	**0.2**
Answer iv): β numerical in the range 0.81-0.97	**0.2**

Part 1(d)	
Answer i): Condition $\beta > f(\phi)$, equation (3.29)	**1.0**
Answer ii): Condition on (β,ϕ), graph	**1.0**

Part 1(e)	
Answer: $(v'_\perp)_{max}$, equation (3.38)	**1.0**

Part 1(f)	
Answer: β in terms of λ-s, by α, equation (3.44)	**1.0**

30th International Physics Olympiad

Padua, Italy

Theoretical competition

Thursday, July 22nd, 1999

Please read this first:

1. The time available is 5 hours for 3 problems.
2. Use only the pen provided.
3. Use only the **front side** of the provided sheets.
4. In addition to the problem texts, that contain the specific data for each problem, a sheet is provided containing a number of general physical constants that may be useful for the problem solutions.
5. Each problem should be answered on separate sheets.
6. In addition to "blank" sheets where you may write freely, for each problem there is an *Answer sheet* where you **must** summarize the results you have obtained. Numerical results must be written with as many digits as appropriate to the given data; don't forget the units.
7. Please write on the "blank" sheets whatever you deem important for the solution of the problem, that you wish to be evaluated during the marking process. However, you should use mainly equations, numbers, symbols, figures, and use *as little text as possible*.
8. **It's absolutely imperative** that you write on top of *each* sheet that you'll use: your name ("**NAME**"), your country ("**TEAM**"), your student code (as shown on the identification tag, "**CODE**"), and additionally on the "blank" sheets: the problem number ("**Problem**"), the progressive number of each sheet (from 1 to *N*, "**Page n.**") and the total number (*N*) of "blank" sheets that you use and wish to be evaluated for that problem ("**Page total**"). It is also useful to write the section you are answering at the beginning of each such section. If you use some sheets for notes that you don't wish to be evaluated by the marking team, just put a large cross through the whole sheet, and don't number it.
9. When you've finished, turn in all sheets in proper order (for each problem: answer sheet first, then used sheets in order; unused sheets and problem text at the bottom) and put them all inside the envelope where you found them; then leave everything on your desk. You are not allowed to take **any** sheets out of the room.

This set of problems consists of 13 pages (including this one, the answer sheets and the page with the physical constants)

These problems have been prepared by the Scientific Committee of the 30th IPhO, including professors at the Universities of Bologna, Naples, Turin and Trieste.

Problem 1

Absorption of radiation by a gas

A cylindrical vessel, with its axis vertical, contains a molecular gas at thermodynamic equilibrium. The upper base of the cylinder can be displaced freely and is made out of a glass plate; let's assume that there is no gas leakage and that the friction between glass plate and cylinder walls is just sufficient to damp oscillations but doesn't involve any significant loss of energy with respect to the other energies involved. Initially the gas temperature is equal to that of the surrounding environment. The gas can be considered as perfect within a good approximation. Let's assume that the cylinder walls (including the bases) have a very low thermal conductivity and capacity, and therefore the heat transfer between gas and environment is very slow, and can be neglected in the solution of this problem.

Through the glass plate we send into the cylinder the light emitted by a constant power laser; this radiation is easily transmitted by air and glass but is completely absorbed by the gas inside the vessel. By absorbing this radiation the molecules reach excited states, where they quickly emit infrared radiation returning in steps to the molecular ground state; this infrared radiation, however, is further absorbed by other molecules and is reflected by the vessel walls, including the glass plate. The energy absorbed from the laser is therefore transferred in a very short time into thermal movement (molecular chaos) and thereafter stays in the gas for a sufficiently long time.

We observe that the glass plate moves upwards; after a certain irradiation time we switch the laser off and we measure this displacement.

1. Using the data below and - if necessary - those on the sheet with physical constants, compute the temperature and the pressure of the gas after the irradiation. *[2 points]*
2. Compute the mechanical work carried out by the gas as a consequence of the radiation absorption. *[1 point]*
3. Compute the radiant energy absorbed during the irradiation. *[2 points]*
4. Compute the power emitted by the laser that is absorbed by the gas, and the corresponding number of photons (and thus of elementary absorption processes) per unit time. *[1.5 points]*
5. Compute the efficiency of the conversion process of optical energy into a change of mechanical potential energy of the glass plate. *[1 point]*

Thereafter the cylinder axis is slowly rotated by 90°, bringing it into a horizontal direction. The heat exchanges between gas and vessel can still be neglected.

6. State whether the pressure and/or the temperature of the gas change as a consequence of such a rotation, and - if that is the case – what is its/their new value. *[2.5 points]*

Data

Room pressure: $p_0 = 101.3$ kPa
Room temperature: $T_0 = 20.0$°C
Inner diameter of the cylinder: $2r = 100$ mm
Mass of the glass plate: $m = 800$ g
Quantity of gas within the vessel: $n = 0.100$ mol
Molar specific heat at constant volume of the gas: $c_V = 20.8$ J/(mol·K)
Emission wavelength of the laser: $\lambda = 514$ nm
Irradiation time: $\Delta t = 10.0$ s
Displacement of the movable plate after irradiation: $\Delta s = 30.0$ mm

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Problem	1
Page n.	A
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Answer sheet

In this problem you are requested to give your results both as analytical expressions and with numerical data and units: write expressions first and then data (e.g. $A=bc$=1.23 m^2).

1. Gas temperature after the irradiation …………………………………………………...

Gas pressure after the irradiation ……………………………………………………….

2. Mechanical work carried out ………………………………………………………………...

3. Overall optical energy absorbed by the gas …………………………………………………

4. Optical laser power absorbed by the gas ……………………………………………………

Absorption rate of photons (number of absorbed photons per unit time) …………………...

5. Efficiency in the conversion of optical energy into change of mechanical potential energy of the glass plate ………………………………………………………………….

6. Owing to the cylinder rotation, is there a pressure change? YES ☐ NO ☐

If yes, what is its new value? ……………………………………………………………

Owing to the cylinder rotation, is there a temperature change? YES ☐ NO ☐

If yes, what is its new value? ……………………………………………………………

Physical constants and general data

In addition to the numerical data given within the text of the individual problems, the knowledge of some general data and physical constants may be useful, and you may find them among the following ones. These are nearly the most accurate data currently available, and they have thus a large number of digits; you are expected, however, to write your results with a number of digits that must be appropriate for each problem.

Speed of light in vacuum: $c = 299792458\ \text{m}\cdot\text{s}^{-1}$
Magnetic permeability of vacuum: $\mu_0 = 4\pi\cdot10^{-7}\ \text{H}\cdot\text{m}^{-1}$
Dielectric constant of vacuum: $\varepsilon_0 = 8.8541878\ \text{pF}\cdot\text{m}^{-1}$
Gravitational constant: $G = 6.67259\cdot10^{-11}\ \text{m}^3/(\text{kg}\cdot\text{s}^2)$
Gas constant: $R = 8.314510\ \text{J}/(\text{mol}\cdot\text{K})$
Boltzmann's constant: $k = 1.380658\cdot10^{-23}\ \text{J}\cdot\text{K}^{-1}$
Stefan's constant: $\sigma = 56.703\ \text{nW}/(\text{m}^2\cdot\text{K}^4)$
Proton charge: $e = 1.60217733\cdot10^{-19}\ \text{C}$
Electron mass: $m_e = 9.1093897\cdot10^{-31}\ \text{kg}$
Planck's constant: $h = 6.6260755\cdot10^{-34}\ \text{J}\cdot\text{s}$
Base of centigrade scale: $T_K = 273.15\ \text{K}$
Sun mass: $M_S = 1.991\cdot10^{30}\ \text{kg}$
Earth mass: $M_E = 5.979\cdot10^{24}\ \text{kg}$
Mean radius of Earth: $r_E = 6.373\ \text{Mm}$
Major semiaxis of Earth orbit: $R_E = 1.4957\cdot10^{11}\ \text{m}$
Sidereal day: $d_S = 86.16406\ \text{ks}$
Year: $y = 31.558150\ \text{Ms}$
Standard value of the gravitational field at the Earth surface: $g = 9.80665\ \text{m}\cdot\text{s}^{-2}$
Standard value of the atmospheric pressure at sea level: $p_0 = 101325\ \text{Pa}$
Refractive index of air for visibile light, at standard pressure and 15 °C: $n_{air} = 1.000277$
Solar constant: $S = 1355\ \text{W}\cdot\text{m}^{-2}$
Jupiter mass: $M = 1.901\cdot10^{27}\ \text{kg}$
Equatorial Jupiter radius: $R_B = 69.8\ \text{Mm}$
Average radius of Jupiter's orbit: $R = 7.783\cdot10^{11}\ \text{m}$
Jovian day: $d_J = 35.6\ \text{ks}$
Jovian year: $y_J = 374.32\ \text{Ms}$
π: 3.14159265

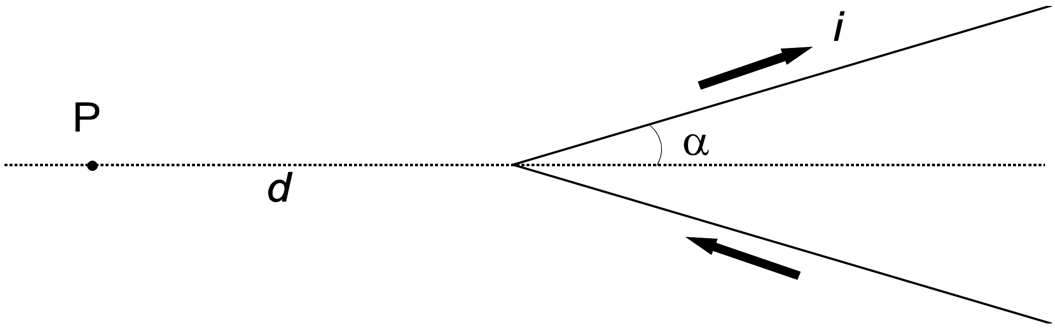
Problem 2

Magnetic field with a V-shaped wire

Among the first successes of the interpretation by Ampère of magnetic phenomena, we have the computation of the magnetic field **B** *generated by wires carrying an electric current, as compared to early assumptions originally made by Biot and Savart.*

A particularly interesting case is that of a very long thin wire, carrying a constant current i, made out of two rectilinear sections and bent in the form of a "V", with angular half-span[1] α (see figure). According to Ampère's computations, the magnitude B of the magnetic field in a given point P lying on the axis of the "V", outside of it and at a distance d from its vertex, is proportional to $\tan\left(\frac{\alpha}{2}\right)$.

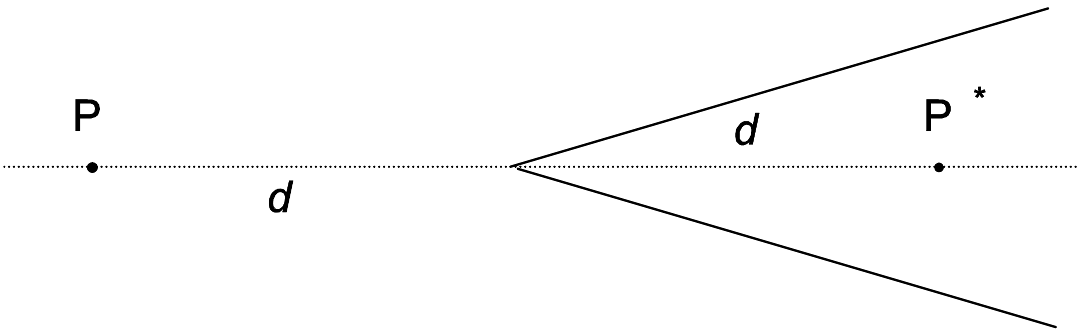
Ampère's work was later embodied in Maxwell's electromagnetic theory, and is universally accepted.



Using our contemporary knowledge of electromagnetism,

1. Find the direction of the field **B** in P. *[1 point]*
2. Knowing that the field is proportional to $\tan\left(\frac{\alpha}{2}\right)$, find the proportionality factor k in $\left|\mathbf{B}(\mathrm{P})\right| = k\tan\left(\frac{\alpha}{2}\right)$. *[1.5 points]*
3. Compute the field **B** in a point P^* symmetric to P with respect to the vertex, *i.e.* along the axis and at the same distance d, but inside the "V" (see figure). *[2 points]*

[1] Throughout this problem α is expressed in radians



4. In order to measure the magnetic field, we place in P a small magnetic needle with moment of inertia I and magnetic dipole moment $\boldsymbol{\mu}$; it oscillates around a fixed point in a plane containing the direction of **B**. Compute the period of small oscillations of this needle as a function of B. *[2.5 points]*

In the same conditions Biot and Savart had instead assumed that the magnetic field in P might have been (we use here the modern notation) $B(\mathrm{P}) = \dfrac{i\,\mu_0\,\alpha}{\pi^2 d}$, where μ_0 is the magnetic permeability of vacuum. In fact they attempted to decide with an experiment between the two interpretations (Ampère's and Biot and Savart's) by measuring the oscillation period of the magnetic needle as a function of the "V" span. For some α values, however, the differences are too small to be easily measurable.

5. If, in order to distinguish experimentally between the two predictions for the magnetic needle oscillation period T in P, we need a difference by at least 10%, namely $T_1 > 1.10\, T_2$ (T_1 being the Ampere prediction and T_2 the Biot-Savart prediction) state in which range, approximately, we must choose the "V" half-span α for being able to decide between the two interpretations. *[3 points]*

Hint

Depending on which path you follow in your solution, the following trigonometric equation might be useful: $\tan\left(\dfrac{\alpha}{2}\right) = \dfrac{\sin\alpha}{1+\cos\alpha}$

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Problem	2
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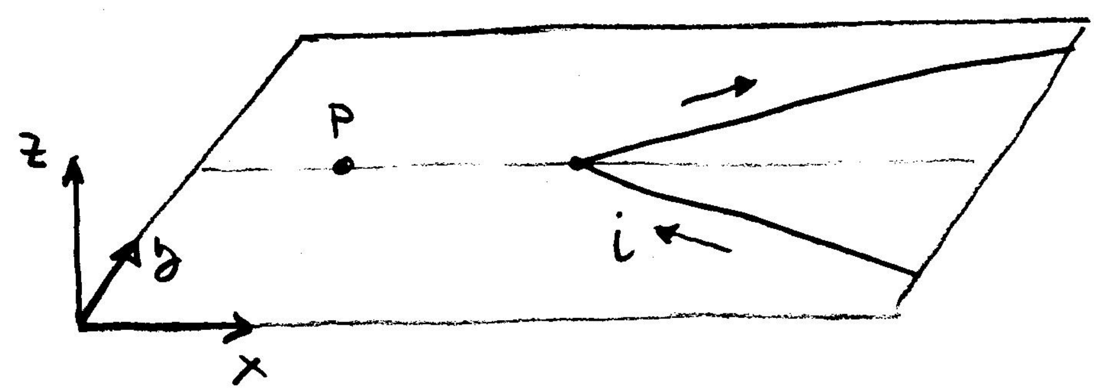
Answer sheet

In this problem write the requested results as analytic expressions, not as numerical values and units, unless explicitly indicated.

1. Using the following sketch draw the direction of the **B** field (the length of the vector is not important). The sketch is a spatial perspective view.



2. Proportionality factor k ……………………….

3. Absolute value of the magnetic field intensity at the point P^*, as described in the text………………………..………………

Draw the direction of the **B** field in the above sketch

4. Period of the small angle oscillations of the magnet ……………………………

5. Write for which range of α values (indicating here the numerical values of the range limits) the ratio between the oscillation periods, as predicted by Ampère and by Biot and Savart, is larger than 1.10:

…………………….. ……………………….

Problem 3

A space probe to Jupiter

We consider in this problem a method frequently used to accelerate space probes in the desired direction. The space probe flies by a planet, and can significantly increase its speed and modify considerably its flight direction, by taking away a very small amount of energy from the planet's orbital motion. We analyze here this effect for a space probe passing near Jupiter.

The planet Jupiter orbits around the Sun along an elliptical trajectory, that can be approximated by a circumference of average radius R; in order to proceed with the analysis of the physical situation we must first:

1. Find the speed V of the planet along its orbit around the Sun. *[1.5 points]*
2. When the probe is between the Sun and Jupiter (on the segment Sun-Jupiter), find the distance from Jupiter where the Sun's gravitational attraction balances that by Jupiter. *[1 point]*

A space probe of mass m = 825 kg flies by Jupiter. For simplicity assume that the trajectory of the space probe is entirely in the plane of Jupiter's orbit; in this way we neglect the important case in which the space probe is expelled from Jupiter's orbital plane.

We only consider what happens in the region where Jupiter's attraction overwhelms all other gravitational forces.

In the reference frame of the Sun's center of mass the initial speed of the space probe is $v_0 = 1.00 \cdot 10^4$ m/s (along the positive y direction) while Jupiter's speed is along the negative x direction (see figure 1); by "initial speed" we mean the space probe speed when it's in the interplanetary space, still far from Jupiter but already in the region where the Sun's attraction is negligible with respect to Jupiter's. We assume that the encounter occurs in a sufficiently short time to allow neglecting the change of direction of Jupiter along its orbit around the Sun. We also assume that the probe passes behind Jupiter, i.e. the x coordinate is greater for the probe than for Jupiter when the y coordinate is the same.

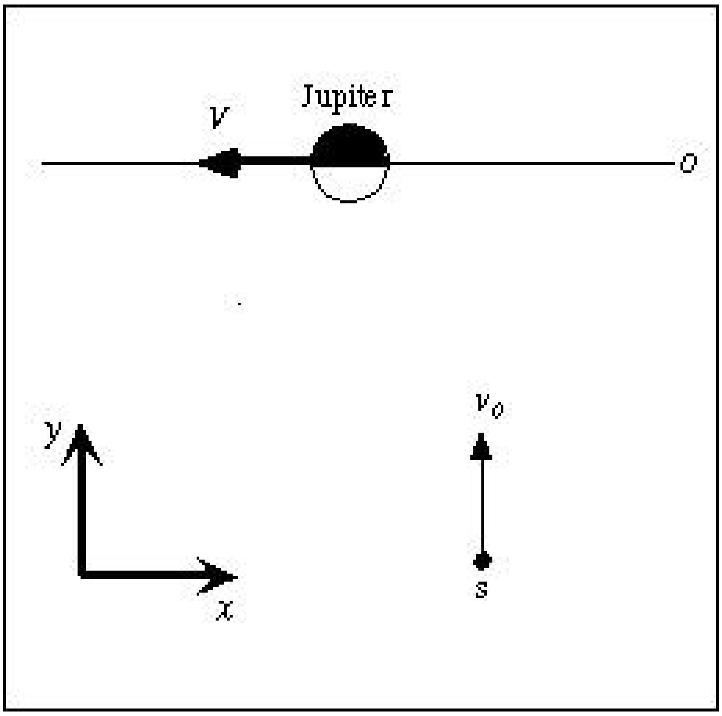


Figure 1: View in the Sun center of mass system. O denotes Jupiter's orbit, s is the space probe.

3. Find the space probe's direction of motion (as the angle φ between its direction and the x axis) and its speed v' in Jupiter's reference frame, when it's still far away from Jupiter. *[2 points]*
4. Find the value of the space probe's total mechanical energy E in Jupiter's reference frame, putting – as usual – equal to zero the value of its potential energy at a very large distance, in this case when it is far enough to move with almost constant velocity owing to the smallness of all gravitational interactions. *[1 point]*

The space probe's trajectory in the reference frame of Jupiter is a hyperbola and its equation in polar coordinates in this reference frame is

$$\frac{1}{r} = \frac{GM}{v'^2 b^2}\left(1 + \sqrt{1 + \frac{2Ev'^2 b^2}{G^2 M^2 m}}\cos\theta\right) \qquad (1)$$

where b is the distance between one of the asymptotes and Jupiter (the so called *impact parameter*), E is the probe's total mechanical energy in Jupiter's reference frame, G is the gravitational constant, M is the mass of Jupiter, r and θ are the polar coordinates (the radial distance and the polar angle).

Figure 2 shows the two branches of a hyperbola as described by equation (1); the asymptotes and the polar co-ordinates are also shown. Note that equation (1) has its origin in the "attractive focus" of the hyperbola. The space probe's trajectory is the attractive trajectory (the emphasized branch).

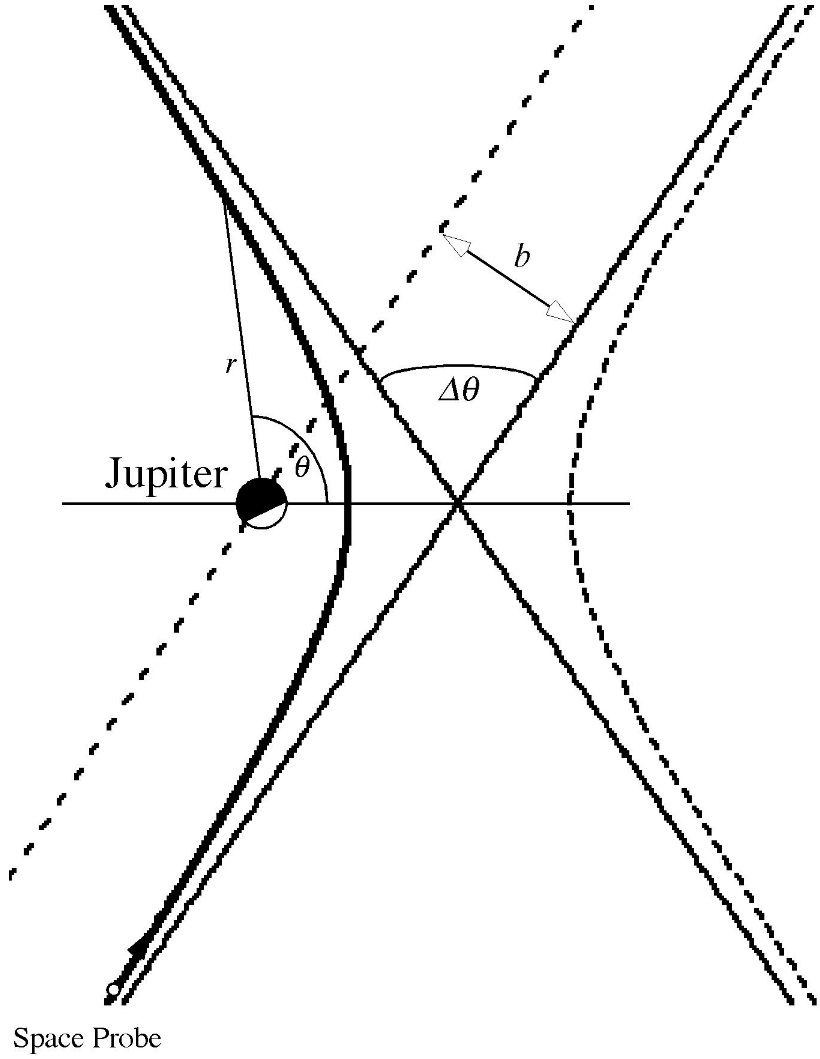


Figure 2

5. Using equation (1) describing the space probe's trajectory, find the total angular deviation $\Delta\theta$ in Jupiter's reference frame (as shown in figure 2) and express it as a function of initial speed v' and impact parameter b. *[2 points]*
6. Assume that the probe cannot pass Jupiter at a distance less than three Jupiter radii from the center of the planet; find the minimum possible impact parameter and the maximum possible angular deviation. *[1 point]*
7. Find an equation for the final speed v'' of the probe in the Sun's reference frame as a function only of Jupiter's speed V, the probe's initial speed v_0 and the deviation angle $\Delta\theta$. *[1 point]*
8. Use the previous result to find the numerical value of the final speed v'' in the Sun's reference frame when the angular deviation has its maximum possible value. *[0.5 points]*

Hint

Depending on which path you follow in your solution, the following trigonometric formulas might be useful:

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$

$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$

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Problem	3
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Answer sheet

Unless explixitly requested to do otherwise, in this problem you are supposed to write your results both as analytic equations (first) and then as numerical results and units (e.g. A=bc=1.23 m^2).

1.Speed V of Jupiter along its orbit ……………………………

2.Distance from Jupiter where the two gravitational attractions balance each other …………………………………….

3.Initial speed v' of the space probe in Jupiter's reference frame ………………………………... and the angle φ its direction forms with the x axis, as defined in figure 1, …………………..………

4.Total energy E of the space probe in Jupiter's reference frame …………………………………...

5.Write a formula linking the probe's deviation $\Delta\theta$ in Jupiter's reference frame to the impact parameter b, the initial speed v' and other known or computed quantities ………………………………………………………………………………… …………..

6.If the distance from Jupiter's center can't be less than three Jovian radii, write the minimum impact parameter and the maximum angular deviation: b = …………………………………;

$\Delta\theta$ = ……………………………………………………..

7.Equation for the final probe speed v'' in the Sun's reference frame as a function of V, v_0 and $\Delta\theta$ …...…………………………..……………………………………...………

8.Numerical value of the final speed in the Sun's reference frame when the angular deviation has its maximum value as computed in step 6 ………….……………………………………………

Final

Solution

1. At equilibrium the pressure p inside the vessel must be equal to the room pressure p_0 plus the pressure induced by the weight of the movable base: $p = p_0 + \dfrac{mg}{\pi r^2}$. This is true before and after irradiation. Initially the gas temperature is room temperature. Owing to the state equation of perfect gases, the initial gas volume V_1 is $V_1 = \dfrac{nRT_0}{p}$ (where R is the gas constant) and therefore the height h_1 of the cylinder which is occupied by the gas is $h_1 = \dfrac{V_1}{\pi r^2} = \dfrac{nRT_0}{p_0 \pi r^2 + mg}$. After irradiation, this height becomes $h_2 = h_1 + \Delta s$, and therefore the new temperature is
$$T_2 = T_0\left(1 + \frac{\Delta s}{h_1}\right) = T_0 + \frac{\Delta s(p_0 \pi r^2 + mg)}{nR}.$$
Numerical values: $p = 102.32$ kPa; $T_2 = 322$ K $= 49$°C

2. The mechanical work made by the gas against the plate weight is $mg\Delta s$ and against the room pressure is $p_0 \pi r^2 \Delta s$, therefore the total work is $L = (mg + p_0 \pi r^2)\Delta s = 24.1$ J

3. The internal energy, owing to the temperature variation, varies by an amount $\Delta U = nc_V(T_2 - T_0)$. The heat introduced into the system during the irradiation time Δt is $Q = \Delta U + L = nc_V \dfrac{T_0 \Delta s}{h_1} + (mg + p_0 \pi r^2)\Delta s = \Delta s\left(p_0 \pi r^2 + mg\right)\left(\dfrac{c_V}{R} + 1\right)$. This heat comes exclusively from the absorption of optical radiation and coincides therefore with the absorbed optical energy, $Q = 84$ J.

The same result can also be obtained by considering an isobaric transformation and remembering the relationship between molecular heats:
$$Q = nc_p(T_2 - T_0) = n(c_V + R)\left[\frac{\Delta s(p_0 \pi r^2 + mg)}{nR}\right] = \Delta s(p_0 \pi r^2 + mg)\left(\frac{c_V}{R} + 1\right)$$

4. Since the laser emits a constant power, the absorbed optical power is $W = \dfrac{Q}{\Delta t} = \left(\dfrac{c_V}{R} + 1\right)\dfrac{\Delta s}{\Delta t}(p_0 \pi r^2 + mg) = 8.4$ W. The energy of each photon is hc/λ, and thus the number of photons absorbed per unit time is $\dfrac{W\lambda}{hc} = 2.2 \cdot 10^{19}\ \mathrm{s}^{-1}$

5. The potential energy change is equal to the mechanical work made against the plate weight, therefore the efficiency η of the energy transformation is

$$\frac{mg\Delta s}{Q} = \frac{1}{\left(1+\frac{p_0\pi r^2}{mg}\right)\left(1+\frac{c_V}{R}\right)} = 2.8\cdot 10^{-3} \approx 0.3\%$$

6. When the cylinder is rotated and its axis becomes horizontal, we have an adiabatic transformation where the pressure changes from p to p_0, and the temperature changes therefore to a new value T_3. The equation of the adiabatic transformation $pV^{\gamma} = \text{constant}$ may now be written in the form $T_3 = T_2\left(\frac{p_0}{p}\right)^{\frac{\gamma-1}{\gamma}}$, where $\gamma = \frac{c_p}{c_V} = \frac{c_V + R}{c_V} = 1 + \frac{R}{c_V} = 1.399$. Finally $T_3 = 321\text{ K} = 48°\text{C}$

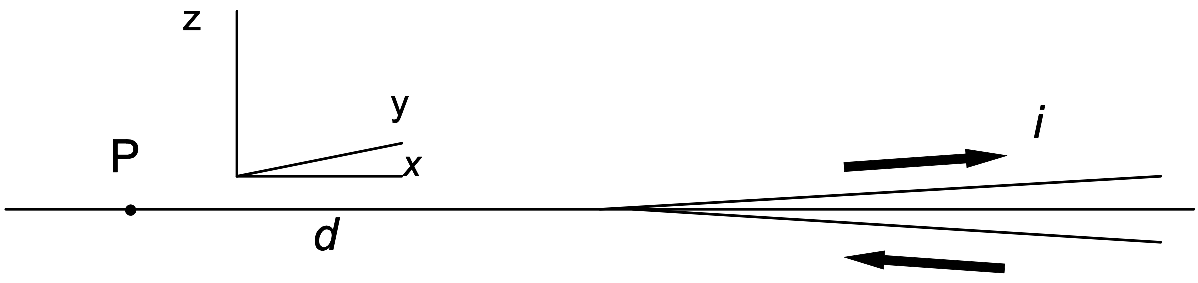
Grading guidelines

1.	0.5	Understanding the relationship between inner and outer pressure
	0.7	Proper use of the plate displacement
	0.2+0.2	Correct results for final pressure
	0.2+0.2	Correct results for final temperature
2.	0.6	Understanding that the work is made both against plate weight and against atmospheric pressure
	0.2+0.2	Correct results for work
3.	1	Correct approach
	0.5	Correct equation for heat
	0.3	Understanding that the absorbed optical energy equals heat
	0.2	Correct numerical result for optical energy
4.	0.2+0.2	Correct results for optical power
	0.5	Einstein's equation
	0.3+0.3	Correct results for number of photons
5.	0.6	Computation of the change in potential energy
	0.2+0.2	Correct results for efficiency
6.	0.8	Understanding that the pressure returns to room value
	0.4	Understanding that there is an adiabatic transformation
	0.4	Equation of adiabatic transformation
	0.5	Derivation of γ from the relationship between specific heats
	0.2+0.2	Correct results for temperature

For "correct results" two possible marks are given: the first one is for the analytical equation and the second one for the numerical value.
For the numerical values a full score cannot be given if the number of digits is incorrect (more than one digit more or less than those given in the solution) or if the units are incorrect or missing.
No bonus can be given for taking into account the gas weight

Solution

1. The contribution to **B** given by each leg of the "V" has the same direction as that of a corresponding infinite wire and therefore - if the current proceeds as indicated by the arrow - the magnetic field is orthogonal to the wire plane taken as the *x-y* plane. If we use a right-handed reference frame as indicated in the figure, **B**(P) is along the positive *z* axis.



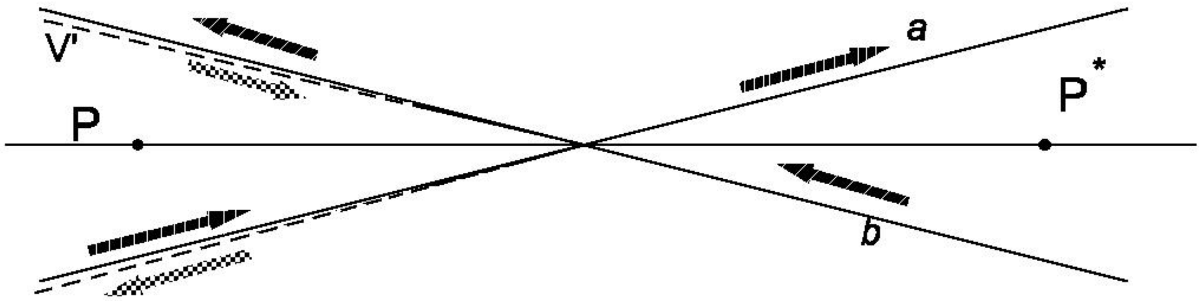
For symmetry reasons, the total field is twice that generated by each leg and has still the same direction.

2A. When $\alpha = \pi/2$ the "V" becomes a straight infinite wire. In this case the magnitude of the field B(P) is known to be $B = \dfrac{i}{2\pi\varepsilon_0 c^2 d} = \dfrac{i\mu_0}{2\pi d}$, and since $\tan(\pi/4) = 1$, the factor k is $\dfrac{i\mu_0}{2\pi d}$.

The following solution is equally acceptable:

2B. If the student is aware of the equation $B = \dfrac{\mu_0 i}{4\pi}\dfrac{\cos\theta_1 - \cos\theta_2}{h}$ for a finite stretch of wire lying on a straight line at a distance h from point P and whose ends are seen from P under the angles θ_1 and θ_2, he can find that the two legs of the "V" both produce fields $\dfrac{\mu_0 i}{4\pi}\dfrac{1-\cos\alpha}{d\,\sin\alpha}$ and therefore the total field is $B = \dfrac{i\mu_0}{2\pi\,d}\dfrac{1-\cos\alpha}{\sin\alpha} = \dfrac{i\mu_0}{2\pi\,d}\tan\left(\dfrac{\alpha}{2}\right)$. This is a more complete solution since it also proves the angular dependence, but it is not required.

3A. In order to compute $\mathbf{B}(P^*)$ we may consider the "V" as equivalent to two crossed infinite wires (*a* and *b* in the following figure) plus another "V", symmetrical to the first one, shown in the figure as V', carrying the same current *i*, in opposite direction.



Then $B(\mathrm{P}^*) = B_a(\mathrm{P}^*) + B_b(\mathrm{P}^*) + B_{\mathrm{V'}}(\mathrm{P}^*)$. The individual contributions are:

$B_a(\mathrm{P}^*) = B_b(\mathrm{P}^*) = \dfrac{i\mu_0}{2\pi d \sin\alpha}$, along the negative z axis;

$B_{\mathrm{V'}}(\mathrm{P}^*) = \dfrac{i\mu_0}{2\pi d}\tan\left(\dfrac{\alpha}{2}\right)$, along the positive z axis.

Therefore we have $B(\mathrm{P}^*) = \dfrac{i\mu_0}{2\pi d}\left[\dfrac{2}{\sin\alpha} - \tan\left(\dfrac{\alpha}{2}\right)\right] = k\left(\dfrac{1+\cos\alpha}{\sin\alpha}\right) = k\cot\left(\dfrac{\alpha}{2}\right)$, and the field is along the negative z axis.

The following solutions are equally acceptable:

3B. The point P^* inside a "V" with half-span α can be treated as if it would be on the outside of a "V" with half-span $\pi - \alpha$ carrying the same current but in an opposite way, therefore the field is $B(\mathrm{P}^*) = k\tan\left(\dfrac{\pi-\alpha}{2}\right) = k\tan\left(\dfrac{\pi}{2} - \dfrac{\alpha}{2}\right) = k\cot\left(\dfrac{\alpha}{2}\right)$; the direction is still that of the z axis but it is along the negative axis because the current flows in the opposite way as previously discussed.

3C. If the student follows the procedure outlined under 2B., he/she may also find the field value in P* by the same method.

4.The mechanical moment **M** acting on the magnetic needle placed in point P is given by $\mathbf{M} = \boldsymbol{\mu} \wedge \mathbf{B}$ (where the symbol $\wedge$ is used for vector product). If the needle is displaced from its equilibrium position by an angle β small enough to approximate $\sin\beta$ with β, the angular momentum theorem gives $M = -\mu B\beta = \dfrac{dL}{dt} = I\dfrac{d^2\beta}{dt^2}$, where there is a minus sign because the mechanical momentum is always opposite to the displacement from equilibrium. The period T of the small oscillations is therefore given by $T = \dfrac{2\pi}{\omega} = 2\pi\sqrt{\dfrac{I}{\mu B}}$.

Writing the differential equation, however, is not required: the student should recognise the same situation as with a harmonic oscillator.

5.If we label with subscript A the computations based on Ampère's interpretation, and with subscript BS those based on the other hypothesis by Biot and Savart, we have

$$B_A = \frac{i\mu_0}{2\pi d}\tan(\frac{\alpha}{2}) \qquad B_{BS} = \frac{i\mu_0}{\pi^2 d}\alpha$$

$$T_A = 2\pi\sqrt{\frac{2\pi Id}{\mu_0\mu i\tan(\frac{\alpha}{2})}} \qquad T_{BS} = 2\pi\sqrt{\frac{\pi^2 Id}{\mu_0\mu i\alpha}}$$

$$\frac{T_A}{T_{BS}} = \sqrt{\frac{2\alpha}{\pi\tan(\frac{\alpha}{2})}}$$

For $\alpha = \pi/2$ (maximum possible value) $T_A = T_{BS}$; and for $\alpha \to 0$ $T_A \to \frac{2}{\sqrt{\pi}}T_{BS} \approx 1.128\ T_{BS}$. Since within this range $\frac{\tan(\alpha/2)}{\alpha/2}$ is a monotonically growing function of α, $\frac{T_A}{T_{BS}}$ is a monotonically decreasing function of α; in an experiment it is therefore not possible to distinguish between the two interpretations if the value of α is larger than the value for which $T_A = 1.10\ T_{BS}$ (10% difference), namely when $\tan\left(\frac{\alpha}{2}\right) = \frac{4}{1.21\pi}\frac{\alpha}{2} = 1.05\frac{\alpha}{2}$. By looking into the trigonometry tables or using a calculator we see that this condition is well approximated when $\alpha/2 = 0.38$ rad; α must therefore be smaller than 0.77 rad $\approx 44°$.

A graphical solution of the equation for α is acceptable but somewhat lengthy. A series development, on the contrary, is not acceptable.

Grading guidelines

1. 1 for recognising that each leg gives the same contribution
0.5 for a correct sketch

2. 0.5 for recognising that $\alpha=\pi/2$ for a straight wire, or for knowledge of the equation given in 2B.
0.25 for correct field equation (infinite or finite)
0.25 for value of k

3. 0.7 for recognising that the V is equivalent to two infinite wires plus another V
0.3 for correct field equation for an infinite wire
0.5 for correct result for the intensity of the required field
0.5 for correct field direction

alternatively
0.8 for describing the point as outside a V with π-α half-amplitude and opposite current
0.7 for correct analytic result
0.5 for correct field direction

alternatively
0.5 for correctly using equation under 2B
1 for correct analytic result
0.5 for correct field direction

4. 0.5 for correct equation for mechanical moment **M**
0.5 for doing small angle approximation $\sin\beta \approx \beta$
1 for correct equation of motion, including sign, or for recognizing analogy with harmonic oscillator
0.5 for correct analytic result for T

5. 0.3 for correct formulas of the two periods
0.3 for recognising the limiting values for α
0.4 for correct ratio between the periods
1 for finding the relationship between α and tangent
0.5 for suitable approximate value of α
0.5 for final explicit limiting value of α

For the numerical values a full score cannot be given if the number of digits is incorrect (more than one digit more or less than those given in the solution) or if the units are incorrect or missing

Solution

1A. Assuming – as outlined in the text – that the orbit is circular, and relating the radial acceleration $\frac{V^2}{R}$ to the gravitational field $\frac{GM_S}{R^2}$ (where M_S is the solar mass) we obtain Jupiter's orbital speed $V = \sqrt{\frac{GM_S}{R}} \approx 1.306 \cdot 10^4$ m/s.

The following alternative solution is also acceptable:

1B. Since we treat Jupiter's motion as circular and uniform, $V = \omega R = \frac{2\pi R}{y_J}$, where y_J is the revolution period of Jupiter, which is given in the list of the general physical constants.

2. The two gravitational forces on the space probe are equal when

$$\frac{GMm}{\rho^2} = \frac{GM_S m}{(R-\rho)^2} \qquad (2)$$

(where ρ is the distance from Jupiter and M is Jupiter's mass), whence

$$\sqrt{M}\,(R-\rho) = \rho\sqrt{M_S} \qquad (3)$$

and

$$\rho = \frac{\sqrt{M}}{\sqrt{M_S}+\sqrt{M}} R = 0.02997R = 2.333 \cdot 10^{10}\,\text{m} \qquad (4)$$

and therefore the two gravitational attractions are equal at a distance of about 23.3 million kilometers from Jupiter (about 334 Jupiter radii).

3. With a simple Galilean transformation we find that the velocity components of the probe in Jupiter's reference frame are

$$\begin{cases} v'_x = V \\ v'_y = v_0 \end{cases}$$

and therefore - in Jupiter's reference frame – the probe travels with an angle $\theta_0 = \arctan\frac{v_0}{V}$ with respect to the x axis and its speed is $v' = \sqrt{v_0^2 + V^2}$ (we also note that

$\cos\theta_0 = \dfrac{V}{\sqrt{v_0^2 + V^2}} = \dfrac{V}{v'}$ and $\sin\theta_0 = \dfrac{v_0}{\sqrt{v_0^2 + V^2}} = \dfrac{v_0}{v'}$).

Using the given values we obtain ${}_0 = 0.653$ rad 37.4° and $v' = 1.65 \cdot 10^4$ m/s .

4. Since the probe trajectory can be described only approximately as the result of a two-body gravitational interaction (we should also take into account the interaction with the Sun and other planets) we assume a large but not infinite distance from Jupiter and we approximate the total energy in Jupiter's reference frame as the probe's kinetic energy at that distance:

$$E \approx \frac{1}{2} m v'^2 \tag{5}$$

The corresponding numerical value is $E = 112$ GJ.

5. Equation (1) shows that the radial distance becomes infinite, and its reciprocal equals zero, when

$$1 + \sqrt{1 + \frac{2Ev'^2 b^2}{G^2 M^2 m}} \cos\theta = 0 \tag{7}$$

namely when

$$\cos\theta = -\frac{1}{\sqrt{1 + \frac{2Ev'^2 b^2}{G^2 M^2 m}}} \tag{8}$$

We should also note that the radial distance can't be negative, and therefore its acceptable values are those satisfying the equation

$$1 + \sqrt{1 + \frac{2Ev'^2 b^2}{G^2 M^2 m}} \cos\theta \ge 0 \tag{9}$$

or

$$\cos\theta \ge -\frac{1}{\sqrt{1 + \frac{2Ev'^2 b^2}{G^2 M^2 m}}} \tag{10}$$

The solutions for the limiting case of eq. (10) (i.e. when the equal sign applies) are:

$$\theta_\pm = \pm \arccos\left[-\left(1 + \frac{2Ev'^2 b^2}{G^2 M^2 m} \right)^{-1/2} \right] = \pm \left(\pi - \arccos \frac{1}{\sqrt{1 + \frac{2Ev'^2 b^2}{G^2 M^2 m}}} \right) \tag{11}$$

and therefore the angle $\Delta\theta$ (shown in figure 2) between the two hyperbola asymptotes is given by:

$$\begin{aligned}\Delta\theta &= (\theta_+ - \theta_-) - \pi \\ &= \pi - 2\arccos\frac{1}{\sqrt{1+\dfrac{2Ev'^2 b^2}{G^2M^2m}}} \qquad (12) \\ &= \pi - 2\arccos\frac{1}{\sqrt{1+\dfrac{v'^4 b^2}{G^2M^2}}}\end{aligned}$$

In the last line, we used the value of the total energy as computed in the previous section.

6. The angular deviation is a monotonically decreasing function of the impact parameter, whence the deviation has a maximum when the impact parameter has a minimum. From the discussion in the previous section we easily see that the point of nearest approach is when $\theta = 0$, and in this case the minimum distance between probe and planet center is easily obtained from eq. (1):

$$r_{\min} = \frac{v'^2 b^2}{GM}\left(1+\sqrt{1+\frac{v'^4 b^2}{G^2M^2}}\right)^{-1} \qquad (13)$$

By inverting equation (13) we obtain the impact parameter

$$b = \sqrt{r_{\min}^2 + \frac{2GM}{v'^2} r_{\min}} \qquad (14)$$

We may note that this result can alternatively be obtained by considering that, due to the conservation of angular momentum, we have

$$L = mv'b = mv'_{\min}\, r_{\min}$$

where we introduced the speed corresponding to the nearest approach. In addition, the conservation of energy gives

$$E = \frac{1}{2}mv'^2 = \frac{1}{2}mv'^2_{\min} - \frac{GMm}{r_{\min}}$$

and by combining these two equations we obtain equation (14) again.

The impact parameter is an increasing function of the distance of nearest approach; therefore, if the probe cannot approach Jupiter's surface by less than two radii (and thus $r_{\min} = 3R_B$, where R_B is Jupiter's body radius), the minimum acceptable value of the impact parameter is

$$b_{\min} = \sqrt{9R_B^2 + \frac{6GM}{v'^2}R_B} \qquad (15)$$

From this equation we finally obtain the maximum possible deviation:

$$\Delta\theta_{\max} = \pi - 2\arccos\frac{1}{\sqrt{1+\frac{v'^4 b_{\min}^2}{G^2M^2}}} = \pi - 2\arccos\frac{1}{\sqrt{1+\frac{v'^4}{G^2M^2}\left(9R_B^2 + \frac{6GM}{v'^2}R_B\right)}} \qquad (16)$$

and by using the numerical values we computed before we obtain:

$b_{\min} = 4.90\cdot10^8$ m $7.0\ R_B$ and $\Delta\theta_{\max} = 1.526$ rad $87.4°$

7. The final direction of motion with respect to the x axis in Jupiter's reference frame is given by the initial angle plus the deviation angle, thus $\theta + \Delta\theta$ if the probe passes behind the planet. The final velocity components in Jupiter's reference frame are therefore:

$$\begin{cases} v'_x = v'\cos(\theta_0 + \Delta\theta) \\ v'_y = v'\sin(\theta_0 + \Delta\theta) \end{cases}$$

whereas in the Sun reference frame they are

$$\begin{cases} v''_x = v'\cos(\theta_0 + \Delta\theta) - V \\ v''_y = v'\sin(\theta_0 + \Delta\theta) \end{cases}$$

Therefore the final probe speed in the Sun reference frame is

$$\begin{aligned} v'' &= \sqrt{(v'\cos(\theta_0+\Delta\theta)-V)^2+(v'\sin(\theta_0+\Delta\theta))^2} \\ &= \sqrt{v_0^2+2V^2-2v'V\cos(\theta_0+\Delta\theta)} \\ &= \sqrt{v_0^2+2V^2-2v'V\left(\cos\theta_0\cos\Delta\theta-\sin\theta_0\sin\Delta\theta\right)} \qquad (17) \\ &= \sqrt{v_0^2+2V^2-2V\left(V\cos\Delta\theta-v_0\sin\Delta\theta\right)} \\ &= \sqrt{v_0\left(v_0+2V\sin\Delta\theta\right)+2V^2\left(1-\cos\Delta\theta\right)} \end{aligned}$$

8. Using the value of the maximum possible angular deviation, the numerical result is $v'' = 2.62\cdot 10^4$ m/s.

Grading guidelines

1.	0.4	Law of gravitation, or law of circular uniform motion
	0.4	Correct approach
	0.4+0.3	Correct results for velocity of Jupiter
2.	0.3	Correct approach
	0.4+0.3	Correct results for distance from Jupiter
3.	1	Correct transformation between reference frames
	0.3+0.2	Correct results for probe speed in Jupiter reference frame
	0.3+0.2	Correct results for probe angle
4.	0.8	Understanding how to handle the potential energy at infinity
	0.2	Numerical result for kinetic energy
5.	0.6	Correct approach
	0.6	Equation for the orientation of the asymptotes
	0.8	Equation for the probe deflection angle
6.	0.3+0.2	Correct results for minimum impact parameter
	0.3+0.2	Correct results for maximum deflection angle
7.	0.5	Equation for velocity components in the Sun reference frame
	0.5	Equation for speed as a function of angular deflection
8.	0.5	Numerical result for final speed

For "correct results" two possible marks are given: the first one is for the analytical equation and the second one for the numerical value.
For the numerical values a full score cannot be given if the number of digits is incorrect (more than one digit more or less than those given in the solution) or if the units are incorrect or missing.

Question 1

A *Bungee Jumper*

(a) The jumper comes to rest when

lost gravitational potential energy = stored strain energy

$$mgy = \tfrac{1}{2} k (y-L)^2$$

$$ky^2 - 2y(kL + mg) + kL^2 = 0$$

This is solved as a quadratic.

$$y = \frac{2(kL+mg) \pm \sqrt{4(kL+mg)^2 - 4k^2L^2}}{2k}$$

$$= \frac{kL + mg \pm \sqrt{2mgkL + m^2g^2}}{k}$$

Need positive root; lower position of rest (other root after initial rise).

(b) The maximum speed is attained when the acceleration is zero and forces balance; i.e. when $mg = kx$

Also kinetic energy = lost potential energy – strain energy within elastic rope

$$\tfrac{1}{2} m\upsilon^2 = mg(L + x) - \tfrac{1}{2} kx^2$$

$$x = \frac{mg}{k}$$

$$\upsilon^2 = 2g(L + \frac{mg}{k}) - \frac{mg^2}{k}$$

$$\upsilon = \sqrt{2gL + \frac{mg^2}{k}}$$

(c) Time to come to rest = time in free fall + time in SHM of rope to stop stretching

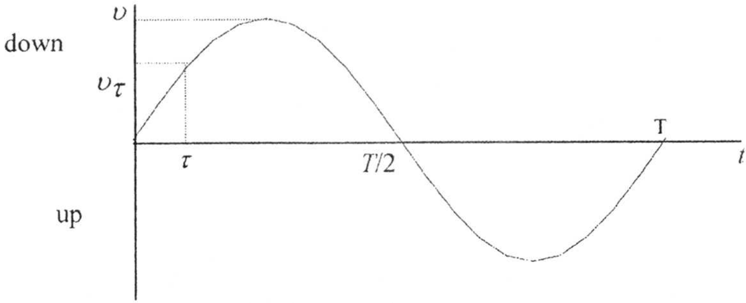
Length of free fall = $L = \tfrac{1}{2} g t_f^2$

Therefore $t_f = \sqrt{\dfrac{2L}{g}}$

The jumper enters the SHM with free fall velocity $= gt_f = \sqrt{2gL} = \upsilon_\tau$

$$\text{Period of SHM} = 2\pi\sqrt{\frac{m}{k}} = T$$

We represent a full SHM cycle by

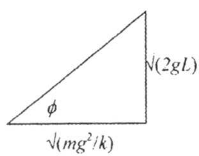


The jumper enters the SHM at time τ given by

$$\tau = \frac{1}{\omega}\sin^{-1}\frac{\upsilon_\tau}{\upsilon} = \frac{1}{\omega}\sin^{-1}\frac{\sqrt{2gL}}{\upsilon}$$

Jumper comes to rest at one half cycle of the SHM at total time given by

$$= t_f + (T/2 - \tau)$$



$$= \sqrt{\frac{2L}{g}} + \pi\sqrt{\frac{m}{k}} - \frac{1}{\omega}\sin^{-1}\frac{\sqrt{2gL}}{\upsilon}$$

$$= \sqrt{\frac{2L}{g}} + \pi\sqrt{\frac{m}{k}} - \frac{1}{\omega}\sin^{-1}\frac{\sqrt{2gL}}{\sqrt{2gL + mg^2/k}}$$

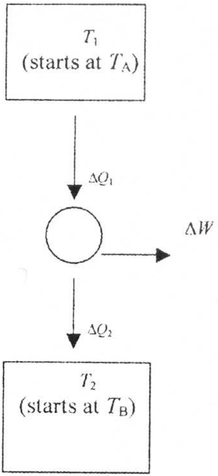
$$= \sqrt{\frac{2L}{g}} + \sqrt{\frac{m}{k}}\left\{\pi - \sin^{-1}\frac{\sqrt{2gL}}{\sqrt{2gL + mg^2/k}}\right\}$$

This is the same as

$$= \sqrt{\frac{2L}{g}} + \sqrt{\frac{m}{k}}\left\{\frac{\pi}{2} + \cos^{-1}\frac{\sqrt{2gL}}{\sqrt{2gL + mg^2/k}}\right\}$$

$$= \sqrt{\frac{2L}{g}} + \sqrt{\frac{m}{k}}\tan^{-1}\left\{-\sqrt{\frac{2kL}{mg}}\right\}$$

B Heat Engine Question



In calculating work obtainable,
we assume no loss (friction etc.) in engine working.

ΔQ_1 = energy from body A
$= -ms\Delta T_1$ (ΔT_1 - ve)

$\Delta Q_2 = ms\Delta T_2$ (ΔT_2 +ve)

(a) *For maximum amount of mechanical energy assume Carnot engine*

$$\frac{\Delta Q_1}{T_1} = \frac{\Delta Q_2}{T_2}$$ throughout operation (second law)

But $\Delta Q_1 = -ms\Delta T_1$ and $\Delta Q_2 = ms\Delta T_2$

$$-ms\int_{T_A}^{T_0}\frac{dT_1}{T_1} = ms\int_{T_B}^{T_0}\frac{dT_2}{T_2}$$

$$\ln\frac{T_A}{T_0} = \ln\frac{T_0}{T_B}$$

$$T_0^2 = T_A T_B$$
$$T_0 = \sqrt{T_A T_B}$$

$$Q_1 = -ms \int_{T_A}^{T_0} dT_1 = ms(T_A - T_0)$$

$$Q_2 = ms \int_{T_B}^{T_0} dT_2 = ms(T_0 - T_B)$$

$$W = Q_1 - Q_2$$

$$W = ms(T_A - T_0 - T_0 + T_B) = ms(T_A + T_B - 2T_0) = ms\,(T_A + T_B - 2\sqrt{T_A T_B}\,)$$

or $\quad ms\,(\sqrt{T_A} - \sqrt{T_B}\,)^2$

(d) Numerical example:

Mass = volume × density

$$W = 2.50 \times 1.00 \times 10^3 \times 4.19 \times 10^3 \times (350 + 300 - 2\sqrt{350 \times 300}\,)\ \text{J}$$
$$= 20 \times 10^6\ \text{J}$$
$$= 20\ \text{MJ}$$

C Radioactivity and age of the Earth

(a) $N = N_0 e^{-\lambda t}$ $\quad\quad$ N_0 = original number

$n = N_0(1 - e^{-\lambda t})$

Therefore $n = N e^{\lambda t}(1 - e^{-\lambda t}) = N(e^{\lambda t} - 1)$

So $n = N(2^{t/\tau} - 1)$ where τ is half-life

or as $\lambda = \dfrac{\ln 2}{T} = \dfrac{0.6931}{T}$, $n = N(e^{\frac{0.6931t}{T}} - 1)$

$^{206}n = {}^{238}N(2^{t/4.50} - 1)$ or $^{206}n = {}^{238}N(e^{0.1540t} - 1)$ where time t is in 10^9 years

(b) $^{207}n = {}^{235}N(2^{t/0.710} - 1)$ or $^{207}n = {}^{235}N(e^{0.9762t} - 1)$

(c) In mixed uranium (i.e. containing Pb of both natural and radioactive origin)

204 : 206 : 207 have proportions	1.00 : 29.6 : 22.6
In pure lead (no radioactivity)	1.00 : 17.9 : 15.5

Therefore for radioactively produced lead by subtraction

204 : 206 : 207 have proportions	1.00 : 29.6 : 22.6
In pure lead (no radioactivity)	1.00 : 17.9 : 15.5

Therefore for radioactivity produced lead by subtraction

206 : 207	11.7 : 7.1

Dividing equations from (a) and (b) gives

$$\frac{^{206}n}{^{207}n} = \frac{^{238}N}{^{235}N}\left\{\frac{2^{t/4.50} - 1}{2^{t/0.710} - 1}\right\} \text{ or } \frac{^{206}n}{^{207}n} = \frac{^{238}N}{^{235}N}\left\{\frac{e^{0.1540t} - 1}{e^{0.9762t} - 1}\right\}$$

$$\frac{11.7}{7.1} = 137\left\{\frac{2^{T/4.50} - 1}{2^{T/0.710} - 1}\right\} \text{ or } \frac{11.7}{7.1} = 137\left\{\frac{e^{0.1540T} - 1}{e^{0.9762T} - 1}\right\}$$

$$0.0120\,\{2^{T/0.710} - 1\} = \{2^{T/4.50} - 1\}$$

$$\text{or } 0.0120\,\{e^{0.9762T} - 1\} = \{e^{0.1540T} - 1\}$$

(d) Assume $T >> 4.50 \times 10^9$ and ignore 1 in both brackets:

$$0.0120\,\{2^{T/0.710}\} = \{2^{T/4.50}\} \text{ or } 0.0120\,\{e^{0.97627}\} = \{e^{0.15407}\}$$

$$0.0120 = \{2^{T/4.50 - T/0.710}\} = 2^{T(0.222-1.4084)} = 2^{-1.18627}$$

$$T = -\frac{\log 0.0120}{\log 2 \times 1.1862} = 5.38$$

$$T = 5.38 \times 10^9 \text{ years}$$

or $0.0120 = e^{-0.8222T}$ $\quad T = \dfrac{\ln 0.0120}{-0.8222} = \dfrac{-4.4228}{-0.8222} = 5.38$

$$T = 5.38 \times 10^9 \text{ years}$$

(e) T is not $>> 4.50 \times 10^9$ years but is $> 0.71 \times 10^9$ years

We can insert the approximate value for T (call it $T^* = 5.38 \times 10^9$ years) in the $2^{T/4.50}$ term and obtain a better value by iteration in the rapidly changing $2^{T/0.710}$ term). We now leave in the -1's, although the -1 on the right-hand side has little effect and may be omitted).

Either $\quad 0.0120((2^{T/0.710} - 1) = 2^{T^*/4.50} - 1$

$$2^{T/0.710} - 1 = \frac{2^{1.1956} - 1}{0.0120} = \frac{2.2904 - 1}{0.0120} = 107.5$$

$$T = 0.710\frac{\log 108.5}{\log 2} = 4.80(0)$$

Put T* = 4.80(0) × 10⁹ years

$$2^{T/0.710} = \frac{2^{1.0668} - 1}{0.0120} = \frac{2.0948 - 1}{0.0120} = 91.2$$

$$T = 0.710\frac{\log 91.2}{\log 2} = 4.62(3)$$

Further iteration gives 4.52

So more accurate answer for **T** ***to be in range 4.6 × 10⁹ years to 4.5 × 10⁹ years (either acceptable).***

D Spherical charge

(a) Charge density = $\rho = \dfrac{Q}{\frac{4}{3}\pi R^3}$ within sphere

$x \leq R$ Field at distance x:

$$E = \frac{\frac{4}{3}\pi x^3 \rho}{4\pi\varepsilon_0 x^2} = \frac{Qx}{4\pi\varepsilon_0 R^3}$$

$x > R$ Field at distance x from the centre: $E = \dfrac{Q}{4\pi\varepsilon_0 x^2}$

(b) Method 1

Energy density is $\frac{1}{2}\varepsilon_0 E^2$.

$x \leq R$

Energy in a thin shell of thickness δx at radius x is given by

$$= \frac{1}{2}\varepsilon_0 E^2\, 4\pi x^2 \delta x = \frac{1}{2} 4\pi\varepsilon_0 \frac{Q^2 x^2}{(4\pi\varepsilon_0)^2 R^6} x^2\, \delta x$$

Energy within the spherical volume = $\dfrac{1}{2}\dfrac{Q^2}{(4\pi\varepsilon_0)R^6}\displaystyle\int_{x=0}^{x=R} x^4\,dx = \dfrac{1}{40}\dfrac{Q^2}{\pi\varepsilon_0}\dfrac{1}{R}$

$x > R$

Energy within spherical shell = $\dfrac{1}{2}\varepsilon_0 E^2\, 4\pi x^2 \delta x = \dfrac{1}{2} 4\pi\varepsilon_0 \dfrac{Q^2}{(4\pi\varepsilon_0)^2 x^4} x^2\, \delta x$

Energy within the spherical volume for $x > R$

$$= \frac{1}{2}\frac{Q^2}{(4\pi\varepsilon_0)}\int_{x=R}^{x=\infty}\frac{1}{x^2}dx = \frac{1}{8}\frac{Q^2}{\pi\varepsilon_0}\frac{1}{R}$$

Total energy associated with the charge distribution = $\dfrac{1}{40}\dfrac{Q^2}{\pi\varepsilon_0}\dfrac{1}{R}$

$+ \dfrac{1}{8}\dfrac{Q^2}{\pi\varepsilon_0}\dfrac{1}{R}$

$$= \frac{3}{20}\frac{Q^2}{\pi\varepsilon_0}\frac{1}{R}$$

Method 2

A shell with charge $4\pi x^2 \delta x \rho$ moves from ∞ to the surface of a sphere radius x where the electric potential is

$$\frac{\frac{4}{3}\pi x^3 \rho}{4\pi\varepsilon_0 x} = \frac{x^2 \rho}{3\varepsilon_0}$$

and will therefore gain electrical potential energy $\left(\frac{x^2\rho}{3\varepsilon_0}\right)\left(4\pi r^2 \rho\right)\delta x$

Total energy of complete sphere = $\displaystyle\int_{x=0}^{x=R} \frac{4\pi\rho^2 x^4}{3\varepsilon_0} dx = \frac{4}{15}\frac{\pi\rho^2 R^5}{\varepsilon_0}$

Putting Q = charge on sphere = $\frac{4}{3}\pi R^3 \rho$, $\rho = \frac{3Q}{4\pi R^3}$

So that total energy is = $\frac{4}{15}\pi\left(\frac{9Q^2}{16\pi^2 R^6}\right)\frac{R^5}{\varepsilon} = \frac{3}{20}\frac{Q^2}{\pi\varepsilon_0 R}$

(c) Binding energy $E_{\text{binding}} = E_{\text{electric}} - E_{\text{nuclear}}$

Binding energy is a negative energy

Therefore $-8.768 = E_{\text{electric}} - 10.980$ MeV per nucleon

$E_{\text{electric}} = 2.212$ MeV per nucleon

Radius of cobalt nucleus is given by $R = \frac{3}{20}\frac{Q^2}{\pi\varepsilon_0 E_{electric}^{total}}$

$$= \frac{3 \times 27^2 \times (1.60 \times 10^{-19})^2}{20 \times \pi \times 8.85 \times 10^{-12} \times 2.212 \times 10^6 \times 57 \times 1.60 \times 10^{-19}} \text{ m}$$

$= 5.0 \times 10^{-15}$ m

$$T = \frac{4\rho d \ln 2}{B^2} = \frac{4 \times 1.70 \times 10^{-8} \times 8.90 \times 10^3 \times 0.6931}{(44.5 \times 10^{-6} \times 0.4384)^2} \text{ s}$$

$= 1.10(2) \times 10^6$ s (=306 hr = 12 days 18 hr)

Method 1 Equating energy

Horizontal component of magnetic field B inducing emf in ring:
$B = 44.5 \times 10^{-6} \cos 64°$
Magnetic flux through ring at angle $\theta = B\pi a^2 \sin\theta$
where a = radius of ring

Instantaneous emf = $\dfrac{d\phi}{dt} = B\pi a^2 \dfrac{d \sin \omega t}{dt}$ where ω = angular velocity

$$= B\pi a^2 \omega \cos \omega t = B\pi a^2 \omega \cos \theta$$

R.m.s. emf over 1 revolution = $\dfrac{B\pi a^2\omega}{\sqrt{2}}$

Average resistive heating of ring = $\dfrac{B^2\eth^2 a^4\omega^2}{2R}$

Moment of inertia = $\dfrac{1}{2}ma^2$

Rotational energy = $\dfrac{1}{4}ma^2\omega^2$ where m = mass of ring

Power producing change in $\omega = \dfrac{d}{dt}\left\{\dfrac{1}{4}ma^2\omega^2\right\} =$

$\dfrac{1}{4}ma^2 2\omega \;\; \dfrac{d\omega}{dt}$

Equating: $$\frac{1}{2}ma^2\omega \;\; \frac{d\omega}{dt} = -\frac{B^2\eth^2 a^4\omega^2}{2R}$$

$$\frac{d\omega}{\omega} = -\frac{B^2\pi^2 a^2}{mR}dt$$

If T is time for angular velocity to halve,

$$\int_{\omega}^{\omega/2}\frac{d\omega}{\omega} = -\int_0^T \frac{B^2\pi^2 a^2}{mR}dt$$

$$\ln 2 = \frac{B^2\eth^2 a^2}{mR}T$$

But R = $\dfrac{2\eth a\rho}{A}$ where A is cross-sectional area of copper ring

$m = 2\pi ad A$ $\quad$ (d = density)

$$\ln 2 = \frac{B^2\eth^2 a^2 T}{\dfrac{2\eth a\rho}{A}2\eth adA} = \frac{B^2 T}{4\rho d}$$

$$T = \frac{4\rho d \ln 2}{B^2} = \frac{4\times1.70\times10^{-8}\times8.90\times10^3\times0.6931}{(44.5\times10^{-6}\times0.4384)^2}\text{s}$$

$$= 1.10(2)\times10^6 \text{ s} \;\; (=306 \text{ hr} = 12 \text{ days } 18 \text{ hr})$$

Method 2 Back Torque

Horizontal component of magnetic field = B = $44.5 \times 10^{-6} \cos 64^{\circ}$

Cross-section of area of ring is A

Radius of ring = a

Density of ring = d

Resistivity = ρ

ω = angular velocity (ω positive when clockwise)

Resistance $R = \rho \dfrac{2\pi a}{A}$

Mass of ring m = $2\pi a A d$

Moment of inertia = $M = \dfrac{1}{2} m a^2$

Magnetic flux through ring at angle $\theta = B\pi a^2 \sin\theta$

Instantaneous emf = $\dfrac{d\phi}{dt} = B\pi a^2 \dfrac{d \sin \omega t}{dt} = B\pi a^2 \omega \cos \omega t = B\pi a^2 \omega \cos\theta$

Induced current = I = $B\pi a^2 \cos\theta / R$

Torque opposing motion = $(B\pi a^2 \cos\theta)\, I = \dfrac{1}{R}\left(B\pi\, a^2\right)^2 \omega \cos^2\theta$

Work done in small $\delta\theta = \dfrac{1}{R}\left(B\pi\, a^2\right)^2 \omega \dfrac{1}{2}(\cos 2\theta + 1)\delta\theta$

Average torque = (work done in 2π revolution)/2π

$$= \frac{1}{2\pi R}\left(B\pi\pi^{\,2}\right)^2 \omega \frac{1}{2} 2\pi = \frac{1}{2R}\left(B\pi\pi^{\,2}\right)^2 \omega$$

This equals $M\dfrac{d\omega}{dt}$ so that $M\dfrac{d\omega}{dt} = -\dfrac{B(\pi a^2)B(\pi a^2)\dfrac{1}{2}}{(\rho / A)(2\pi a)}\omega$

$$\frac{1}{2}(2\pi a A d)a^2 \frac{d\omega}{dt} = -\frac{B^2(\pi a^2)^2 A}{4\rho\pi a}\omega$$

$$\frac{d\omega}{dt} = -\frac{B^2}{4\rho d}\omega$$

$$\int_{\omega}^{\omega/2} \frac{d\omega}{\omega} = \int_0^T \frac{B^2}{4\rho d} dt$$

$$\ln 2 = \frac{B^2 T}{4\rho d}$$

$$T = \frac{4\rho d \ln 2}{B^2} = \frac{4 \times 1.70 \times 10^{-8} \times 8.90 \times 10^3 \times 0.6931}{(44.5 \times 10^{-6} \times 0.4384)^2}\ \text{s}$$

$= 1.10(2) \times 10^6$ s = 306 hr = 12 days 18 hr

Question Two ~ Solution

(a) Focusing occurs for one "cyclotron" orbit of the electron.

Angular velocity $\omega = \mathrm{e}\, B / \mathrm{m}$; so time for one orbit $T = 2\,\pi\,\mathrm{m} / \mathrm{e}\, B$

Speed of electron $u = (2\,\mathrm{e}\, V / \mathrm{m})^{1/2}$

Distance travelled $D = T\, u \cos\beta \approx T\, u = (2^{3/2}\, \pi / B)\, (V\, \mathrm{m} / \mathrm{e})^{1/2}$

Thus charge to mass ratio $= \mathrm{e} / \mathrm{m} = 8\ \mathrm{V} \times\ (\pi / B\, D)^2$

(b) Consider condition (ii) - Force due to electric field acts upwards

In region A force due magnetic field acts upwards as well, electron hits upper plate and does not reach the film.

In region B, force due magnetic field acts downwards, and *if* force is equal and opposite to the electrostatic force, there will be no unbalanced force, and electron will emerge from plates to expose film.

Piece was taken from region B.

(c) We require forces to balance. Electric force given by $\mathrm{e}V / t$, magnitude of magnetic force given by $\mathrm{e}\, u\, B \sin\phi$, with u the speed of the electron.

For these to balance we require $u = V / t\, B\, |\sin\phi|$

Maximum u corresponds to minimum ϕ - at 23°

Therefore $u = 2.687 \times 10^8$ m/s = 0.896 c.

Relativistic $\gamma = (1 - v^2/c^2)^{-1/2} = 2.255$,
so kinetic energy of electron $= (\gamma\text{-}1)\ \mathrm{m}\, c^2 = 641$ keV.

(d) After emerging from region between plates, electrons experience force due to magnetic field only. We approximate this by a vertical force, because angle of electron to horizontal remains small.

Acceleration caused by this force $a = B\,\mathrm{e}\,u \sin\phi / \gamma \mathrm{m}$

Initial horizontal speed is u, therefore time taken to reach the film after emerging from the region between the plates $t = s / u$.

Change in vertical displacement during this time $= y / 2 = ½\, a\, (s / u)^2$

$y = B\,\mathrm{e}\,s^2 \sin\phi / \gamma \mathrm{m}\, u$

From part (f), for electron to have emerged from plate,
we also know $u = V / t\,B\,|\sin\phi|$.

Therefore we eliminate u to obtain:

$$y^2 = (\mathrm{e}\,B\,s \sin\phi / \mathrm{m})^2 \;\{(B\,s\,t \sin\phi / V)^2 - (s / \mathrm{c})^2\}$$

and we plot VERTICAL $(y / B\,s \sin\phi)^2$

HORIZONTAL $(B\,s\,t \sin\phi / V)^2$

Therefore we have a gradient $(\mathrm{e} / \mathrm{m})^2$

and a vertical-axis intercept $-(\mathrm{e}\,s / \mathrm{m\,c})^2$

The intercept is read as -537.7 $(\mathrm{C\,s / kg})^2$, giving e/m = 1.70×10^{11} C / kg

The gradient is read as 2.826×10^{22} $(\mathrm{C/kg})^2$, giving e/m = 1.68×10^{11} C / kg.

A **a)** $\Delta x_t = ae^{-\mu t}\cos(\omega t + \phi)$, $0.8 = e^{-50\mu} \Rightarrow \mu = 4.5 \times 10^{-3}\ \text{s}^{-1}$.

b) $\upsilon = (E/\rho)^{1/2} = (7.1 \times 10^{10}/2700)^{1/2} = 5100\ \text{m.s}^{-1}$.
At fundamental $\lambda_{rod} = 4l = 4$ m.
$f = 5100 / 4 = 1.3 \times 10^3$ Hz.
$\omega = 2\pi f = 8.1 \times 10^3\ \text{rad.s}^{-1}$.

c) $\upsilon = f\lambda_{rod}$, $\delta\lambda_{rod} / \lambda_{rod} = (-)\delta f / f \Rightarrow \delta l / l$.
$\delta l = l.(\delta f / f)$.
[0.6]
$\delta l = 1 \times (5.0 \times 10^{-3}/1.3 \times 10^3) = 3.8 \times 10^{-6}$ m.

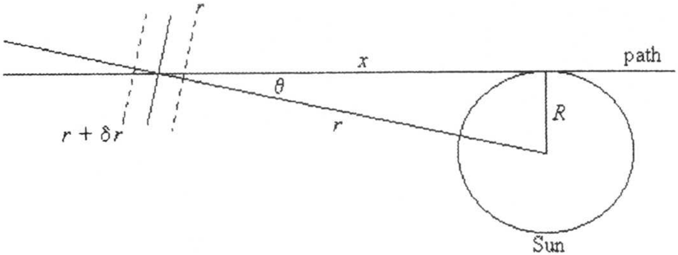
d) Change in gravitational force on rod at a distance x from the free end $= m\Delta g$ and $m = \rho x A$,
where A is the cross-sectional area of the rod.
Change in stress $= m\Delta g/A = \rho x \Delta g$.
Change in strain $= \delta(dx)/dx = \rho x \Delta g/E$;
that is, $dx \rightarrow (1 + \rho x \Delta g/E)dx \Rightarrow \Delta l = (\rho\Delta g/2E)l^2$.

e) At fundamental $\lambda_{rod} = 4l \Rightarrow \Delta l = \Delta\lambda_{rod}/4$,
for $\Delta\lambda_{rod} = 656\ \text{nm}/10^4 \Rightarrow \Delta l = 656\ \text{nm}/(4 \times 10^4)$.
$\Delta l = 656\ \text{nm}/(4 \times 10^4) = (\rho\Delta g/2E)l^2$
[0.1]
$\Delta l = (2700 \times 10^{-19}/ 14 \times 10^{10})\, l^2 \Rightarrow l = 9.2 \times 10^7$ m.

B **a)** $mc^2 = hf \Rightarrow m = hf/c^2$,
[0.3]
$hf' = hf - GMm/R$,
$\Rightarrow hf' = hf(1 - GM/Rc^2)$, $\therefore f' = f(1 - GM/Rc^2)$.

b) $n_r = c / c(1 - GM/rc^2)^2$,
$n_r = 1 + 2GM/rc^2$, for small GM/rc^2; i.e. $\alpha = 2$.

c)



Diagram

By Snell's law: $n(r + \delta r) \sin \theta = n(r) \sin (\theta - \delta\xi)$,

$(n(r) + (dn/dr)\, \delta r) \sin \theta = n(r) \sin \theta - n(r) \cos \theta\, \delta\xi$.

$(dn/dr)\, \delta r \sin \theta = -\, n(r) \cos \theta\, \delta\xi$.

Now $n(r) = 1 + 2GM/rc^2$, so $(dn/dr) = -\, 2GM/c^2r^2$,

and $(2GM/c^2r^2) \sin \theta\, \delta r = n(r) \cos \theta\, \delta\xi$.

Hence $\delta\xi = (2GM/c^2r^2) \tan \theta\, (\delta r/n) \approx (2GM \tan \theta\, /c^2r^2)\delta r$.

Now $r^2 = x^2 + R^2$, so $rdr = xdx$.

$$\int d\xi = \frac{2GM}{c^2}\int \frac{\tan\theta dr}{r^2} = \frac{2GM}{c^2}\int \frac{\tan\theta r dr}{r^3} = \frac{2GMR}{c^2}\int_{-\infty}^{+\infty} \frac{dx}{(x^2 + R^2)^{3/2}}$$

$$\xi = \frac{4GM}{Rc^2}\, radians = 8.4 \times 10^{-6}\, radians.$$

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31st International Physics Olympiad

Leicester, U.K.

Experimental Competition

Wednesday, July 12th, 2000

Please read this first:

1. The time available is 2 ½ hours for each of the 2 experimental questions. Answers for your first question will be collected after 2 ½ hours.
2. Use only the pen issued in your back pack.
3. Use only the front side of the sheets of paper provided. Do not use the side marked with a cross.
4. Each question should be answered on separate sheets of paper.
5. For each question, in addition to the ***blank writing sheets*** where you may write, there is an ***answer sheet*** where you *must* summarise the results you have obtained. Numerical results should be written with as many digits as are appropriate to the given data. Do not forget to state the units
6. Write on the blank sheets of paper the results of all your measurements and whatever else you consider is required for the solution of the question and that you wish to be marked. However you should use mainly equations, numbers, symbols, graphs and diagrams. Please use *as little text as possible*.
7. *It is absolutely essential* that you enter in the boxes at the top of each sheet of paper used your ***Country*** and your student number (***Student No***.). In addition, on the blank sheets of paper used for each question, you should enter the number of the question ***(Question No.)***, the progressive number of each sheet ***(Page No.)*** and the total number of blank sheets that you have used and wish to be marked for each question ***(Total No. of pages)***. It is also helpful to write the question number and the section label of the part you are answering at the beginning of each sheet of writing paper. If you use some blank sheets of paper for notes that you do not wish to be marked, put a large cross through the whole sheet and do not include it in your numbering.
8. When you have finished, arrange all sheets *in proper order* (for *each* question put answer sheets first, then used sheets in order, followed by the sheets you do not wish to be marked. Put unused sheets and the printed question at the bottom). Place the papers for each question inside the envelope labelled with the appropriate question number, and leave everything on your desk. You are not allowed to take *any* sheets of paper out of the room.

CDROM SPECTROMETER

In this experiment, you are NOT expected to indicate uncertainties in your measurements.

The aim is to produce a graph showing how the conductance* of a light-dependent resistor (LDR) varies with wavelength across the visible spectrum.

*conductance G = 1/resistance (units: siemens, 1 S = 1 W $^{-1}$)

There are five parts to this experiment:

- *Using a concave reflection grating (made from a strip of CDROM) to produce a focused first order spectrum of the light from bulb A (12 V 50W tungsten filament).*
- *Measuring and plotting the conductance of the LDR against wavelength as it is scanned through this first order spectrum.*
- *Showing that the filament in bulb A behaves approximately as an ideal black body.*

Finding the temperature of the filament in bulb A when it is connected to the 12 V supply.

- *Correcting the graph of conductance against wavelength to take account of the energy distribution within the spectrum of light emitted by bulb A.*

Precautions

- *Beware of hot surfaces.*
- ***Bulb B should not be connected to any potential difference greater than 2.0 V.***
- *Do not use the multimeter on its resistance settings in any live circuit.*

Procedure

(a) The apparatus shown in Figure 1 has been set up so that light from bulb A falls normally on the curved grating and the LDR has been positioned in the focused **first order** spectrum. Move the LDR through this **first-order** spectrum and observe how its resistance (*measured by multimeter X*) changes with position.

(b) (i) Measure and record the resistance R of the LDR at different positions within this first-order spectrum. Record your data in the blank table provided.

(ii) Plot a graph of the conductance G of the LDR against wavelength l using the graph paper provided.

Note The angle q between the direction of light of wavelength l in the first-order spectrum and that of the white light reflected from the grating (see Figure 1) is given by:

sin q = l /d where d is the separation of lines in the grating.

The grating has 620 lines per mm.

The graph plotted in (b)(ii) does not represent the sensitivity of the LDR to different wavelengths correctly as the emission characteristics of bulb A have not been taken into account. These characteristics are investigated in parts (c) and (d) leading to a corrected curve plotted in part (e).

- ***Note for part (c) that three multimeters are connected as ammeters. These should NOT be adjusted or moved. Use the fourth multimeter (labelled X) for all voltage measurements.***

(c) If the filament of a 50 W bulb acts as a black-body radiator it can be shown that the potential difference V across it should be related to the current I through it by the expression:

$$V^3 = CI^5 \text{ where } C \text{ is a constant.}$$

Measure corresponding values of V and I for bulb A (in the can). *The ammeter is already connected and should not be adjusted.*

(i) Record your data and any calculated values in the table provided on the answer sheet.

(ii) Plot a suitable graph to show that the filament acts as a black-body radiator on the graph paper provided.

(d) To correct the graph in (b)(ii) we need to know the working temperature of the tungsten filament in bulb A. This can be found from the variation of filament resistance with temperature.

- **You are provided with a graph of tungsten resistivity** (m W **cm) against temperature (K).**

If the resistance of the filament in bulb A can be found at a known temperature then its temperature when run from the 12 V supply can be found from its resistance at that operating potential difference. Unfortunately its resistance at room temperature is too small to be measured accurately with this apparatus. However, you are provided with a second smaller bulb, C, which has a larger, *measurable* resistance at room temperature. Bulb C can be used as an intermediary by following the procedure described below. You are also provided with a second 12V 50W bulb (B) identical to bulb A. Bulbs B and C are mounted on the board provided and connected as shown in Figure 2.

(i) Measure the resistance of bulb C when it is unlit at room temperature (*use multimeter X*, and take room temperature to be 300 K). Record this resistance R_{C1} on the answer sheet.

i. Use the circuit shown in Figure 2 to compare the filaments of bulbs B and C. Use the variable resistor to vary the current through bulb C until you can see that overlapping filaments are at the same temperature. If the small filament is cooler than the larger one it appears as a thin black loop. Measure the resistances of bulbs B and C when this condition has been reached and record their values, R_{C2} and R_B, on the answer sheet. *Remember, the ammeters are already connected.*

(iii) Use the graph of resistivity against temperature (supplied) to work out the temperature of the filaments of B and C when they are matched. Record this temperature, T_{2V}, on the answer sheet.

(iv) Measure the resistance of the filament in bulb A (in the can) when it is connected to the 12 V a.c. supply. *Once again the ammeter is already connected and should not be adjusted*. Record this value, R_{12V} on the answer sheet.

(v) Use the values for the resistance of bulb A at 2 V and 12 V and its temperature at 2 V to work out its temperature when run from the 12 V supply. Record this temperature, T_{12V} in the table on the answer sheet.

- **You are provided with graphs that give the relative intensity of radiation from a black-body radiator (Planck curves) at 2000 K, 2250 K, 2500 K, 2750 K, 3000 K and 3250 K.**

(e) Use these graphs and the result from (d)(v) to plot a corrected graph of LDR conductance (arbitrary units) versus wavelength using the graph paper provided. Assume that the conductance of the LDR at any wavelength is directly proportional to the intensity of radiation at that wavelength (This assumption is reasonable at the low intensities falling on the LDR in this experiment). Assume also that the grating diffracts light equally to all parts of the first order spectrum.

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Figure 1 - Experimental arrangement for (a)

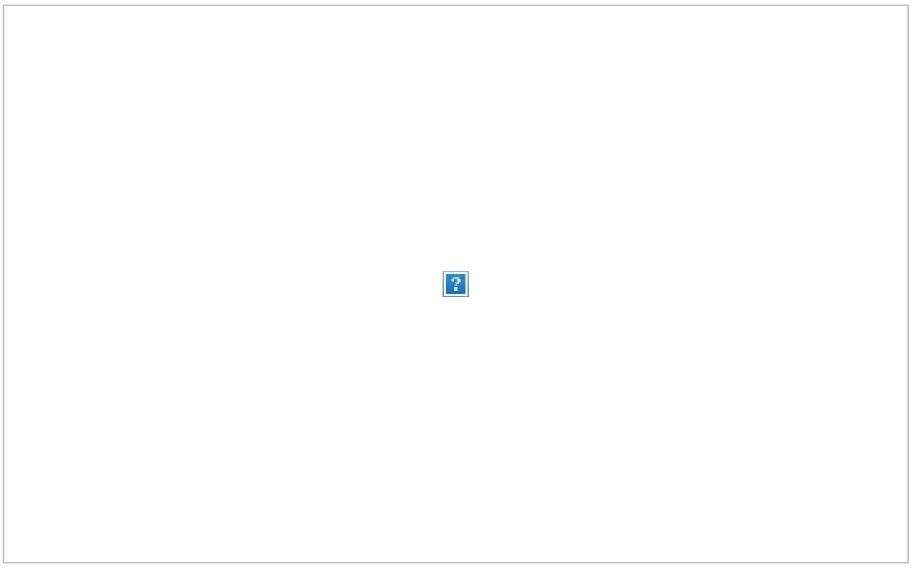

Figure 1: Detail - the grating:

Figure 1: Detail - LDR and Multimeter:

Figure 2

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The Magnetic Puck

July 2000

2.5 hours

In this experiment you ARE expected to indicate uncertainties in your measurements, results and graphs

Aim

To investigate the forces on a puck when it slides down the slope.

Warning

Do ***not touch*** *the circular flat faces of the puck or the paper surface of the slope with your hands. Use the glove provided. The faces have different coloured paper stickers for convenience but the frictional characteristics of the paper faces may be assumed to be the same.*

Timing

The sensors underneath the track trigger electronic gates in the box and the green LED will light when the puck is between the sensors. The multimeter measures the potential difference across a capacitor, which is connected to a constant-current source (whose current is proportional to the voltage of the battery) whilst the green light is on. The reading of the multimeter is therefore a measure of the time during which the puck is between the sensors. This reading can give a value for the speed of the puck in arbitrary units.

Operating the timer

i) Press and hold down the black push button on the side of the box. This switches the electronics on.

ii) If the green light goes on slide the puck (light face up) past the lower sensor. The green light should go off.

iii) The potential difference across the capacitor can be reduced to zero before the puck is released by pressing the red button for at least 10s.

iv) The battery potential difference can be measured by connecting the multimeter across the terminals marked with the cell symbol.

Definitions

(i) A moving body sliding down an inclined plane experiences a tangential retarding force *F* and a normal reaction *N*. Define

(ii) When the retarding force is due to friction alone, x equals m_s and is called the dynamic coefficient of friction for the surface. It is independent of speed.

(iii) When the blue (dark) side is in contact with the plane define

where the tangential force F_d is partly due to the surface friction and partly due to magnetic effects.

(iv) The variable x_{ds} which gives the magnetic effects only is defined by

Important hints and advice

(i) You will find it helpful initially to investigate the behaviour of the puck qualitatively.

(ii) Think about the physics before you do a quantitative investigation. Remember to use graphical presentation where possible.

(iii) Do not attempt to take too many experimental readings unless you have plenty of time.

(iv) You are measuring the potential difference across an electrolytic capacitor. This does not behave quite like a simple air capacitor. Slow leakage of charge is normal and the potential difference will not remain completely steady.

(v) You are given one puck and one 9.0 V battery. Conserve the battery! The constant current filling the capacitor is proportional to the battery potential difference. It is therefore advisable to monitor the battery potential difference. In addition, the sensors may not be reliable if the potential difference of the battery falls below 8.4 V. If this happens, ask for another battery.

(vi) Your answer pack contains 4 sides of graph paper only. You will not be given further sheets. You may keep the puck at the end of your experiment.

(vii) If you have trouble operating the multimeters ask an invigilator.

Data

- Weight of puck = $5.84'10^{-2}$ N
- The voltmeter reading indicates the time of travel of the puck. When the potential difference of the battery is 9.0 V then 1V corresponds to 0.213 s
- Distance between sensors = 0.294 m

Experiment

Using only the apparatus provided investigate how x_{ds} depends on the speed v_q of the puck for track inclinations q to the horizontal.

State on the answer sheet the algebraic equations/relations used in analysing your results and in plotting your graphs.

Suggest a quantitative model to explain your results. Use the data which you collect to justify your model.

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Theoretical Problem 1

Part A

A bungee jumper is attached to one end of a long elastic rope. The other end of the elastic rope is fixed to a high bridge. The jumper steps off the bridge and falls, from rest, towards the river below. He does not hit the water. The mass of the jumper is m, the unstretched length of the rope is L, the rope has a force constant (force to produce 1 m extension) of k and the gravitational field strength is g.

You may assume that

- the jumper can be regarded as a point mass m attached to the end of the rope,
- the mass of the rope is negligible compared to m,
- the rope obeys Hooke's law,
- air resistance can be ignored throughout the fall of the jumper.

Obtain expressions for the following and insert on the answer sheet:

- the distance y dropped by the jumper before coming instantaneously to rest for the first time,
- the maximum speed u attained by the jumper during this drop,
- the time t taken during the drop before coming to rest for the first time.

Part B

A heat engine operates between two identical bodies at different temperatures T_A and T_B ($T_A > T_B$), with each body having mass m and constant specific heat capacity s. The bodies remain at constant pressure and undergo no change of phase.

1. Showing full working, obtain an expression for the final temperature T_0 attained by the two bodies A and B if the heat engine extracts from the system the maximum amount of mechanical work that is theoretically possible.

   Write your expression for the final temperature T_0 on the answer sheet.

2. Hence, obtain and write on the answer sheet an expression for this maximum amount of work available.

   The heat engine operates between two tanks of water each of volume 2.50 m^3. One tank is at 350 K and the other is at 300 K.

3. Calculate the maximum amount of mechanical energy obtainable. Insert the value on the answer sheet.

   Specific heat capacity of water = 4.19 ´ 10^3 J kg^{-1} K^{-1}

   Density of water = 1.00 ´ 10^3 kg m^{-3}

Part C

It is assumed that when the earth was formed the isotopes and were present but not their decay products. The decays of and are used to establish the age of the earth, *T*.

a. The isotope decays with a half-life of 4.50 ′ 10^9 years. The decay products in the resulting radioactive series have half-lives short compared to this; to a first approximation their existence can be ignored. The decay series terminates in the stable lead isotope

   Obtain and insert on the answer sheet an expression for the number of atoms, denoted , produced by radioactive decay with time t, in terms of the present number of atoms, denoted , and the half- life time of . (You may find it helpful to work in units of 10^9 years.)

b. Similarly, decays with a half-life of 0.710 ′ 10^9 years through a series of shorter half-life products to give the stable isotope .

   Write down on the answer sheet an equation relating to and the half-life of .

c. A uranium ore, mixed with a lead ore, is analysed with a mass spectrometer. The relative concentrations of the three lead isotopes , and are measured and the number of atoms are found to be in the ratios 1.00 : 29.6 : 22.6 respectively. The isotope is used for reference as it is <u>not</u> of radioactive origin. Analysing a pure lead ore gives ratios of 1.00 : 17.9 : 15.5.

   Given that the ratio : is 137 : 1, derive and insert on the answer sheet an equation involving *T*.

d. Assume that *T* is much greater than the half lives of both uranium isotopes and hence obtain an approximate value for *T*.

e. This approximate value is clearly not significantly greater than the longer half life, but can be used to obtain a much more accurate value for *T*.
   Hence, or otherwise, estimate a value for the age of the earth correct to within 2%.

Part D

Charge *Q* is uniformly distributed *in vacuo* throughout a spherical volume of radius *R*.

a. Derive expressions for the electric field strength at distance *r* from the centre of the sphere for *r* £ *R* and *r* > *R*.

b. Obtain an expression for the total electric energy associated with this distribution of charge.

Insert your answers to (a) and (b) on the answer sheet.

Part E

A circular ring of thin copper wire is set rotating about a vertical diameter at a point within the Earth's magnetic field. The magnetic flux density of the Earth's magnetic field at this point is 44.5 mT directed at an angle of 64^{o} below the horizontal. Given that the density of copper is 8.90 ′ 10^3 kg m^{-3} and its resistivity is 1.70 ′ 10^{-8} W m, calculate how long it will take for the angular velocity of the ring to halve. Show the steps of your working and insert the value of the time on the answer sheet. This time is much longer than the time for one revolution.

You may assume that the frictional effects of the supports and air are negligible, and for the purposes of this question you should ignore self-inductance effects, although these would not be negligible.

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Theoretical Problem 2

a. A cathode ray tube (CRT), consisting of an electron gun and a screen, is placed within a uniform constant magnetic field of magnitude **B** such that the magnetic field is parallel to the beam axis of the gun, as shown in figure 2.1.

Figure 2.1

The electron beam emerges from the anode of the electron gun on the axis, but with a divergence of up to 5° from the axis, as illustrated in figure 2.2. In general a diffuse spot is produced on the screen, but for certain values of the magnetic field a sharply focused spot is obtained.

Figure 2.2

By considering the motion of an electron initially moving at an angle b (where 0 £ b £ 5°) to the axis as it leaves the electron gun, and considering the components of its motion parallel and perpendicular to the axis, derive an expression for the charge to mass ratio e/m for the electron in terms of the following quantities:

- the smallest magnetic field for which a focused spot is obtained,
- the accelerating potential difference across the electron gun V (note that $V < 2$ kV),
- D, the distance between the anode and the screen.

Write your expression in the box provided in section 2a of the answer sheet.

b. Consider another method of evaluating the charge to mass ratio of the electron. The arrangement is shown from a side view and in plan view (from above) in figure 2.3, with the direction of the magnetic field marked **B**. Within this uniform magnetic field **B** are placed two brass circular plates of radius r which are separated by a very small distance t. A potential difference V is maintained between them. The plates are mutually parallel and co-axial, however their axis is perpendicular to the magnetic field. A photographic film, covers the inside of the curved surface of a cylinder of radius r + s, which is held co-axial with the plates. In other words, the film is at a radial distance s from the edges of the plates. The entire arrangement is placed *in vacuo*. Note that t

is very much smaller than both *s* and r.

A point source of b particles, which emits the b particles uniformly in all directions with a range of velocities, is placed between the centres of the plates, and the *same piece of film* is exposed under three different conditions:

- firstly with $B = 0$, and $V = 0$,
- secondly with $B = B_0$, and $V = V_0$, and
- thirdly with $B = -B_0$, and $V = -V_0$;

where V_0 and B_0 are positive constants. Please note that the upper plate is positively charged when $V>0$ (negative when $V<0$), and that the magnetic field is in the direction defined by figure 2.5 when $B>0$ (in the opposite direction when $B<0$). For this part you may assume the gap is negligibly small.

Two regions of the film are labelled A and B on figure 2.3. After exposure and development, a sketch of one of these regions is given in figure 2.4. From which region was this piece taken (on your answer sheet write A or B)? Justify your answer by showing the directions of the forces acting on the electron.

c. After exposure and development, a sketch of the film is given in figure 2.4. Measurements are made of the separation of the two outermost traces with a microscope, and this distance (y) is also indicated for one particular angle on figure 2.4. The results are given in the table below, the angle f being defined in figure 2.3 as the angle between the magnetic field and a line joining the centre of the plates to the point on the film.

   Numerical values of the system parameters are given below:

   $B_0 = 6.91$ mT $V_0 = 580$ V $t = 0.80$ mm $s = 41.0$ mm

   In addition, you may take the speed of light in vacuum to be 3.00×10^8 m s^{-1}, and the rest mass of the electron to be 9.11×10^{-31} kg.

   Determine the maximum b particle kinetic energy observed.

   Write the maximum kinetic energy as a numerical result in eV in the box on the answer sheet, section 2c.

d. Using the information given in part (c), obtain a value for the charge to rest mass ratio of the electron. This should be done by plotting an appropriate graph on the paper provided.

   Indicate *algebraically* the quantities being plotted on the horizontal and vertical axes both on the graph itself *and* on the answer sheet in the boxes provided in section 2d.

   Write your value for the charge to mass ratio of the electron in the box provided on the answer sheet, section 2d.

   Please note that the answer you obtain may not agree with the accepted value because of a systematic error in the observations.

Additional Figures

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Theoretical Problem 3

Part A

This part is concerned with the difficulties of detecting gravitational waves generated by astronomical events. It should be realised that the explosion of a distant supernova may produce fluctuations in the gravitational field strength at the surface of the Earth of about 10^{-19} N kg^{-1}.A model for a gravitational wave detector (see figure 3.1) consists of two metal rods each 1m long, held at right angles to each other. One end of each rod is polished optically flat and the other end is held rigidly. The position of one rod is adjusted so there is a minimum signal received from the photocell (see figure 3.1).

Figure 3.1

The rods are given a short sharp longitudinal impulse by a piezoelectric device. As a result the free ends of the rods oscillate with a longitudinal displacement Dx_t, where

and a, m, w and f are constants.

(a) If the amplitude of the motion is reduced by 20% during a 50s interval determine a value for m.

(b) Given that longitudinal wave velocity,v = Ö(E/r), determine also the lowest value for w, given that the rods are made of aluminium with a density (r) of 2700 kg.m^{-3} and a Young modulus (E) of 7.1 x 10^{10} Pa.

(c) It is impossible to make the rods exactly the same length so the photocell signal has a beat frequency of 0.005 Hz. What is the difference in length of the rods?

(d) For the rod of length l, derive an algebraic expression for the change in length, Dl, due to a change, Dg, in the gravitational field strength, g, in terms of l and other constants of the rod material. The response of the detector to this change takes place in the direction of one of the rods.

(e) The light produced by the laser is monochromatic with a wavelength of 656nm. If the minimum fringe shift that can be detected is 10^{-4} of the wavelength of the laser, what is the minimum value of l necessary if such a system were to be capable of detecting variations in g of 10^{-19} N kg^{-1}?

Part B

This part is concerned with the effect of a gravitational field on the propagation of light in space.

(a) A photon emitted from the surface of the Sun (mass M, radius R) is red-shifted. By assuming a rest-mass equivalent for the photon energy, apply Newtonian gravitational theory to show that the effective (or measured) frequency of the photon at infinity is reduced (red-shifted) by the factor ($1 - GM/Rc^2$).

(b) A reduction of the photon's frequency is equivalent to an increase in its time period, or, using the photon as a standard clock, a dilation of time. In addition, it may be shown that a time dilation is always accompanied by a contraction in the unit of length by the same factor.

We will now try to study the effect that this has on the propagation of light near the Sun. Let us first define an effective refractive index n_r. at a point r from the centre of the Sun. Let

where c is the speed of light as measured by a co-ordinate system far away from the Sun's gravitational influence (r ® ¥), and c_r¢ is the speed of light as measured by a co-ordinate system at a distance r from the centre of the Sun.

Show that n_r. may be approximated to:

for small GM/rc^2, where a is a constant that you determine.

(c) Using this expression for n_r, calculate in radians the deflection of a light ray from its straight path as it passes the edge of the Sun.

Data:

Gravitational constant, G = 6.67 ´ 10^{-11} N m^2 kg^{-2}.

Mass of Sun, M = 1.99 ´ 10^{30} kg.

Radius of Sun, R = 6.95 ´ 10^8 m.

Theoretical Competition

Monday, July 2nd, 2001

Please read this first:

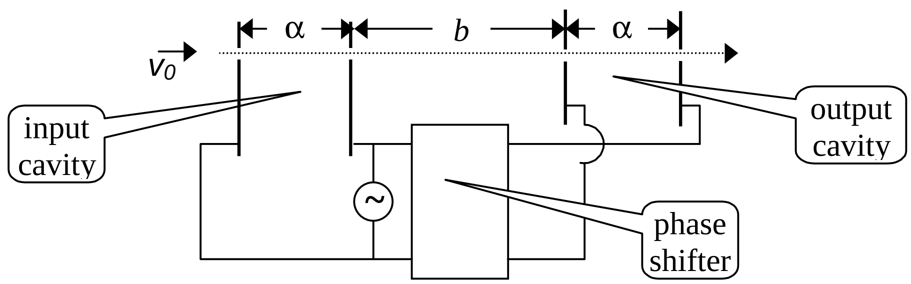
1. The time available is 5 hours for the theoretical competition.
2. Use only the pen provided.
3. Use only the front side of the paper.
4. Begin each part of the problem on a separate sheet.
5. For each question, in addition to the ***blank sheets*** where you may write, there is an **a*nswer form*** where you *must* summarize the results you have obtained. Numerical results should be written with as many digits as are appropriate to the given data.
6. Write on the blank sheets of paper whatever you consider is required for the solution of the question. Please use *as little text as possible*; express yourself primarily in equations, numbers, figures, and plots.
7. Fill in the boxes at the top of each sheet of paper used by writing your ***Country No*** and ***Country Code,*** your student number (***Student No***), the number of the question ***(Question No)***, the progressive number of each sheet ***(Page No),*** and the total number of blank sheets used for each question ***(Total No of pages).*** Write the question number and the section letter of the part you are answering at the top of each sheet. If you use some blank sheets of paper for notes that you do not wish to be marked, put a large X across the entire sheet and do not include it in your numbering.
8. At the end of the exam, arrange all sheets for each problem *in the following order;*
   - answer form
   - used sheets in order
   - the sheets you do not wish to be marked
   - unused sheets and the printed question

   Place the papers inside the envelope and leave everything on your desk. You are not allowed to take *any* sheets of paper out of the room.

Question 1

1a) KLYSTRON

Klystrons are devices used for amplifying very high-frequency signals. A klystron basically consists of two identical pairs of parallel plates (cavities) separated by a distance b, as shown in the figure.



An electron beam with an initial speed v_0 traverses the entire system, passing through small holes in the plates. The high-frequency voltage to be amplified is applied to both pairs of plates with a certain phase difference (where period T corresponds to 2π phase) between them, producing horizontal, alternating electric fields in the cavities. The electrons entering the input cavity when the electric field is to the right are retarded and vice versa, so that the emerging electrons form bunches at a certain distance. If the output cavity is placed at the bunching point, the electric field in this cavity will absorb power from the beam provided that its phase is appropriately chosen. Let the voltage signal be a square wave with period T=1.0x10^{-9} s, changing between V=±0.5 volts. The initial velocity of the electrons is v_0=2.0x10^6 m/s and the charge to mass ratio is e/m=1.76x10^{11} C/kg. The distance α is so small that the transit time in the cavities can be neglected. Keeping 4 significant figures, calculate;

a) the distance b, where the electrons bunch. Copy your result onto the **answer form.** *[1.5 pts]*

b) the necessary phase difference to be provided by the phase shifter. Copy your result onto the **answer form.** *[1.0 pts]*

1b) INTERMOLECULAR DISTANCE

Let d_L and d_V represent the average distances between molecules of water in the liquid phase and in the vapor phase, respectively. Assume that both phases are at 100 °C and atmospheric pressure, and the vapor behaves like an ideal gas. Using the following data, calculate the ratio d_V/d_L and copy your result onto the **answer form.** *[2.5 pts]*

Density of water in liquid phase: ρ_L=1.0x10^3 kg/m^3,

Molar mass of water: M=1.8x10^{-2} kg/mol
Atmospheric pressure: P_a=1.0x10^5 N/m^2
Gas constant: R=8.3 J/mol· K
Avagadro's number: N_A=6.0x10^{23} /mol

1c) SIMPLE SAWTOOTH SIGNAL GENERATOR

A sawtooth voltage waveform V_0 can be obtained across the capacitor C in Fig. 1. R is a variable resistor, V_i is an ideal battery, and SG is a spark gap consisting of two electrodes with an adjustable distance between them. When the voltage across the electrodes exceeds the firing voltage V_f, the air between the electrodes breaks down, hence the gap becomes a short circuit and remains so until the voltage across the gap becomes very small.

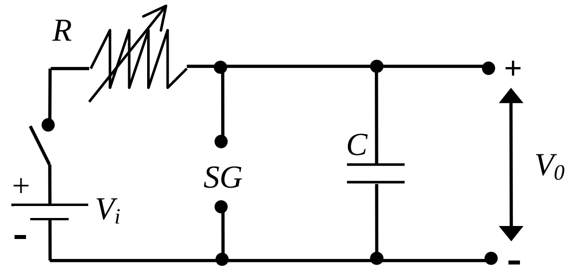


Figure 1

a) Draw the voltage waveform V_0 versus time t, after the switch is closed. *[0.5 pts]*

b) What condition must be satisfied in order to have an almost linearly varying sawtooth voltage waveform V_0? Copy your result onto the **answer form**. *[0.2 pts]*

c) Provided that this condition is satisfied, derive a simplified expression for the period T of the waveform. Copy your result onto the **answer form.** *[0.4 pts]*

d) What should you vary(R and/or SG) to change the period only? Copy your result onto the **answer form.** *[0.2 pts]*

e) What should you vary (R and/or SG) to change the amplitude only? Copy your result onto the **answer form.** *[0.2 pts]*

f) You are given an additional, adjustable DC voltage supply. Design and draw a new circuit indicating the terminals where you would obtain the voltage waveform V_0' described in Fig. 2. *[1.0 pts]*

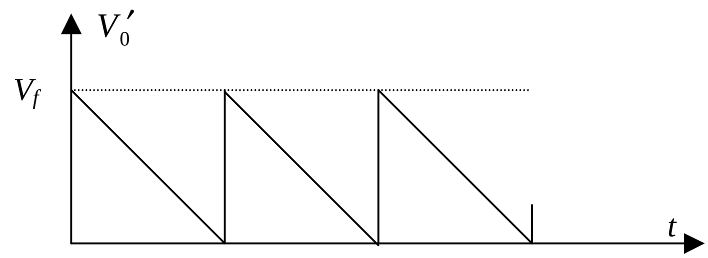
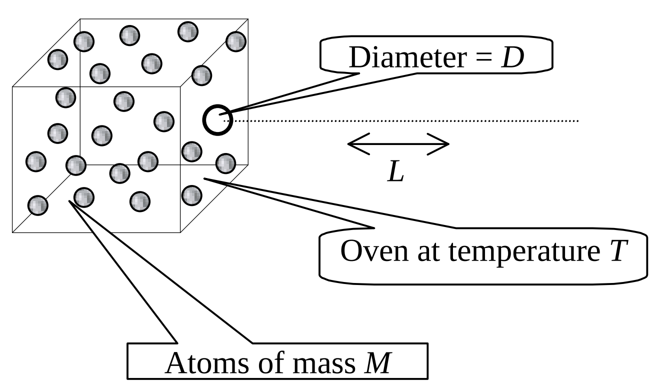


Figure 2

1d) ATOMIC BEAM

An atomic beam is prepared by heating a collection of atoms to a temperature T and allowing them to emerge horizontally through a small hole (of atomic dimensions) of diameter D in one side of the oven. Estimate the diameter of the beam after it has traveled a horizontal length L along its path. The mass of an atom is M. Copy your result onto the **answer form.** *[2.5 pts]*



Question 2

BINARY STAR SYSTEM

a) It is well known that most stars form binary systems. One type of binary system consists of an ordinary star with mass m_0 and radius R, and a more massive, compact neutron star with mass M, rotating around each other. In all the following ignore the motion of the earth. Observations of such a binary system reveal the following information:

- The maximum angular displacement of the ordinary star is $\Delta\theta$, whereas that of the neutron star is $\Delta\phi$ (see Fig. 1).
- The time it takes for these maximum displacements is τ.
- The radiation characteristics of the ordinary star indicate that its surface temperature is T and the radiated energy incident on a unit area on earth's surface per unit time is P.
- The calcium line in this radiation differs from its normal wavelength λ_0 by an amount $\Delta\lambda$, due only to the gravitational field of the ordinary star. (For this calculation the photon can be considered to have an effective mass of $h/c\lambda$.)

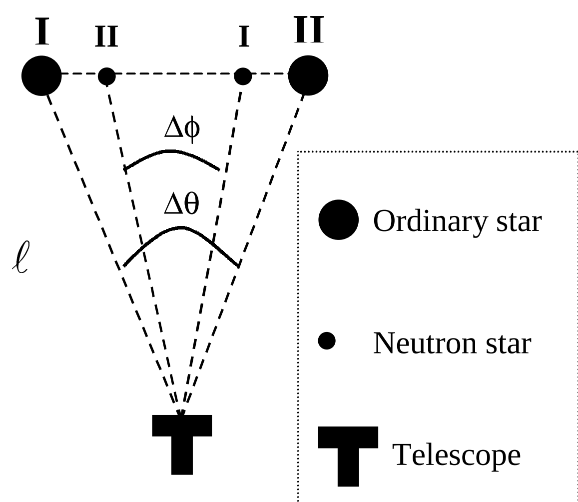


Fig. 1

Find an expression for the distance ℓ from earth to this system, only in terms of the observed quantities and universal constants. Copy your result onto the **answer form**. *[7 pts]*

b) Assume that $M >> m_0$, so that the ordinary star is basically rotating around the neutron star in a circular orbit of radius r_0. Assume that the ordinary star starts emitting gas toward the neutron star with a speed v_0, relative to the ordinary star (see Fig. 2). Assuming that the neutron star is the dominant gravitational force in this problem and neglecting the orbital changes of the ordinary star find the distance of closest approach r_f shown in Fig. 2. Copy your result onto the **answer form**. *[3pts]*

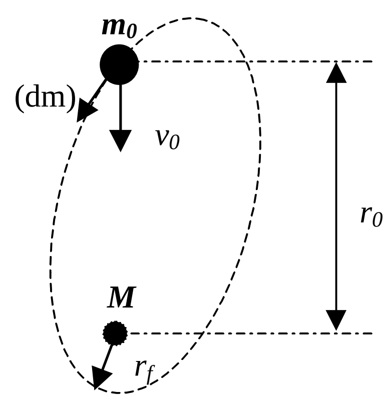
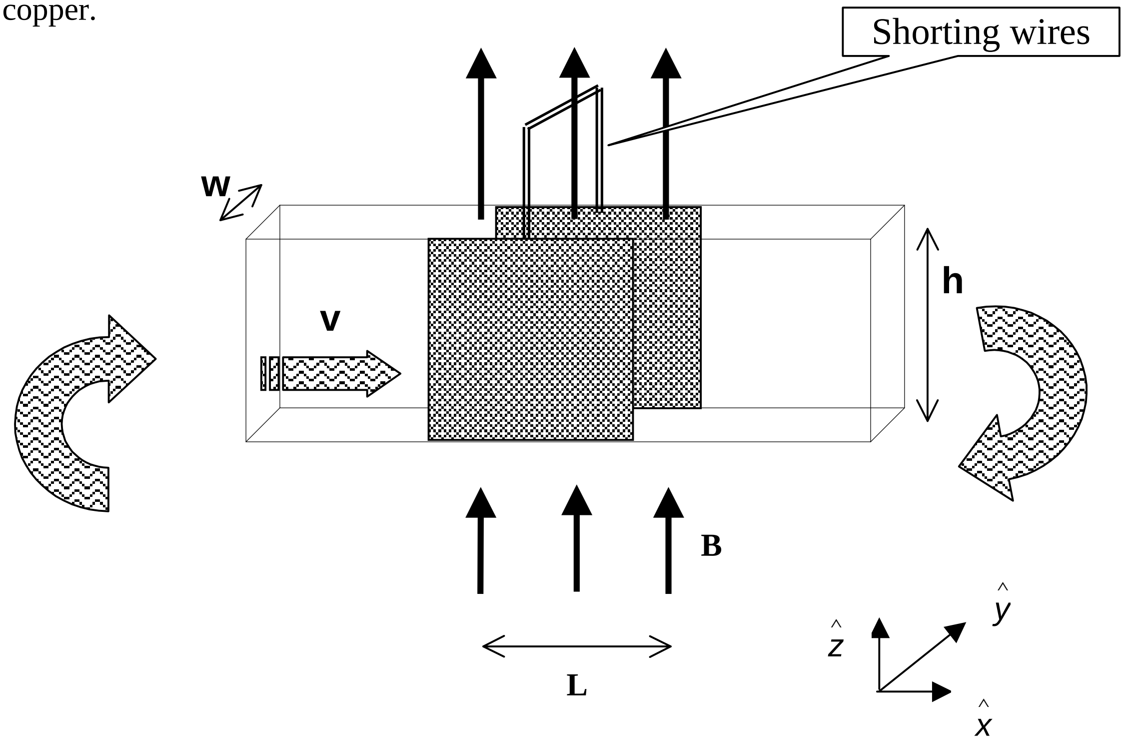


Fig. 2

Question 3

MAGNETOHYDRODYNAMIC (MHD) GENERATOR

A horizontal rectangular plastic pipe of width w and height h, which closes upon itself, is filled with mercury of resistivity ρ. An overpressure P is produced by a turbine which drives this fluid with a constant speed v_0. The two opposite vertical walls of a section of the pipe with length L are made of copper.



The motion of a real fluid is very complex. To simplify the situation we assume the following:

- Although the fluid is viscous, its speed is uniform over the entire cross section.
- The speed of the fluid is always proportional to the net external force acting upon it.
- The fluid is incompressible.

These walls are electrically shorted externally and a uniform, magnetic field **B** is applied vertically upward only in this section. The set up is illustrated in the figure above, with the unit vectors $\hat{x}$, $\hat{y}$, $\hat{z}$ to be used in the solution.

a) Find the force acting on the fluid due to the magnetic field (in terms of L, B, h, w, ρ and the new velocity v) *[2.0 pts]*

b) Derive an expression for the new speed v of the fluid (in terms of v_0, P, L, B and ρ) after the magnetic field is applied. *[3.0 pts]*

c) Derive an expression for the additional power that must be supplied by the turbine to increase the speed to its original value v_0. Copy your result onto the **answer form.** *[2.0 pts]*

d) Now the magnetic field is turned off and mercury is replaced by water flowing with speed v_0. An electromagnetic wave with a single frequency is sent along the section with length L in the direction of the flow. The refractive index of water is n, and $v_0 \ll c$. Derive an expression for the contribution of the fluid's motion to the phase difference between the waves entering and leaving section L. Copy your result onto the **answer form.** *[3.0 pts]*

Country no	Country code	Student No.	Question No.	Page No.	Total No. of pages

ANSWER FORM

1A

a)

$b =$

b)

Phase difference=

1B

$\dfrac{d_V}{d_L} =$

Country no	Country code	Student No.	Question No.	Page No.	Total No. of pages

1C

b)

c)

T=

d)

e)

1D

New diameter of the beam =

Country no	Country code	Student No.	Question No.	Page No.	Total No. of pages

ANSWER FORM

2a)

$\ell =$

2b)

$r_f =$

Country no	Country code	Student No.	Question No.	Page No.	Total No. of pages

ANSWER FORM

3a)

3b)

v =

3c)

Power =

3d)

Phase difference =

Solution

Part 1a

a. $v_{ret} = \sqrt{v_0^2 - 2(e/m)V} = 1.956 \times 10^6 \, m/s$ (0.5 pts)

$v_{acc} = \sqrt{v_0^2 + 2(e/m)V} = 2.044 \times 10^6 \, m/s$

$x_{ret} = v_{ret} t, \qquad \mathrm{x}_{\mathrm{acc}} = v_{acc}(t - T/2)$ (0.5 pts)

$x_{ret} = x_{acc} \rightarrow \mathrm{t}_{\mathrm{bunch}} = \dfrac{v_{acc} T}{2(v_{acc} - v_{ret})} = 11.61T$ (0.3 pts)

$b = v_{ret} t_{bunch} = 2.272 \times 10^{-2} \, m.$ (0.2 pts)

b. The phase difference:

$\Delta\varphi = \pm(\dfrac{t_{bunch}}{T} - n)2\pi = \pm 0.61 \times 2\pi = \pm 220°.$ (1.0 pts)

OR

$\Delta\varphi = \pm 140°$

Part 1b

$\rho_L = n_L \dfrac{M}{N_A}$ (0.3 pts)

where n_L is the number of molecules per cubic meter in the liquid phase

Average distance between the molecules of water in the liquid phase:

$d_L = (n_L)^{-1/3} = (\dfrac{M}{\rho_L N_A})^{\frac{1}{3}}$ (0.2 pts)

$P_a V = nRT$,
where n is the number of moles (0.6 pts)

$P_a = \dfrac{nM}{V}\dfrac{RT}{M} = \rho_V \dfrac{RT}{M} = \dfrac{n_V M}{N_A}\dfrac{RT}{M}$

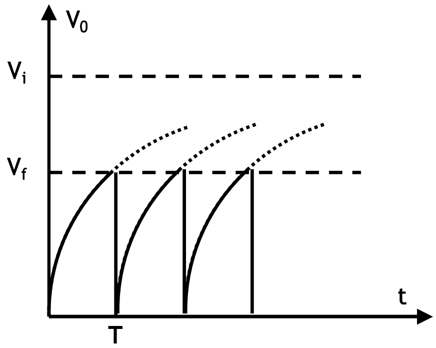
where n_V is the number of molecules per cubic meter in the vapor phase. (0.9 pts)

$d_V = (n_V)^{-1/3} = (\dfrac{RT}{P_a N_A})^{\frac{1}{3}}$ (0.2 pts)

$\dfrac{d_V}{d_L} = (\dfrac{RT\rho_L}{P_a M})^{\frac{1}{3}} = 12$ (0.3 pts)

Part 1c

a. (0.5 pts.)



b. $V_i >> V_f$ (0.2 pts)

c. $V_f = V_i\left(1 - e^{-T/RC}\right)$ (0.2 pts)

If

$V_i >> V_f$, $T/RC << 1$, $e^{-T/RC} \approx 1 - (T/RC)$

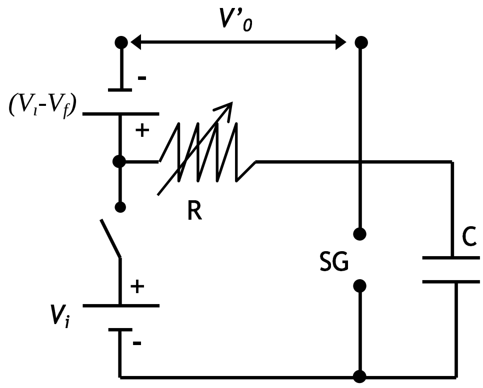
then

$T = (V_f / V_i)\, RC$ (0.2 pts)

d. R (0.2 pts)

e. SG and R (0.2 pts)

f. Correct circuit (0.4 pts)



V_0' (0.3 pts)

$V_i' - V_f$ with the correct polarity (0.3 pts)

Total (1.0 pts)

Part 1d

As the beam passes through a hole of diameter D the resulting uncertainty in the y-component of the momentum;

$$\Delta p_y \approx \frac{\hbar}{D} \qquad \text{(0.6 pts)}$$

and the corresponding velocity component;

$$\Delta v_y \approx \frac{\hbar}{MD} \qquad \text{(0.4 pts)}$$

Diameter of the beam grows larger than the diameter of the hole by an amount

$\Delta D = \Delta v_y . t$,

where t is the time of travel. (0.2 pts)

If the oven temperature is T, a typical atom leaves the hole with kinetic energy

$$KE = \frac{1}{2}Mv^2 = \frac{3}{2}kT \qquad \text{(0.4 pts)}$$

$$v = \sqrt{\frac{3kT}{M}} \qquad \text{(0.2 pts)}$$

Beam travels the horizontal distance L at speed v in time

$t = \frac{L}{v}$, so (0.2 pts)

$$\Delta D = t\Delta v_y \approx \frac{L}{v}\frac{\hbar}{MD} = \frac{L\hbar}{MD\sqrt{\frac{3kT}{M}}} = \frac{L\hbar}{D\sqrt{3MkT}} \qquad \text{(0.4 pts)}$$

Hence the new diameter after a distance L will be;

$$D_{new} = D + \frac{L\hbar}{D\sqrt{3MkT}} \qquad \text{(0.1 pts)}$$

Part 2a

The total energy radiated per second = $4\pi R^2\sigma T^4$, where σ is the Stephan-Boltzmann constant. The energy incident on a unit area on earth per second is;

$$P = \frac{4\pi R^2 \sigma T^4}{4\pi\ell^2} \text{ yielding, } R = \left(P/\sigma T^4\right)^{1/2} \ell \quad (1) \qquad (0.8 \text{ pts})$$

The energy of a photon is $hf=hc/\lambda$. The equivalent mass of a photon is $h/c\lambda$. Conservation of photon energy:

$$\frac{hc}{\lambda_0} - \frac{Gm_0}{R}\cdot\frac{h}{c\lambda_0} = \frac{hc}{\lambda} \qquad (0.8 \text{ pts})$$

yielding

$$R = \frac{Gm_0(\lambda_0 + \Delta\lambda)}{c^2\Delta\lambda} \quad (2)$$

and (2) yields,

$$m_0 = \frac{c^2\Delta\lambda\left(P/\sigma T^4\right)^{1/2}}{G(\lambda_0 + \Delta\lambda)}\ell \quad (3) \qquad (0.2 \text{ pts})$$

The stars are rotating around the center of mass with equal angular speeds:

$\omega = (2\pi/2\tau) = \pi/\tau$ (4) (0.2 pts)

The equilibrium conditions for the stars are;

$$\frac{GMm_0}{(r_1 + r_2)^2} = m_0 r_1\omega^2 = Mr_2\omega^2 \quad (5) \qquad (0.8 \text{ pts})$$

with

$$r_1 = \ell\frac{\Delta\theta}{2},\ r_2 = \ell\frac{\Delta\phi}{2} \quad (6) \qquad (0.4 \text{ pts})$$

Substituting (3), (4) and (6) into (5) yields

$$\ell = \left(\frac{8c^2\Delta\lambda\left(P/\sigma T^4\right)^{1/2}}{\Delta\phi(\pi/\tau)^2(\lambda_0 + \Delta\lambda)(\Delta\theta + \Delta\phi)^2}\right)^{1/2}. \qquad (0.8 \text{ pts})$$

Part 2b

Conservation of angular momentum for the ordinary star;

$$mr^2\omega = m_0 r_0^2\omega_0 \quad (7) \qquad (0.6 \text{ pts.})$$

Conservation of angular momentum for dm:

$$r^2\omega\, dm = r_f^2\omega_f\, dm \quad (8) \qquad (0.6 \text{ pts})$$

where ω_f is the angular velocity of the ring. Equilibrium in the original state yields,

$$\omega_0 = \left(\frac{GM}{r_0^3}\right)^{1/2} \quad (9) \qquad (0.8 \text{ pts})$$

and (7), (8) and (9) give,

$$\omega = \frac{m_0 r_0}{m r^2}\left(\frac{GM}{r_0}\right)^{1/2}, \ \omega_f = \frac{m_0 r_0}{m r_f^2}\left(\frac{GM}{r_0}\right)^{1/2} \quad (10) \qquad (0.4 \text{ pts})$$

Conservation of energy for dm;

$$\frac{1}{2} dm\left(v_0^2 + r^2\omega^2\right) - \frac{GM \text{ dm}}{\text{r}} = \frac{1}{2} dm\, r_f{}^2 \omega_f{}^2 - \frac{GM \text{ dm}}{\text{r}_\text{f}} \quad (11) \qquad (1.2 \text{ pts})$$

Substituting (10);

$$v_0^2 + \frac{m_0^2 r_0 GM}{m^2}\left(\frac{1}{r^2} - \frac{1}{r_f^2}\right) - 2GM\left(\frac{1}{r} - \frac{1}{r_f}\right) = 0 \quad (12)$$

Since $r_0 >> r_f$, if $r > r_0$, r^{-1} and r^{-2} terms can be neglected. Hence,

$$r_f = \frac{GM}{v_0^2}\left(\left(1 + \frac{m_0^2 r_0 v_0^2}{GMm^2}\right)^{1/2} - 1\right). \qquad (0.8 \text{ pts})$$

To show that $r > r_0$ change in the linear momentum of the ordinary star in its reference frame:

$$-\frac{GMm}{r^2} + mr\omega^2 - m\frac{dv_r}{dt} = -v_0 \frac{dm_{gas}}{dt} \quad (13) \qquad (0.8 \text{ pts})$$

and (13) implies the existence of an outward force initially and hence r starts growing. Using (7) one can write

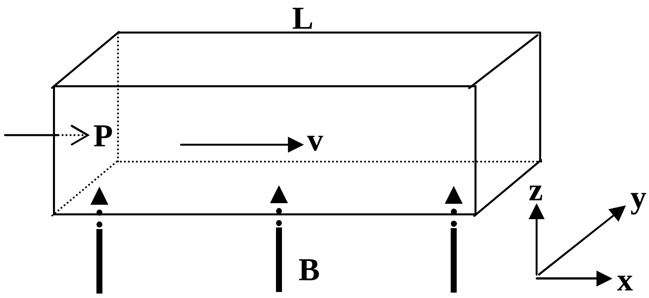
$$mr\omega^2 = \frac{m_0^2 r_0^4 \omega_0^2}{mr^3} .$$

Hence, $\dfrac{Gravitational\ force}{Centrifugal\ force} \alpha\, \text{m}^2 r$. (0.4 pts)

where m is definitely decreasing. If r starts decreasing at some time also, this ratio starts decreasing, which is a contradiction.

So $r > r_0$. (0.4 pts)

Part 3a



The net force on a charged particle must be zero in the steady state

$$\vec{F} = 0 = q\vec{E} + q\vec{v}x\vec{B}$$

$$\vec{E} = -\vec{v}x\vec{B} = vB\,\hat{y}$$ (0.4 pts)

$$V_H = vBw$$

$$I = \frac{V_H}{R} = \frac{V_H}{\frac{\rho w}{Lh}} = \frac{vBwLh}{\rho w} = \frac{vBLh}{\rho}$$, direction: $-\hat{y}$ (0.6 pts)

$$\vec{F} = I\vec{\ell}x\vec{B} = \frac{vB^2Lhw}{\rho}$$, direction: **(**$-\hat{y}\,x\,\hat{z} = -\hat{x}$**)**

Force is in the -x direction (0.8 pts)

This creates a back pressure P_b

$$P_b = \frac{vB^2Lhw}{\rho hw} = \frac{vB^2L}{\rho}$$ (0.6 pts)

$F_{net}=(P-P_b)hw$, (0.6 pts)

$v=\alpha F_{net}$ (0.4 pts)

$v=\alpha(P-P_b)hw= \alpha\,(P - \frac{vB^2L}{\rho})\frac{v_0}{\alpha P} = v_0 - \frac{vv_0B^2L}{P\rho}$

$$v(1+\frac{v_0B^2L}{P\rho}) = v_0$$

$$v = v_0\left(1+\frac{v_0B^2L}{P\rho}\right)^{-1}$$

$$v = v_0\,\frac{P\rho}{P\rho + v_0B^2L}$$ (0.6 pts)

Part 3b

From conservation of energy:

$$\Delta Power = V_H I = \frac{v_0^2 B^2 whL}{\rho}$$

or,

to recover v_0 the pump must supply an additional pressure $\Delta P=P_b$ **(1.0 pts)**

$$\Delta Power = \Delta P hwv_0 = P_b hwv_0 = \frac{v_0^2 B^2 whL}{\rho}$$

Part 3c

1. $u = \frac{c}{n}$ $\qquad u' = \frac{\frac{c}{n}+v}{1+\frac{c}{n}\frac{v}{c^2}} = \frac{\frac{c}{n}+v}{1+\frac{v}{cn}}$ (0.5 pts)

   For small v (v<<c);

   neglect the terms containing $\frac{v^2}{c^2}$ in the expansion of $(1+\frac{v}{cn})^{-1}$

   $$u' = (\frac{c}{n}+v)\frac{1}{1+\frac{v}{cn}} \approx (\frac{c}{n}+v)(1-\frac{v}{cn}) \approx \frac{c}{n}+v(1-\frac{1}{n^2})$$

   $\Delta u = u' - u \approx v(1-\frac{1}{n^2})$ (0.5 pts)

   $\Delta\phi = 2\pi f \Delta T$, $T = \frac{L}{u}$, $\Delta \mathrm{T} = \frac{\Delta \mathrm{u}}{\mathrm{u}^2} L \approx \frac{Lv}{c^2}(n^2-1)$ (0.5 pts)

   $v=v_0$ so that, $\Delta\phi = 2\pi f \frac{L}{c^2}(n^2-1)v_0$ (0.5 pts)

2. $\Delta\phi = 2\pi f \frac{L}{c^2}(n^2-1)v_0$

   a phase of $\pi/36$ results in (0.4 pts)

   $v_0 = \frac{c^2}{72L(n^2-1)f}$ (0.2 pts)

   $v_0 = \frac{9x10^{16}}{72x10^{-1}x(2.56-1)x25} = 3.2x10^{14}\, m/s$ which is not physical. (0.4 pts)

3. For v=20 m/s, f≈$4x10^{14}$ Hz. But for this value of f, skin depth is about 25 nm. This means that amplitude of the signal reaching the end of the tube is practically zero. Therefore mercury should be replaced with water. (0.6 pts)

   On the other hand if water is used instead of mercury, at 25 Hz $\delta \approx 3x10^{5}$ m. Signal reaches to the end but v≈$6x10^{14}$ m/s, is still nonphysical. Therefore frequency should be readjusted. (0.6 pts)

   For v=20 m/s electromagnetic wave of f≈$8x10^{14}$ Hz has a skin depth of about $\delta \approx 5.6$ cm in water and the emerging wave is out of phase by $\pi/36$ with respect to the incident wave. (The amplitude of the wave reaching to the end of the section is about 17% of the incident amplitude). (0.6 pts)

   Therefore mercury should be replaced with water and frequency should be adjusted to f≈$8x10^{14}$ Hz. The correct choice is (iii) (0.2 pts)

I. Ground-Penetrating Radar

Ground-penetrating radar (GPR) is used to detect and locate underground objects near the surface by means of transmitting electromagnetic waves into the ground and receiving the waves reflected from those objects. The antenna and the detector are directly on the ground and they are located at the same point.

A linearly polarized electromagnetic plane wave of angular frequency ω propagating in the z direction is represented by the following expression for its field:

$$E = E_0\, e^{-\alpha z} \cos(\omega t - \beta z)\ , \qquad (1)$$

where E_o is constant, α is the attenuation coefficient and β is the wave number expressed respectively as follows

$$\alpha = \omega\left\{\frac{\mu\varepsilon}{2}\left[\left(1+\frac{\sigma^2}{\varepsilon^2\omega^2}\right)^{1/2}-1\right]\right\}^{1/2}\ ,\quad \beta = \omega\left\{\frac{\mu\varepsilon}{2}\left[\left(1+\frac{\sigma^2}{\varepsilon^2\omega^2}\right)^{1/2}+1\right]\right\}^{1/2} \qquad (2)$$

with μ, ε, and σ denoting the magnetic permeability, the electrical permittivity, and the electrical conductivity respectively.

The signal becomes undetected when the amplitude of the radar signal arriving at the object drops below 1/e (≈ 37%) of its initial value. An electromagnetic wave of variable frequency (10 MHz – 1000 MHz) is usually used to allow adjustment of range and resolution of detection.

The performance of GPR depends on its resolution. The resolution is given by the minimum separation between the two adjacent reflectors to be detected. The minimum separation should give rise to a minimum phase difference of 180^o between the two reflected waves at the detector.

Questions:

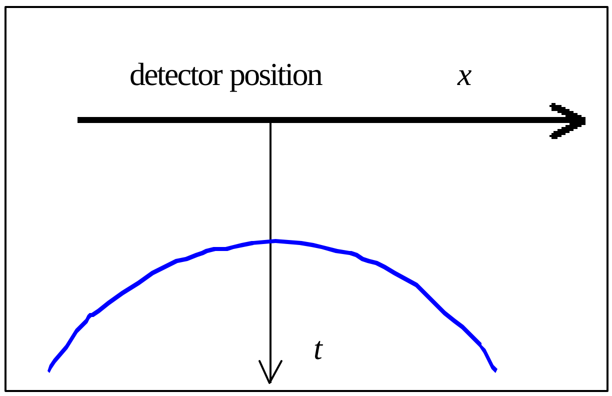
(Given : $\mu_0 = 4\pi \times 10^{-7}$ H/m and $\varepsilon_0 = 8.85 \times 10^{-12}$ F/m)

1. Assume that the ground is non-magnetic ($\mu = \mu_0$) satisfying the condition $\left(\frac{\sigma}{\omega\varepsilon}\right)^2 \langle\langle 1$. Derive the expression of propagation speed v in terms of μ and ε, using equations (1) and (2) **[*1.0 pts*]**.

2. Determine the maximum depth of detection of an object in the ground with conductivity of 1.0 mS/m and permittivity of $9\varepsilon_0$, satisfying the condition $\left(\frac{\sigma}{\omega\varepsilon}\right)^2 \langle\langle 1$, (S=ohm^{-1} ; use $\mu=\mu_0$). **[*2.0 pts*]**

3. Consider two parallel conducting rods buried horizontally in the ground. The rods are 4 meter deep. The ground is known to have conductivity of 1.0 mS/m and permittivity of $9\varepsilon_0$. Suppose the GPR measurement is carried out at a position aproximately above one of the rod. Assume point detector is used. Determine the minimum frequency required to get a lateral resolution of 50 cm **[*3.5 pts*]**.

4. To determine the depth of a buried rod d in the same ground, consider the measurements carried out along a line perpendicular to the rod. The result is described by the following figure:



Graph of traveltime t vs detector position x, t_{min} = 100 ns.

Derive t as a function of x and determine d **[*3.5 pts*]**.

II. Sensing Electrical Signals

Some seawater animals have the ability to detect other creatures at some distance away due to electric currents produced by the creatures during the breathing processes or other processes involving muscular contraction. Some predators use this electrical signal to locate their preys, even when buried under the sands.

The physical mechanism underlying the current generation at the prey and its detection by the predator can be modeled as described by Figure II-1. The current generated by the prey flows between two spheres with positive and negative potential in the prey's body. The distance between the centers of the two spheres is l_s, each having a radius of r_s, which is much smaller than l_s. The seawater resistivity is r. Assume that the resistivity of the prey's body is the same as that of the surrounding seawater, implying that the boundary surrounding the prey in the figure can be ignored.

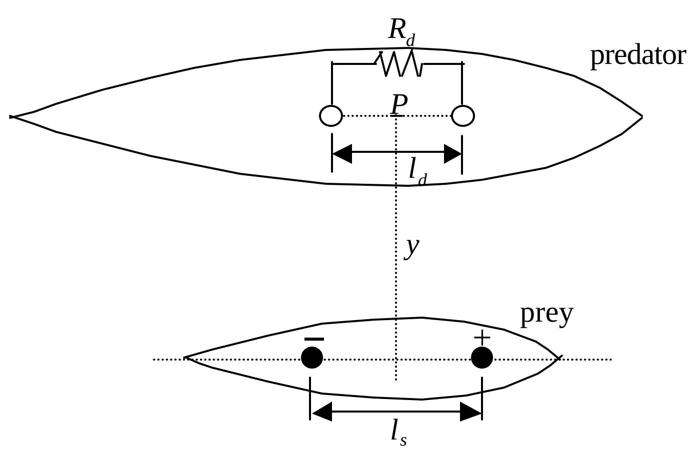


Figure II-1. A model describing the detection of electric power coming from a prey by its predator.

In order to describe the detection of electric power by the predator coming from the prey, the detector is modeled similarly by two spheres on the predator's body and in contact with the surrounding seawater, lying parallel to the pair in the prey's body. They are separated by a distance of l_d, each having a radius of r_d which is much smaller than l_d. In this case, the center of the detector is located at a distance y right above the source and the line connecting the two spheres is parallel to the electric field as shown in Figure II-1. Both l_s and l_d are also much smaller than y. The electric field strength along the line connecting the two spheres is assumed to be constant. Therefore the detector forms a closed circuit system connecting the prey, the surrounding seawater and the predator as described in Figure II-2.

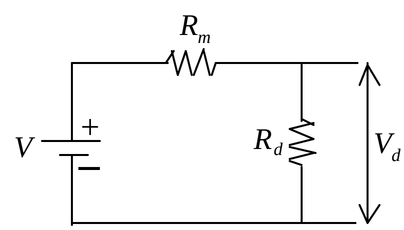


Figure II-2. The equivalent closed circuit system involving the sensing predator, the prey and the surrounding seawater.

In the figure, V is the voltage difference between the detector's spheres due to the electric field induced by the prey, R_m is the inner resistance due to the surrounding sea water. Further, V_d and R_d are respectively the voltage difference between the detecting spheres and the resistance of the detecting element within the predator.

Questions:

1. Determine the current density vector $\vec{j}$ (current per unit area) caused by a point current source I_s at a distance r in an infinite medium. **[*1.5 pts*]**

2. Based on the law $\vec{E} = r\vec{j}$, determine the electric field strength $\vec{E}_p$ at the middle of the detecting spheres (at point P) for a given current I_s that flows between two spheres in the prey's body **[*2.0 pts*]**.

3. Determine for the same current I_s, the voltage difference between the source spheres (V_s) in the prey **[*1.5 pts*]**. Determine the resistance between the two source spheres (R_s) **[*0.5 pts*]** and the power produced by the source (P_s) **[*0.5 pts*]**.

4. Determine R_m **[*0.5 pts*], V_d [*1.0 pts*]** in Figure II-2 and calculate also the power transferred from the source to the detector (P_d) **[*0.5 pts*]**.

5. Determine the optimum value of R_d leading to maximum detected power **[*1.5 pts*]** and determine also the maximum power **[*0.5 pts*]**.

III. A Heavy Vehicle Moving on An Inclined Road

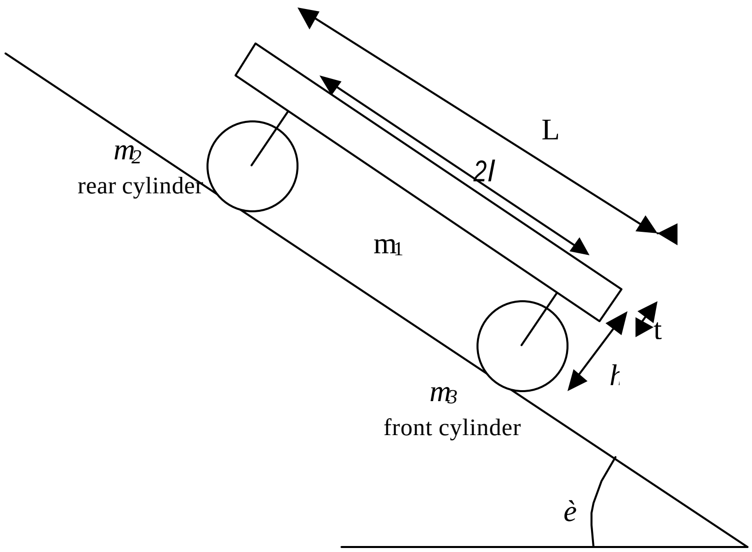


Figure III-1: A simplified model of a heavy vehicle moving on an inclined road.

The above figure is a simplified model of a heavy vehicle (road roller) with one rear and one front cylinder as its wheels on an inclined road with inclination angle of θ as shown in Figure III-1. Each of the two cylinders has a total mass $M(m_2=m_3=M)$ and consists of a cylindrical shell of outer radius R_o , inner radius $R_i = 0.8\ R_o$ and eight number of spokes with total mass 0.2 M. The mass of the undercarriage supporting the vehicle's body is negligible. The cylinder can be modeled as shown in Figure III-2. The vehicle is moving down the road under the influence of gravitational and frictional forces. The front and rear cylinder are positioned symmetrically with respect to the vehicle.

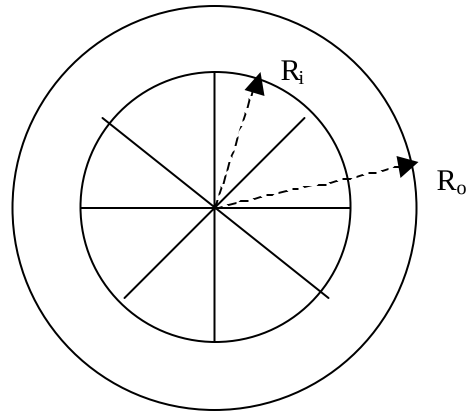


Figure III-2: A simplified model of the cylinders.

The static and kinetic friction coefficients between the cylinder *and* the road are m_s and m_k respectively. The body of the vehicle has a mass of 5M , length of L and thickness of t . The distance between the front and the rear cylinder is $2l$ while the distance from the center of cylinder to the base of the vehicle's body is h. Assume that the rolling friction between the cylinder and its axis is negligible.

Questions:

1. Calculate the moment of inertia of either cylinder **[*1.5 pts*]**.

2. Draw all forces that act on the body, the front cylinder, and the rear one. Write down equations of motion for each part of them **[*2.5 pts*]**.

3. The vehicle is assummed to move from rest, then freely move under gravitational influence. State all the possible types of motion of the system and derive their accelerations in terms of the given physical quantities **[*4.0 pts*]**.

4. Assume that after the vehicle travels a distance d by pure rolling from rest the vehicle enters a section of the road with all the friction coefficients drop to smaller constant values m_s' and m_k' such that the two cylinders start to slide. Calculate the linear and angular velocities of each cylinder after the vehicle has traveled a total distance of s meters. Here we assume that d and s is much larger than the dimension of vehicle **[*2.0 pts*]**

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Tuesday, July 23rd, 2002

Solution I: Ground-Penetrating Radar

1. Speed of radar signal in the material v_m:

$$\omega t - \beta z = \text{ constant } \rightarrow \beta z = \text{-constant } + \omega t \quad \text{(0.2 pts)}$$

$$v_m = \frac{\omega}{\beta}$$

$$v_m = \frac{1}{\omega\left\{\frac{\mu\varepsilon}{2}\left[(1+\frac{\sigma^2}{\varepsilon^2\omega^2})^{1/2}+1\right]\right\}^{1/2}} \quad \text{(0.4 pts)}$$

$$v_m = \frac{1}{\left\{\frac{\mu\varepsilon}{2}(1+1)\right\}^{1/2}} = \frac{1}{\sqrt{\mu\varepsilon}} \quad \text{(0.4 pts)}$$

2. The maximum depth of detection (skin depth, δ) of an object in the ground is inversely proportional to the attenuation constant:

(0.5 pts) **(0.3 pts)** **(0.2 pts)**

$$\delta = \frac{1}{\alpha} = \frac{1}{\omega\left\{\frac{\mu\varepsilon}{2}\left[\left(1+\frac{\sigma^2}{\varepsilon^2\omega^2}\right)^{1/2}-1\right]\right\}^{1/2}} = \frac{1}{\omega\left\{\frac{\mu\varepsilon}{2}\left[\left(1+\frac{1}{2}\frac{\sigma^2}{\varepsilon^2\omega^2}\right)-1\right]\right\}^{1/2}} = \frac{1}{\omega\left\{\frac{\mu\varepsilon}{2}.\frac{1}{2}\frac{\sigma^2}{\varepsilon^2\omega^2}\right\}^{1/2}}$$

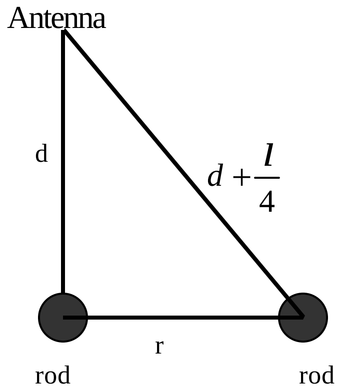
$$\delta = \left(\frac{2}{\sigma}\right)\left(\frac{\varepsilon}{\mu}\right)^{1/2}.$$

Numerically $\delta = \frac{\left(5.31\sqrt{\varepsilon_r}\right)}{\sigma}$ m, where σ is in mS/m. **(0.5 pts)**

For a medium with conductivity of 1.0 mS/m and relative permittivity of 9, the skin depth

$$\delta = \frac{\left(5.31\sqrt{9}\right)}{1.0} = 15.93 \text{ m}$$ **(0.3 pts) + (0.2 pts)**

3. Lateral resolution:



$$r^2+d^2=(d+\frac{\lambda}{4})^2$$

$$r=\left(\frac{\lambda d}{2}+\frac{\lambda^2}{16}\right)^{1/2}$$

(1.0 pts)

r =0.5 m, d =4 m: $\frac{1}{2}=\left(\frac{4\lambda}{2}+\frac{\lambda^2}{16}\right)^{1/2}$, $\lambda^2+32\lambda-4=0$ **(0.5 pts)**

The wavelength is λ=0.125 m. **(0.3 pts) + (0.2 pts)**

The propagation speed of the signal in medium is

$$v_m=\frac{1}{\sqrt{\mu\varepsilon}}=\frac{1}{\sqrt{\mu_o\mu_r\varepsilon_o\varepsilon_r}}=\frac{1}{\sqrt{\mu_o\varepsilon_o}}\frac{1}{\sqrt{\mu_r\varepsilon_r}}$$

$$v_m=\frac{c}{\sqrt{\mu_r\varepsilon_r}}=\frac{0.3}{\sqrt{\varepsilon_r}}\text{ m/ns}\quad,\text{ where } c=\frac{1}{\sqrt{\mu_o\varepsilon_o}}\text{ and } \mu_r=1$$

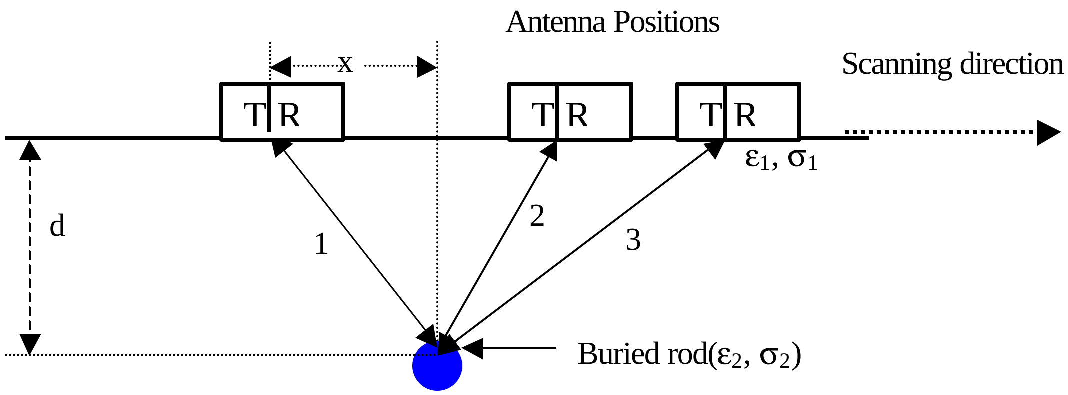
$$v_m=0.1\text{ m/ns}=10^8\text{ m/s}$$ **(0.5 pts)**

The minimum frequency need to distinguish the two rods as two separate objects is

$$f_{\min}=\frac{v}{\lambda}$$ **(0.5 pts)**

$$f_{\min}=\frac{\frac{0.3}{\sqrt{9}}}{0.125}x10^9\text{ Hz}=800\text{ MHz}$$ **(0.3 pts) + (0.20 pts)**

4. Path of EM waves for some positions on the ground surface

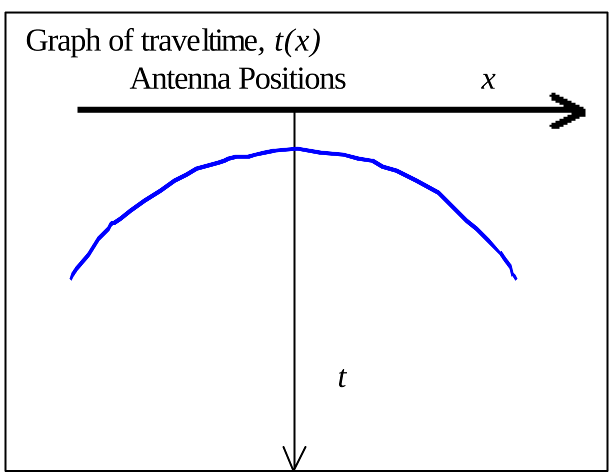


The traveltime as function of x is

$$\left(\frac{t\ v}{2}\right)^2 = d^2 + x^2 ,$$ **(1.0 pts)**

$$t(x) = \sqrt{\frac{4d^2 + 4x^2}{v}}$$ **(1.0 pts)**

$$t(x) = \frac{2\sqrt{e_{1r}}}{0.3}\sqrt{d^2 + x^2}$$



For x =0 **(1.0 pts)**

100 = 2×(3/0.3) d

d = 5 m **(0.5 pts)**

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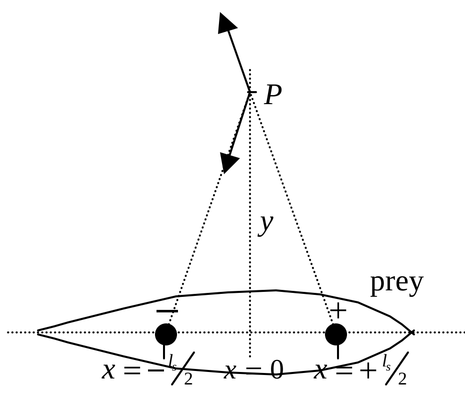
Solution II: Sensing Electrical Signals

1. When a point current source I_s is in infinite isotropic medium, the current density vector at a distance r from the point is

$$\vec{j} = \frac{I_s}{4pr^3}\vec{r}$$

[+1.5 pts] (without vector notation, -0.5 pts)

2.



Assuming that the resistivities of the prey body and that of the surrounding seawater are the same, implying the elimination of the boundary surrounding the prey, the two spheres seem to be in infinite isotropic medium with the resistivity of r. When a small sphere produces current at a rate I_s, the current flux density at a distance r from the sphere's center is also

$$\vec{j} = \frac{I_s}{4pr^3}\vec{r}$$

The seawater resistivity is r, therefore the field strength at r is

$$\vec{E}(\vec{r}) = r\vec{j} = \frac{rI_s}{4pr^3}\vec{r} \qquad \text{[+0.2 pts]}$$

In the model, we have two small spheres. One is at positive voltage relative to the other therefore current I_s flows from the positively charged sphere to the negatively charged sphere. They are separated by l_s. The field strength at P(0,y) is:

$$\vec{E}_p = \vec{E}_+ + \vec{E}_- \qquad \text{[+0.8 pts]}$$

$$= \frac{rI_s}{4p}\left[\frac{1}{\left(\left(\frac{l_s}{2}\right)^2 + y^2\right)^{\frac{3}{2}}}\left(-\frac{l_s}{2}i + yj\right) + \frac{1}{\left(\left(\frac{l_s}{2}\right)^2 + y^2\right)^{\frac{3}{2}}}\left(-\frac{l_s}{2}i - yj\right)\right]$$

$$= \frac{rI_s}{4p}\left[\frac{l_s(-i)}{\left(\left(\frac{l_s}{2}\right)^2 + y^2\right)^{\frac{3}{2}}}\right]$$

$$\vec{E}_p \approx \frac{rI_s l_s}{4py^3}(-i) \quad \text{for} \quad \text{ls} << \text{y} \qquad \text{[+1.0 pts]}$$

3. The field strength along the axis between the two source spheres is:

$$\vec{E}(x) = \frac{rI_s}{4p}\left(\frac{1}{\left(x - \frac{l_s}{2}\right)^2} + \frac{1}{\left(x + \frac{l_s}{2}\right)^2}\right)(-i) \qquad \text{[+0.5 pts]}$$

The voltage difference to produce the given current I_s is

$$V_s = \Delta V = V_+ - V_- = -\int_{\left(-\frac{l_s}{2}+r_s\right)}^{\left(\frac{l_s}{2}-r_s\right)} \vec{E}(x).d\vec{x} = -\frac{rI_s}{4p}\int\left(\frac{1}{\left(x - \frac{l_s}{2}\right)^2} + \frac{1}{\left(x + \frac{l_s}{2}\right)^2}\right)(-i).(idx) \quad \text{[+0.5 pts]}$$

$$= \frac{rI_s}{4p}\left[\frac{1}{-2+1}\left(\frac{1}{\left(\frac{l_s}{2} - r_s - \frac{l_s}{2}\right)} - \frac{1}{\left(-\frac{l_s}{2} + r_s - \frac{l_s}{2}\right)}\right) + \frac{1}{-2+1}\left(\frac{1}{\left(\frac{l_s}{2} - r_s + \frac{l_s}{2}\right)} - \frac{1}{\left(-\frac{l_s}{2} + r_s + \frac{l_s}{2}\right)}\right)\right]$$

$$= \frac{rI_s}{4p}\left(\frac{2}{r_s} - \frac{2}{l_s - r_s}\right) = \frac{2rI_s}{4p}\left(\frac{l_s - r_s - r_s}{(l_s - r_s)r_s}\right) = \frac{rI_s}{2pr_s}\left(\frac{l_s - 2r_s}{l_s - r_s}\right)$$

$$V_s = \Delta V \approx \frac{rI_s}{2pr_s} \quad \text{for} \quad l_s >> r_s. \qquad \text{[+0.5 pts]}$$

The resistance between the two source spheres is:

$$R_s = \frac{V_s}{I_s} = \frac{\rho}{2\pi r_s}$$

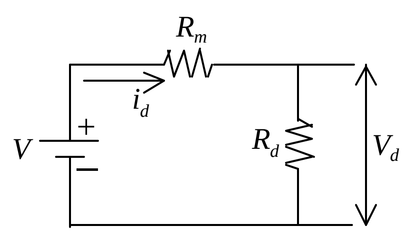
[+0.5 pts]

The power produced by the source is:

$$P = I_s V_s = \frac{\rho I_s^{\,2}}{2\pi r_s}$$

[+0.5 pts]

4.



V is the voltage difference between the detector's spheres due to the electric field induced by the prey, R_m is the inner resistance due to the surrounding sea water. V_d and R_d are respectively the voltage difference between the detecting spheres and the resistance of the detecting element within the predator and i_d is the current flowing in the closed circuit.

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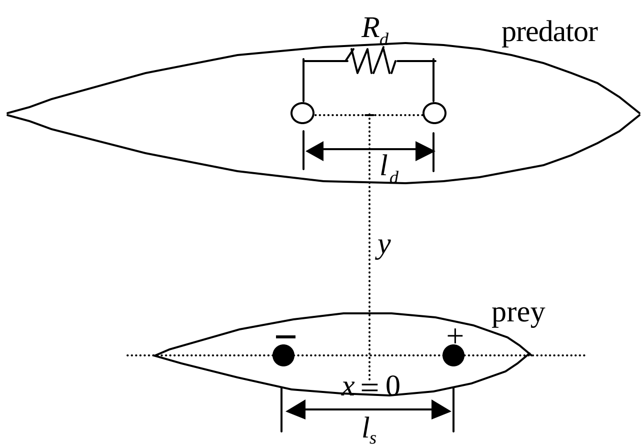


Analog to the resistance between the two source spheres, the resistance of the medium with resistivity ρ between the detector spheres, each having a radius of r_d is:

$$R_m = \frac{\rho}{2\pi r_d}$$

[+0.5 pts]

Since l_d is much smaller than y, the electric field strength between the detector spheres can be assumed to be constant, that is:

$$E = \frac{\rho I_s l_s}{4\pi y^3}$$ [+0.2 pts]

Therefore, the voltage difference present in the medium between the detector spheres is:

$$V = E l_d = \frac{\rho I_s l_s l_d}{4\pi y^3}$$ [+0.3 pts]

The voltage difference across the detector spheres is:

$$V_d = V\frac{R_d}{R_d + R_m} = \frac{rI_s l_s l_d}{4py^3}\frac{R_d}{R_d + \dfrac{r}{2pr_d}}$$

[+0.5 pts]

The power transferred from the source to the detector is:

$$P_d = i_d V_d = \frac{V}{R_d + R_m}V_d = \left(\frac{rI_s l_s l_d}{4py^3}\right)^2 \frac{R_d}{\left(R_d + \dfrac{r}{2pr_d}\right)^2}$$

[+0.5 pts]

5. P_d is maximum when

$$R_t = \frac{R_d}{\left(R_d + \dfrac{r}{2pr_d}\right)^2} = \frac{R_d}{(R_d + R_m)^2} \quad \text{is maximum} \qquad \text{[+0.5 pts]}$$

Therefore,

$$\frac{dR_t}{dR_d} = \frac{1(R_d + R_m)^2 - R_d 2(R_d + R_m)}{(R_d + R_m)^4} = 0 \qquad \text{[+0.5 pts]}$$

$$(R_d + R_m) - 2R_d = 0$$

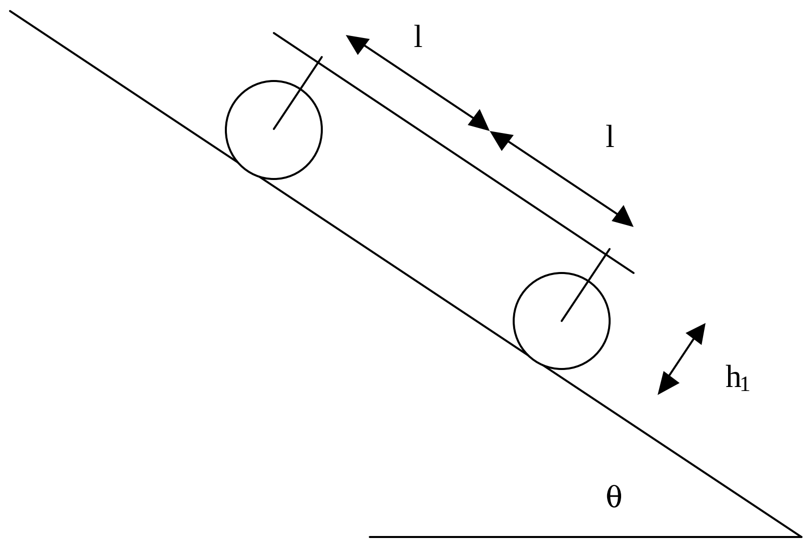
$$R_d^{optimum} = R_m = \frac{r}{2pr_d} \qquad \text{[+0.5 pts]}$$

The maximum power is:

$$P_d^{\max imum} = \left(\frac{rI_s l_s l_d}{4py^3}\right)^2 \frac{pr_d}{2r} = \frac{r(I_s l_s l_d)^2 r_d}{32py^6}$$

[+0.5 pts]

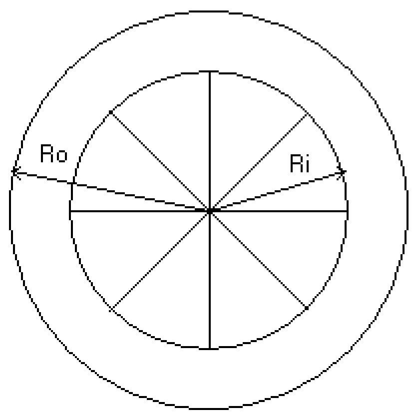
SOLUTION T3 : . **A Heavy Vehicle Moving on An Inclined Road**



To simplify the model we use the above figure with $h_1 = h+0.5\ t$
$R_o = R$

1. **Calculation of the moment inertia of the cylinder**

   $R_i=0.8\ R_o$
   Mass of cylinder part : $m_{cylinder} = 0.8\ M$
   Mass of each rod : $m_{rod} = 0.025\ M$



$$I = \oint\limits_{wholepart} r^2 dm = \oint\limits_{cyl.shell} r^2 dm + \oint\limits_{\mathrm{rod1}} r^2 dm + \ldots + \oint\limits_{rod\,n} r^2 dm \qquad 0.4\ \text{pts}$$

$$\oint\limits_{cyl.shell} r^2 dm = 2\boldsymbol{ps}\int_{Ri}^{Ro} r^3 dr = 0.5\boldsymbol{ps}(R_o^4 - R_i^4) = 0.5 m_{cylinder}(R_o^2 + R_i^2)$$

$$= 0.5(0.8M)R^{\,2}(1 + 0.64) = 0.656MR^{\,2} \qquad 0.5\ \text{pts}$$

$$\oint\limits_{\mathrm{rod}} r^2 dm = \boldsymbol{l}\int_0^{Rin} r^2 dr = \frac{1}{3}\,\boldsymbol{l}R_{in}^3 = \frac{1}{3} m_{rod} R_{in}^2 = \frac{1}{3} 0.025M(0.64R^2) = 0.00533MR^2 \qquad 0.5\ \text{pts}$$

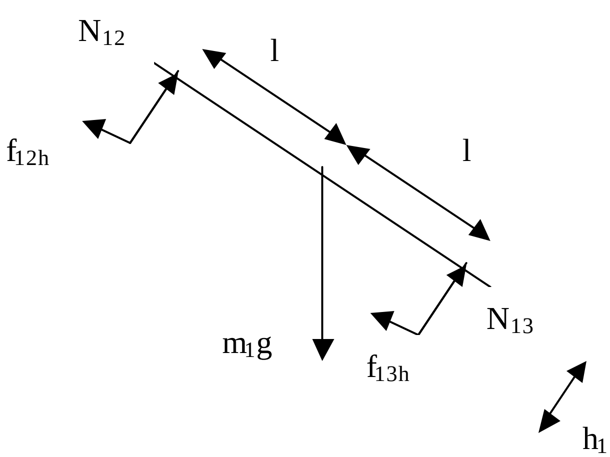
The moment inertia of each wheel becomes

$$I = 0.656MR^2 + 8x0.00533MR^2 = 0.7MR^2 \qquad 0.1\ \text{pts}$$

2. Force diagram and balance equations:

To simplify the analysis we devide the system into three parts: frame (part1) which mainly can be treated as flat homogeneous plate, rear cylinders (two cylinders are treated collectively as part 2 of the system), and front cylinders (two front cylinders are treated collectively as part 3 of the system).

Part 1 : Frame



0.4 pts

The balance equation related to the forces work to this parts are:

Required conditions:
Balance of force in the horizontal axis

$m_1 g \sin \theta - f_{12h} - f_{13h} = m_1 a$ (1) 0.2 pts

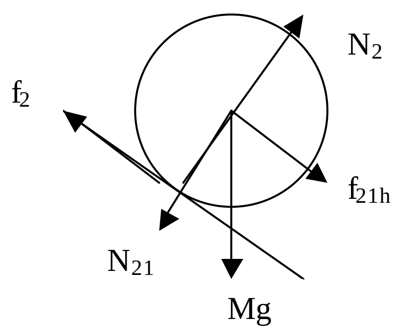
Balance of force in the vertical axis

$m_1 g \cos \theta = N_{12} + N_{13}$ (2) 0.2 pts

Then torsi on against O is zero, so that

$N_{12} l - N_{13} l + f_{12h} h_1 + f_{13h} h_1 = 0$ (3) 0.2 pts

Part two : Rear cylinder



0.25 pts

From balance condition in rear wheel :

$f_{21h} - f_2 + Mg \sin \theta = Ma$ (4) 0.15 pts

$N_2 - N_{21} - Mg \cos \theta = 0$ (5) 0.15 pts

For pure rolling:

$$f_2 R = I\alpha_2 = I \frac{a_2}{R}$$

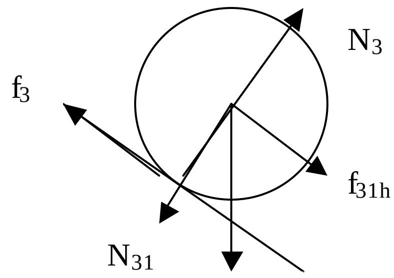
$$\text{or} \quad f_2 = \frac{I}{R^2} a \qquad (6)$$

For rolling with sliding:

$$F_2 = u_k N_2 \qquad (7)$$

0.2 pts

Part Three : Front Cylinder:



0.25 pts

From balance condition in the front whee l :

$f_{31h} - f_3 + Mg\sin\theta = Ma$ (8) 0.15 pts

$N_3 - N_{31} - Mg\cos\theta = 0$ (9) 0.15 pts

For pure rolling:

$$f_3 R = I\alpha_3 = I\frac{a_3}{R}$$

or $f_3 = \frac{I}{R^2}a$ (10)

For rolling with sliding:

$F_3 = u_k N_3$ (11)

0.2 pts

3. From equation (2), (5) and (9) we get

$m_1 g\cos\theta = N_2 - m_2 g\cos\theta + N_3 - m_3 g\cos\theta$

$N_2 + N_3 = (m_1+m_2+m_3)g\cos\theta = 7Mg\cos\theta$ (12)

And from equation (3), (5) and (8) we get

$(N_3 - Mg\cos\theta)l - (N_2 - Mg\cos\theta)l = h_1(f_2 + Ma - Mg\sin\theta + f_3 + Ma - Mg\sin\theta)$

$(N_3 - N_2) = h_1(f_2 + 2Ma - 2Mg\sin\theta + f_3)/l$

Equations 12 and 13 are given **0.25 pts**

CASE ALL CYLINDER IN PURE ROLLING

From equation (4) and (6) we get

f_{21h} = (I/R^2)a +Ma-Mg sinθ (14) 0.2 pts

From equation (8) and (10) we get

f_{31h} = (I/R^2)a +Ma-Mg sinθ (15) 0.2 pts

Then from eq. (1) , (14) and (15) we get

5Mg sinθ - {(I/R^2)a +Ma-Mg sinθ}-{(I/R^2)a +Ma-Mg sinθ}= m_1a

7 Mg sinθ = (2I/R^2 +7M)a

$$a = \frac{7Mg\sin q}{7M + 2\frac{I}{R^2}} = \frac{7Mg\sin q}{7M + 2\frac{0.7MR^2}{R^2}} = 0.833g\sin q \qquad (16) \quad 0.35 \text{ pts}$$

$$N_3 = \frac{7M}{2}g\cos q + \frac{h_1}{l}[(M + \frac{I}{R^2})\times 0.833g\sin q - Mg\sin q]$$

$$= 3.5\mathrm{Mg}\cos q + \frac{h_1}{l}[(M + 0.7M)\times 0.833g\sin q - Mg\sin q]$$

$$= 3.5\ \mathrm{Mg}\cos q + 0.41\frac{h_1}{l}Mg\sin q$$

$$N_2 = \frac{7M}{2}g\cos q - \frac{h_1}{l}[(\frac{I}{R^2} + M)\times 0.833g\sin q - Mg\sin q]$$

$$= 3.5\mathrm{g}\cos q - \frac{h_1}{l}[(0.7M + M)\frac{7Mg\sin q}{0.7M + 7M} - 2Mg\sin q]$$

$$= 3.5\mathrm{g}\cos q - 0.41\frac{h_1}{l}Mg\sin q$$

0.2 pts

The Conditions for pure rolling:

$f_2 \le m_s N_2$ and $f_3 \le m_s N_3$

$\frac{\mathrm{I}_2}{\mathrm{R}_2^2}\mathrm{a} \le m_s N_2$ and $\frac{\mathrm{I}_3}{\mathrm{R}_3^2}\mathrm{a} \le m_s N_3$ 0.2 pts

The left equation becomes

$$0.7M\times 0.833g\sin q \le m_s(3.5\mathrm{Mg}\cos q - 0.41\frac{h_1}{l}Mg\sin q)$$

$$\tan q \le \frac{3.5m_s}{0.5831 + 0.41m_s\frac{h_1}{l}}$$

While the right equation becomes

$$0.7m \times 0.833 g \sin\theta \le \mu_s (3.5 mg\cos\theta + 0.41\frac{h_1}{l} mg\sin\theta)$$

$$\tan\theta \le \frac{3.5\mu_s}{0.5831 - 0.41\mu_s \frac{h_1}{l}}$$

(17) 0.1 pts

CASE ALL CYLINDER SLIDING

From eq. (4) $f_{21h} = Ma + u_k N_2 - Mg\sin\theta$ (18) 0.15 pts

From eq. (8) $f_{31h} = Ma + u_k N_3 - Mg\sin\theta$ (19) 0.15 pts

From eq. (18) and 19 :

$5Mg\sin\theta - (Ma + u_k N_2 - Mg\sin\theta) - (Ma + u_k N_3 - Mg\sin\theta) = m_1 a$

$$a = \frac{7Mg\sin\theta - \mu_k N_2 - \mu_k N_3}{7M} = g\sin\theta - \frac{\mu_k (N_2 + N_3)}{7M} \qquad (20) \qquad 0.2\text{ pts}$$

$$N_3 + N_2 = 7Mg\cos\theta$$

From the above two equations we get :

$$a = g\sin\theta - \mu_k g\cos\theta \qquad 0.25\text{ pts}$$

The Conditions for complete sliding: are the opposite of that of pure rolling

$$f_2 \rangle \mu_s N'_2 \quad \text{and} \quad f_3 \rangle \mu_s N'_3$$

$$\frac{I_2}{R_2^2} a \rangle \mu_s N'_2 \quad \text{and} \quad \frac{I_3}{R_3^2} a \rangle \mu_s N'_3 \qquad (21) \qquad 0.2\text{ pts}$$

Where N_2' and N_3' is calculated in case all cylinder in pure rolling. 0.1 pts

Finally weget

$$\tan\theta \rangle \frac{3.5\mu_s}{0.5831 + 0.41\mu_s \frac{h_1}{l}} \quad \text{and} \quad \tan\theta \rangle \frac{3.5\mu_s}{0.5831 - 0.41\mu_s \frac{h_1}{l}} \qquad 0.2\text{ pts}$$

The left inequality finally become decisive.

CASE ONE CYLINDER IN PURE ROLLING AND ANOTHER IN SLIDING CONDITION

{ For example R_3 (front cylinders) pure rolling while R_2 (Rear cylinders) sliding}

From equation (4) we get

$F_{21h} = m_2 a + u_k N_2 - m_2 g \sin\theta$ (22) 0.15 pts

From equation (5) we get

$f_{31h} = m_3 a + (I/R^2)a - m_3 g \sin\theta$ (23) 0.15 pts

Then from eq. (1) , (22) and (23) we get

$m_1 g \sin\theta - \{ m_2 a + u_k N_2 - m_2 g \sin\theta\} - \{m_3 a + (I/R^2)a - m_3 g \sin\theta\} = m_1 a$

$m_1 g \sin\theta + m_2 g \sin\theta + m_3 \sin\theta - u_k N_2 = (I/R^2 + m_3)a + m_2 a + m_1 a$

$5Mg \sin\theta + Mg \sin\theta + Mg \sin\theta - u_k N_2 = (0.7M + M)a + Ma + 5Ma$

$$a = \frac{7Mg\sin\theta - \mu_k N_2}{7.7M} = 0.9091 g\sin\theta - \frac{\mu_k N_2}{7.7M} \qquad (24) \qquad 0.2 \text{ pts}$$

$$N_3 - N_2 = \frac{h_1}{l}\left(\mu_k N_2 + \frac{I}{R^2}a + 2Ma - 2Mg\sin\theta\right)$$

$$N_3 - N_2 = \frac{h_1}{l}\left(\mu_k N_2 + 2.7M \times 0.9091 g\sin\theta - 2.7\mu_k N_2/7.7 - 2Mg\sin\theta\right)$$

$$N_3 - N_2\left(1 + 0.65\mu_k \frac{h_1}{l}\right) = 0.4546 Mg\sin\theta$$

$$N_3 + N_2 = 7Mg\cos\theta$$

Therefore we get

$$N_2 = \frac{7Mg\cos\theta - 0.4546Mg\sin\theta}{2 + 0.65\mu_k \frac{h_1}{l}}$$

$$N_3 = 7Mg\cos\theta - \frac{7Mg\cos\theta - 0.4546Mg\sin\theta}{2 + 0.65\mu_k \frac{h_1}{l}} \qquad (25) \qquad 0.3 \text{ pts}$$

Then we can substitute the results above into equation (16) to get the following result

$$a = 0.9091 g\sin\theta - \frac{\mu_k N_2}{7.7M} = 0.9091 g\sin\theta - \frac{\mu_k}{7.7}\frac{7g\cos\theta - 0.4546 g\sin\theta}{2 + 0.65\mu_k \frac{h_1}{l}} \qquad (26)$$

0.2 pts

The Conditions for this partial sliding is:

$$f_2 \le m_s N_2' \qquad \text{and} \qquad f_3 \rangle m_s N_3'$$

$$\frac{\mathrm{I}}{\mathrm{R}^2}\mathrm{a} \le m_s N_2' \qquad \text{and} \qquad \frac{\mathrm{I}}{\mathrm{R}^2}\mathrm{a} \rangle m_s N_3' \qquad (27) \qquad 0.25 \text{ pts}$$

where N_2' and N_3' are normal forces for pure rolling condition

4. Assumed that after rolling d meter all cylinder start to sliding until reaching the end of incline road (total distant is s meter). Assummed that ηmeter is reached in t_1 second.

$$v_{t1} = v_o + at_1 = 0 + a_1 t_1 = a_1 t_1$$

$$d = v_o t_1 + \frac{1}{2} a_1 t_1^2 = \frac{1}{2} a_1 t_1^2$$

$$t_1 = \sqrt{\frac{2d}{a_1}}$$

0.5 pts

$$v_{t1} = a_1 \sqrt{\frac{2d}{a_1}} = \sqrt{2da_1} = \sqrt{2d0.833g \sin q} = \sqrt{1.666dg \sin q} \qquad (28)$$

The angular velocity after rolling d meters is same for front and rear cylinders:

$$w_{t1} = \frac{v_{t1}}{R} = \frac{1}{R}\sqrt{1.666dg \sin q} \qquad (29)$$

0.5 pts

Then the vehicle sliding untill the end of declining road. Assumed that the time needed by vehicle to move from d position to the end of the declining road is t_2 second.

$$v_{t2} = v_{t1} + a_2 t_2 = \sqrt{1.666dg \sin q} + a_2 t_2$$

$$s - d = v_{t1} t_2 + \frac{1}{2} a_2 t_2^2$$

$$t_2 = \frac{-v_{t1} + \sqrt{v_{t1}^2 + 2a_2(s-d)}}{a_2} \qquad (30) \qquad 0.4 \text{ pts}$$

$$v_{t2} = \sqrt{1.666dg \sin q} - v_{t1} + \sqrt{v_{t1}^2 + 2a_2(s-d)}$$

Inserting v_{t1} and a_2 from the previous results we get the final results.

For the angular velocity, while sliding they receive torsion:

$$t = m_k NR$$

$$a = \frac{t}{I} = \frac{m_k NR}{I} \qquad (31)$$

$$w_{t2} = w_{t1} + a t_2 = \frac{1}{R}\sqrt{1.666\, dg \sin q} + \frac{m_k NR}{I}\,\frac{-v_{t1} + \sqrt{v_{t1}^2 + 2a_2(s-d)}}{a_2}$$

0.6 pts

Theoretical Question 1

A Swing with a Falling Weight

A rigid cylindrical rod of radius R is held horizontal above the ground. With a string of negligible mass and length L ($L > 2\pi R$), a pendulum bob of mass m is suspended from point A at the top of the rod as shown in Figure 1a. The bob is raised until it is level with A and then released from rest when the string is taut. Neglect any stretching of the string. Assume the pendulum bob may be treated as a mass point and swings only in a plane perpendicular to the axis of the rod. Accordingly, the pendulum bob is also referred to as the *particle*. The acceleration of gravity is $\vec{g}$.

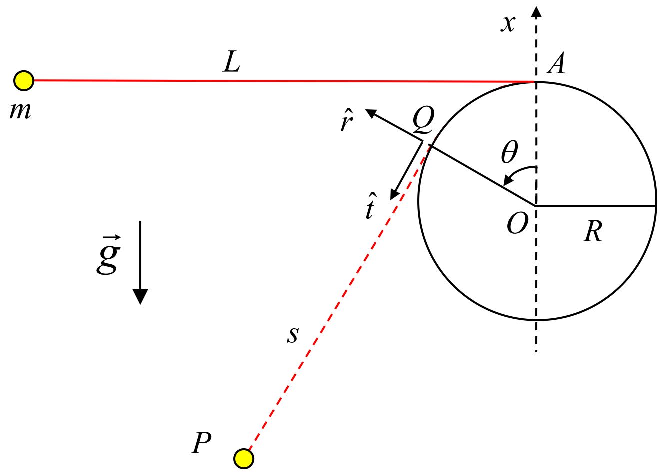


Figure 1a

Let O be the origin of the coordinate system. When the particle is at point P, the string is tangential to the cylindrical surface at Q. The length of the line segment QP is called s. The unit tangent vector and the unit radial vector at Q are given by $\hat{t}$ and $\hat{r}$, respectively. The angular displacement θ of the radius OQ, as measured counterclockwise from the vertical x-axis along OA, is taken to be positive.

When $\theta = 0$, the length s is equal to L and the gravitational potential energy U of the particle is zero. As the particle moves, the instantaneous time rates of change of θ and s are given by $\dot{\theta}$ and $\dot{s}$, respectively.

Unless otherwise stated, all the speeds and velocities are relative to the fixed point O.

Part A

In Part A, the string is taut as the particle moves. In terms of the quantities introduced above (i.e., s, θ, $\dot{s}$, $\dot{\theta}$, R, L, g, $\hat{t}$ and $\hat{r}$), find:

(a) The relation between $\dot{\theta}$ and $\dot{s}$. [0.5 point]

(b) The velocity $\vec{v}_Q$ of the moving point Q relative to O. [0.5 point]

(c) The particle's velocity $\vec{v}\,'$ relative to the moving point Q when it is at P. [0.7 point]

(d) The particle's velocity $\vec{v}$ relative to O when it is at P. [0.7 point]

(e) The $\hat{t}$-component of the particle's *acceleration* relative to O when it is at P. [0.7 point]

(f) The particle's gravitational potential energy U when it is at P. [0.5 point]

(g) The speed v_m of the particle at the lowest point of its trajectory. [0.7 point]

Part B

In Part B, the ratio L to R has the following value:

$$\frac{L}{R} = \frac{9\pi}{8} + \frac{2}{3}\cot\frac{\pi}{16} = 3.534 + 3.352 = 6.886$$

(h) What is the speed v_s of the particle when the string segment from Q to P is both straight and shortest in length? (in terms of g and R) [2.4 points]

(i) What is the speed v_H of the particle at its highest point H when it has swung to the other side of the rod? (in terms of g and R [1.9 points]

Part C

In Part C, instead of being suspended from A, the pendulum bob of mass m is connected by a string over the top of the rod to a heavier weight of mass M, as shown in Figure 1b. The weight can also be treated as a particle.

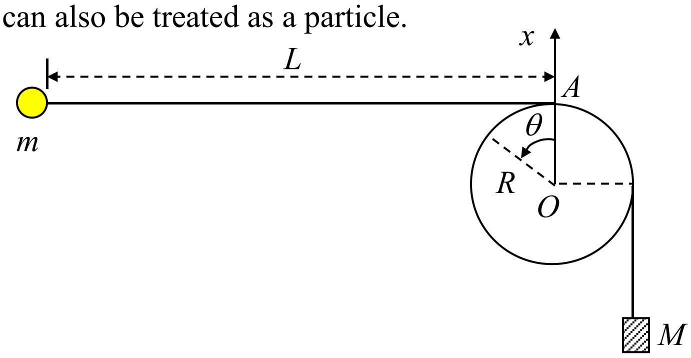


Figure 1b

Initially, the bob is held stationary at the same level as A so that, with the weight hanging below O, the string is taut with a horizontal section of length L. The bob is then released from rest and the weight starts falling. Assume that the bob remains in a vertical plane and can swing past the falling weight without any interruption.

The kinetic friction between the string and the rod surface is negligible. But the static friction is assumed to be large enough so that the weight will remain stationary once it has come to a stop (i.e. zero velocity).

(j) Assume that the weight indeed comes to a stop after falling a distance D and that $(L\text{-}D) >> R$. If the particle can then swing around the rod to $\theta = 2\pi$ while both segments of the string free from the rod remain straight, the ratio $\alpha = D/L$ must not be smaller than a critical value α_c. Neglecting terms of the order R/L or higher, obtain an estimate on α_c in terms of M/m. [3.4 points]

[Answer Sheet] *Theoretical Question 1*

A Swing with a Falling Weight

(a) The relation between $\dot{\theta}$ and $\dot{s}$ is

(b) The velocity of the moving point Q relative to O is

$\vec{v}_Q =$

(c) When at P, the particle's velocity relative to the moving point Q is

$\vec{v}' =$

(d) When at P, the particle's velocity relative to O is

$\vec{v} =$

(e) When at P, the $\hat{t}$ -component of the particle's acceleration relative to O is

(f) When at P, the particle's gravitational potential energy is

$U =$

(g) The particle's speed when at the lowest point of its trajectory is

$v_m =$

(h) When line segment QP is straight with the shortest length, the particle's speed is (Give *expression and value in terms of* g *and* R)

$v_s =$

(i) At the highest point, the particle's speed is (Give *expression and value in terms of* g *and* R)

$v_H =$

(j) In terms of the mass ratio M/m, the critical value α_c of the ratio D/L is

$\alpha_c =$

Theoretical Question 2

A Piezoelectric Crystal Resonator under an Alternating Voltage

Consider a uniform rod of unstressed length ℓ and cross-sectional area A (Figure 2a). Its length changes by $\Delta\ell$ when equal and opposite forces of magnitude F are applied to its ends faces normally. The *stress* T on the end faces is defined to be F/A. The *fractional change* in its length, i.e., $\Delta\ell/\ell$, is called the *strain* S of the rod. In terms of stress and strain, Hooke's law may be expressed as

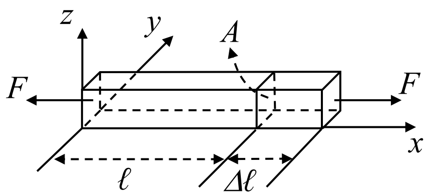
$$T = Y\,S \quad \text{or} \quad \frac{F}{A} = Y\frac{\Delta\ell}{\ell} \tag{1}$$

where Y is called the *Young's modulus* of the rod material. Note that a *compressive* stress T corresponds to $F < 0$ and a decrease in length (i.e., $\Delta\ell < 0$). Such a stress is thus negative in value and is related to the pressure p by $T = -p$.

For a uniform rod of density ρ, the speed of propagation of longitudinal waves (i.e., sound speed) along the rod is given by

$$u = \sqrt{Y/\rho} \tag{2}$$

Figure 2a



The effect of damping and dissipation can be ignored in answering the following questions.

Part A: mechanical properties

A uniform rod of semi-infinite length, extending from $x = 0$ to ∞ (see Figure 2b), has a density ρ. It is initially stationary and unstressed. A piston then steadily exerts a small pressure p on its left face at $x = 0$ for a very short time Δt, causing a pressure wave to propagate with *speed* u to the right.

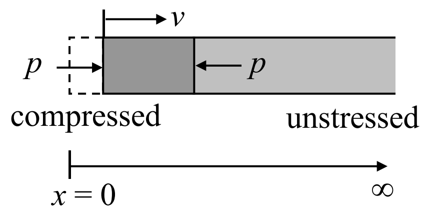


Figure 2b

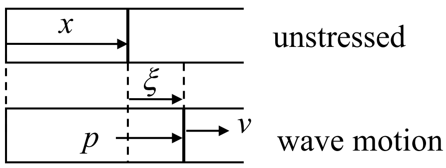


Figure 2c

(a) If the piston causes the rod's left face to move at a constant velocity v (Figure 2b), what are the strain S and pressure p at the left face during the time Δt? *Answers must be given in terms of ρ, u, and v only.* [1.6 points]

(b) Consider a longitudinal wave traveling along the x direction in the rod. For a cross section at x when the rod is unstressed (Figure 2c), let $\xi(x, t)$ be its displacement at time t and assume

$$\xi(x,t) = \xi_0 \sin k(x - ut) \tag{3}$$

where ξ_0 and k are constants. Determine the corresponding velocity $v(x, t)$, strain $S(x, t)$, and pressure $p(x, t)$ as a function of x and t. [2.4 points]

Part B: electromechanical properties (including piezoelectric effect)

Consider a quartz crystal slab of length b, thickness h, and width w (Figure 2d). Its length and thickness are along the x-axis and z-axis. Electrodes are formed by thin metallic coatings at its top and bottom surfaces. Electrical leads that also serve as mounting support (Figure 2e) are soldered to the electrode's centers, which may be assumed to be stationary for longitudinal oscillations along the x direction.

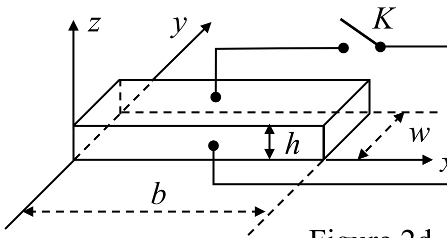


Figure 2d

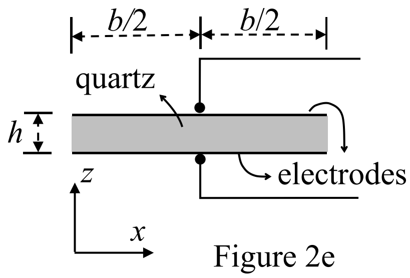


Figure 2e

The quartz crystal under consideration has a density ρ of 2.65×10^3 kg/m^3 and Young's modulus Y of 7.87×10^{10} N/m^2. The length b of the slab is 1.00 cm and the width w and height h of the slab are such that $h \ll w$ and $w \ll b$. With switch K left open, we assume only *longitudinal modes* of standing wave oscillation in the x direction are excited in the quartz slab.

For a standing wave of frequency $f=\omega/2\pi$, the displacement $\xi(x, t)$ at time t of a cross section of the slab with equilibrium position x may be written as

$$\xi(x,t) = 2\xi_0\, g(x)\cos\omega t\,, \qquad (0 \le x \le b) \tag{4a}$$

where ξ_0 is a positive constant and the spatial function $g(x)$ is of the form

$$g(x) = B_1 \sin k(x - \frac{b}{2}) + B_2 \cos k(x - \frac{b}{2}). \tag{4b}$$

g(x) has the maximum value of one and $k=\omega/u$. Keep in mind that the centers of the electrodes are stationary and the left and right faces of the slab are free and must have zero stress (or pressure).

(c) Determine the values of B_1 and B_2 in Eq. (4b) for a longitudinal standing wave in the quartz slab. [1.2 point]

(d) What are the two lowest frequencies at which longitudinal standing waves may be excited in the quartz slab? [1.2 point]

The *piezoelectric* effect is a special property of a *quartz* crystal. Compression or dilatation of the crystal generates an electric voltage across the crystal, and conversely, an external voltage applied across the crystal causes the crystal to expand or contract depending on the polarity of the voltage. Therefore, mechanical and electrical oscillations can be coupled and made to resonate through a quartz crystal.

To account for the piezoelectric effect, let the surface charge densities on the upper and lower electrodes be $-\sigma$ and $+\sigma$, respectively, when the quartz slab is under an electric field E in the z direction. Denote the slab's strain and stress in the x direction by S and T, respectively. Then the piezoelectric effect of the quartz crystal can be described by the following set of equations:

$$S = (1/Y)T + d_p E \tag{5a}$$

$$\sigma = d_p T + \varepsilon_T E \tag{5b}$$

where $1/Y = 1.27\times10^{-11}$ m^2/N is the *elastic compliance* (i.e., inverse of Young's modulus) at constant electric field and $\varepsilon_T = 4.06\times10^{-11}$ F/m is the *permittivity* at constant stress, while $d_p = 2.25\times10^{-12}$ m/V is the *piezoelectric coefficient*.

Let switch K in Fig. 2d be closed. The alternating voltage $V(t) = V_m \cos \omega t$ now acts across the electrodes and a *uniform* electric field $E(t) = V(t)/h$ in the z direction appears in the quartz slab. When a steady state is reached, a longitudinal standing wave oscillation of angular frequency ω appears in the slab in the x direction.

With E being uniform, the wavelength λ and the frequency f of a longitudinal standing wave in the slab are still related by $\lambda = u/f$ with u given by Eq. (2). But, as Eq. (5a) shows, $T = YS$ is no longer valid, although the definitions of strain and stress remain unchanged and the end faces of the slab remain free with zero stress.

(e) Taking Eqs. (5a) and (5b) into account, the surface charge density σ on the lower electrode as a function of x and t is of the form,

$$\sigma(x,t) = \left[D_1 \cos k\left(x - \frac{b}{2}\right) + D_2\right]\frac{V(t)}{h},$$

where $k = \omega/u$. Find the expressions for D_1 and D_2. [2.2 points]

(f) The total surface charge $Q(t)$ on the lower electrode is related to $V(t)$ by

$$Q(t) = [1 + \alpha^2(\frac{2}{kb}\tan\frac{kb}{2} - 1)]C_0 V(t) \tag{6}$$

Find the expression for C_0 and the expression and numerical value of α^2.

[1.4 points]

*[**Answer Sheet**]* *Theoretical Question 2*

A Piezoelectric Crystal Resonator under an Alternating Voltage

***Wherever requested*, give *each answer* as *analytical expressions* followed by *numerical values* and *units*. *For example*:** area of a circle $A = \pi r^2 = 1.23\ \text{m}^2$.

(a) The strain S and pressure p at the left face are (*in terms of* ρ, u, *and* v)

$S =$
$p =$

(b) The velocity $v(x, t)$, strain $S(x, t)$, and pressure $p(x, t)$ are

$v(x, t) =$
$S(x, t) =$
$p(x, t) =$

(c) The values of B_1 and B_2 are

$B_1 =$
$B_2 =$

(d) The lowest two frequencies of standing waves are (*expression and value*)

The Lowest
The Second Lowest

(e) The expressions of D_1 and D_2 are

$D_1 =$
$D_2 =$

(f) The constants α^2 (*expression and value*) and C_0 are (*expression only*)

$\alpha^2 =$
$C_0 =$

Theoretical Question 3

Part A

Neutrino Mass and Neutron Decay

A free neutron of mass m_n decays at rest in the laboratory frame of reference into three non-interacting particles: a proton, an electron, and an anti-neutrino. The rest mass of the proton is m_p, while the rest mass of the anti-neutrino m_ν is assumed to be nonzero and much smaller than the rest mass of the electron m_e. Denote the speed of light in vacuum by c. The measured values of mass are as follows:

$$m_n = 939.56563 \text{ MeV}/c^2,\ m_p = 938.27231 \text{ MeV}/c^2,\ m_e = 0.5109907 \text{ MeV}/c^2$$

In the following, all energies and velocities are referred to the laboratory frame. Let E be the total energy of the electron coming out of the decay.

(a) Find the maximum possible value $E_{\max}$ of E and the speed v_m of the anti-neutrino when $E = E_{\max}$. Both answers must be expressed in terms of the rest masses of the particles and the speed of light. Given that $m_\nu < 7.3 \text{ eV}/c^2$, compute $E_{\max}$ and the ratio v_m/c to 3 significant digits. [4.0 points]

Part B

Light Levitation

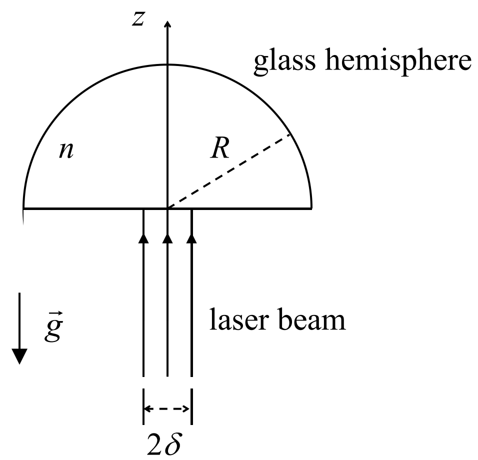
A transparent glass hemisphere with radius R and mass m has an index of refraction n. In the medium outside the hemisphere, the index of refraction is equal to one. A parallel beam of monochromatic laser light is incident uniformly and normally onto the central portion of its planar surface, as shown in Figure 3a. The acceleration of gravity $\vec{g}$ is vertically downwards. The radius δ of the circular cross-section of the laser beam is much smaller than R. Both the glass hemisphere and the laser beam are axially symmetric with respect to the z-axis.

The glass hemisphere does not absorb any laser light. Its surface has been coated with a thin layer of transparent material so that reflections are negligible when light enters and leaves the glass hemisphere. The optical path traversed by laser light passing through the non-reflecting surface layer is also negligible.

(b) Neglecting terms of the order $(\delta/R)^3$ or higher, find the laser power P needed to balance the weight of the glass hemisphere. [4.0 points]

Hint: $\cos\theta \approx 1-\theta^2/2$ when θ is much smaller than one.

Figure 3a



*[**Answer Sheet**]* *Theoretical Question 3*

Wherever requested, **give *each answer* as *analytical expressions* followed by *numerical values* and *units*. *For example*:** area of a circle $A = \pi r^2 = 1.23\ \text{m}^2$.

Neutrino Mass and Neutron Decay

(a) (Give expressions in terms of rest masses of the particles and the speed of light)

The maximum energy of the electron is (*expression and value*)

$E_{\max} =$

The ratio of anti-neutrino's speed at $E = E_{\max}$ to c is (*expression and value*)

$v_m / c =$

Light Levitation

(b) The laser power needed to balance the weight of the glass hemisphere is

$P =$

Solution- Theoretical Question 1

A Swing with a Falling Weight

Part A

(a) Since the length of the string $L = s + R\theta$ is constant, its rate of change must be zero. Hence we have

$$\dot{s} + R\dot{\theta} = 0 \qquad \text{(A1)*}[1]$$

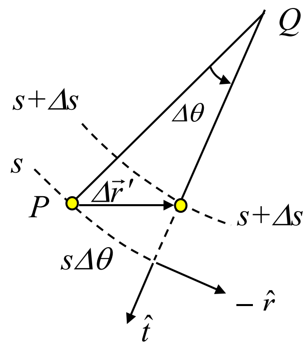
(b) Relative to *O*, *Q* moves on a circle of radius *R* with angular velocity $\dot{\theta}$, so

$$\vec{v}_Q = R\dot{\theta}\,\hat{t} = -\dot{s}\,\hat{t} \qquad \text{(A2)*}$$

(c) Refer to Fig. A1. Relative to *Q*, the displacement of *P* in a time interval Δt is $\Delta\vec{r}' = (s\Delta\theta)(-\hat{r}) + (\Delta s)\hat{t} = [(s\dot{\theta})(-\hat{r}) + \dot{s}\,\hat{t}]\Delta t$. It follows

$$\vec{v}\,' = -s\dot{\theta}\,\hat{r} + \dot{s}\,\hat{t} \qquad \text{(A3)*}$$

Figure A1



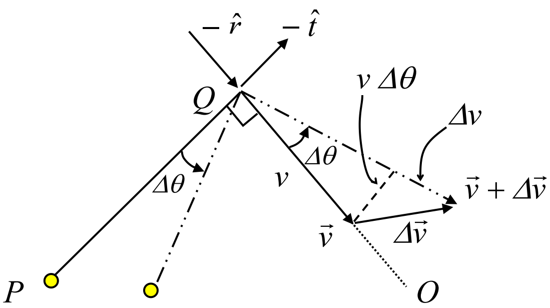
(d) The velocity of the particle relative to *O* is the sum of the two relative velocities given in Eqs. (A2) and (A3) so that

$$\vec{v} = \vec{v}\,' + \vec{v}_Q = (-s\dot{\theta}\,\hat{r} + \dot{s}\,\hat{t}) + R\dot{\theta}\,\hat{t} = -s\dot{\theta}\,\hat{r} \qquad \text{(A4)*}$$

(e) Refer to Fig. A2. The $(-\hat{t})$-component of the velocity change $\Delta\vec{v}$ is given by $(-\hat{t})\cdot\Delta\vec{v} = v\Delta\theta = v\dot{\theta}\,\Delta t$. Therefore, the $\hat{t}$-component of the acceleration $\vec{a} = \Delta\vec{v}/\Delta t$ is given by $\hat{t}\cdot\hat{a} = -v\dot{\theta}$. Since the speed v of the particle is $s\dot{\theta}$ according to Eq. (A4), we see that the $\hat{t}$-component of the particle's acceleration at *P* is given by

$$\vec{a}\cdot\hat{t} = -v\dot{\theta} = -(s\dot{\theta})\dot{\theta} = -s\dot{\theta}^2 \qquad \text{(A5)*}$$

Figure A2



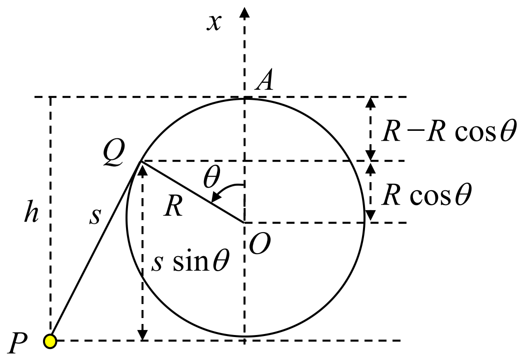
[1] An equation marked with an asterisk contains answer to the problem.

Note that, from Fig. A2, the radial component of the acceleration may also be obtained as $\vec{a}\cdot\hat{r} = -dv/dt = -d(s\dot{\theta})/dt$.

(f) Refer to Fig. A3. The gravitational potential energy of the particle is given by $U = -mgh$. It may be expressed in terms of s and θ as

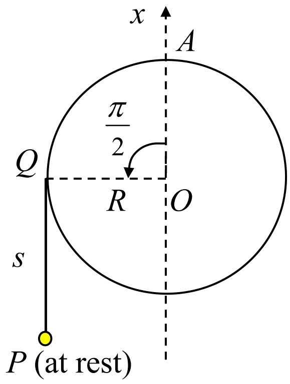
$$U(\theta) = -mg[R(1-\cos\theta) + s\sin\theta] \qquad \text{(A6)*}$$

Figure A3



(g) At the lowest point of its trajectory, the particle's gravitational potential energy U must assume its minimum value U_m. If the particle's mechanical energy E were equal to U_m, its kinetic energy would be zero. The particle would then remain stationary and be in the static equilibrium state shown in Fig. A4. Thus, the potential energy reaches its minimum value when $\theta = \pi/2$ or $s = L - \pi R/2$.

Figure A4



From Fig. A4 or Eq. (A6), the minimum potential energy is then

$$U_m = U(\frac{\pi}{2}) = -mg[R + L - (\pi R/2)]. \qquad \text{(A7)}$$

Initially, the total mechanical energy E is 0. Since E is conserved, the speed v_m of the particle at the lowest point of its trajectory must satisfy

$$E = 0 = \frac{1}{2}mv_m^2 + U_m. \qquad \text{(A8)}$$

From Eqs. (A7) and (A8), we obtain

$$v_m = \sqrt{-2U_m/m} = \sqrt{2g[R + (L - \pi R/2)]}. \qquad \text{(A9)*}$$

Part B

(h) From Eq. (A6), the total mechanical energy of the particle may be written as

$$E = 0 = \frac{1}{2}mv^2 + U(\theta) = \frac{1}{2}mv^2 - mg[R(1-\cos\theta) + s\sin\theta] \qquad \text{(B1)}$$

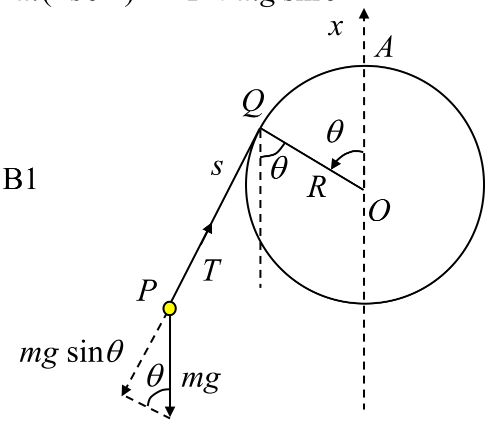
From Eq. (A4), the speed v is equal to $s\dot{\theta}$. Therefore, Eq. (B1) implies

$$v^2 = (s\dot{\theta})^2 = 2g[R(1-\cos\theta) + s\sin\theta] \qquad \text{(B2)}$$

Let T be the tension in the string. Then, as Fig. B1 shows, the $\hat{t}$ -component of the net force on the particle is $-T + mg\sin\theta$. From Eq. (A5), the tangential acceleration of the particle is $(-s\dot{\theta}^2)$. Thus, by Newton's second law, we have

$$m(-s\dot{\theta}^2) = -T + mg\sin\theta \qquad \text{(B3)}$$

Figure B1

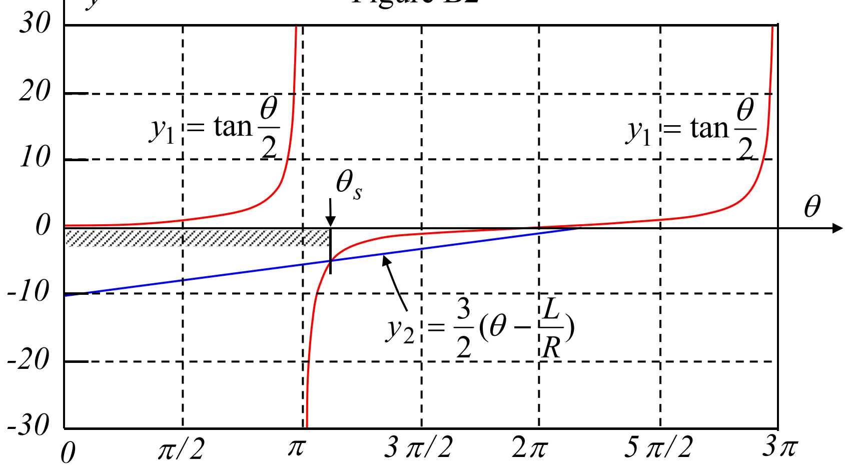


According to the last two equations, the tension may be expressed as

$$\begin{aligned} T &= m(s\dot{\theta}^2 + g\sin\theta) = \frac{mg}{s}[2R(1-\cos\theta) + 3s\sin\theta] \\ &= \frac{2mgR}{s}[\tan\frac{\theta}{2} - \frac{3}{2}(\theta - \frac{L}{R})](\sin\theta) \qquad \text{(B4)} \\ &= \frac{2mgR}{s}(y_1 - y_2)(\sin\theta) \end{aligned}$$

The functions $y_1 = \tan(\theta/2)$ and $y_2 = 3(\theta - L/R)/2$ are plotted in Fig B2.

Figure B2



From Eq. (B4) and Fig. B2, we obtain the result shown in Table B1. The angle at which .$y_2 = y_1$ is called θ_s ($\pi < \theta_s < 2\pi$) and is given by

$$\frac{3}{2}(\theta_s - \frac{L}{R}) = \tan\frac{\theta_s}{2} \tag{B5}$$

or, equivalently, by

$$\frac{L}{R} = \theta_s - \frac{2}{3}\tan\frac{\theta_s}{2} \tag{B6}$$

Since the ratio L/R is known to be given by

$$\frac{L}{R} = \frac{9\pi}{8} + \frac{2}{3}\cot\frac{\pi}{16} = (\pi + \frac{\pi}{8}) - \frac{2}{3}\tan\frac{1}{2}(\pi + \frac{\pi}{8}) \tag{B7}$$

one can readily see from the last two equations that $\theta_s = 9\pi/8$.

Table B1

	$(y_1 - y_2)$	$\sin\theta$	tension T
$0 < \theta < \pi$	positive	positive	positive
$\theta = \pi$	$+\infty$	0	positive
$\pi < \theta < \theta_s$	negative	negative	positive
$\theta = \theta_s$	zero	negative	zero
$\theta_s < \theta < 2\pi$	positive	negative	negative

Table B1 shows that the tension T must be positive (or the string must be taut and straight) in the angular range $0<\theta < \theta_s$. Once θ reaches θ_s, the tension T becomes zero and the part of the string not in contact with the rod will not be straight afterwards. The shortest possible value $s_{\min}$ for the length s of the line segment QP therefore occurs at $\theta = \theta_s$ and is given by

$$s_{\min} = L - R\theta_s = R(\frac{9\pi}{8} + \frac{2}{3}\cot\frac{\pi}{16} - \frac{9\pi}{8}) = \frac{2R}{3}\cot\frac{\pi}{16} = 3.352R \tag{B8}$$

When $\theta = \theta_s$, we have $T = 0$ and Eqs. (B2) and (B3) then leads to $v^2 = -gs\sin\theta$. Hence the speed v_s is

$$v_s = \sqrt{-g s_{\min} \sin\theta_s} = \sqrt{\frac{2gR}{3}\cot\frac{\pi}{16}\sin\frac{\pi}{8}} = \sqrt{\frac{4gR}{3}\cos\frac{\pi}{16}} \qquad \text{(B9)*}$$
$$= 1.133\sqrt{gR}$$

(i) When $\theta \geq \theta_s$, the particle moves like a projectile under gravity. As shown in Fig. B3, it is projected with an initial speed v_s from the position $P = (x_s, y_s)$ in a direction making an angle $\phi = (3\pi/2 - \theta_s)$ with the y-axis.

The speed v_H of the particle at the highest point of its parabolic trajectory is equal to the y-component of its initial velocity when projected. Thus,

$$v_H = v_s \sin(\theta_s - \pi) = \sqrt{\frac{4gR}{3}}\cos\frac{\pi}{16}\sin\frac{\pi}{8} = 0.4334\sqrt{gR} \qquad \text{(B10)*}$$

The horizontal distance H traveled by the particle from point P to the point of maximum height is

$$H = \frac{v_s^2 \sin 2(\theta_s - \pi)}{2g} = \frac{v_s^2}{2g}\sin\frac{9\pi}{4} = 0.4535R \qquad \text{(B11)}$$

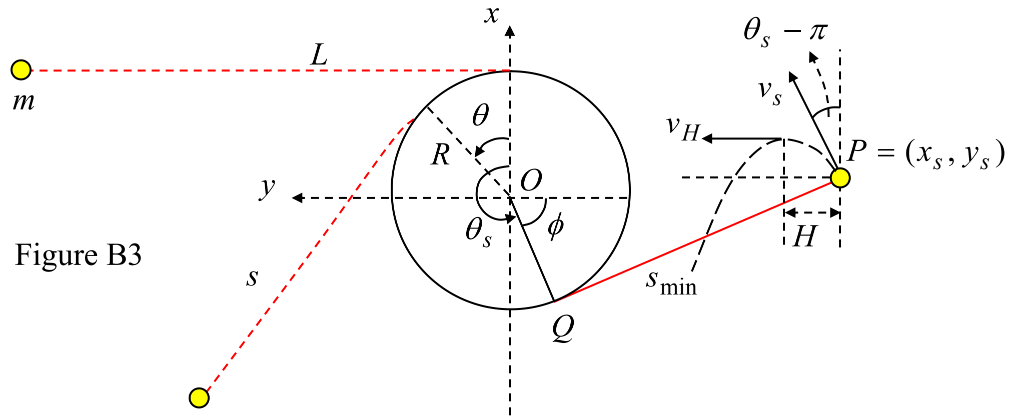


Figure B3

The coordinates of the particle when $\theta = \theta_s$ are given by

$$x_s = R\cos\theta_s - s_{\min}\sin\theta_s = -R\cos\frac{\pi}{8} + s_{\min}\sin\frac{\pi}{8} = 0.358R \qquad \text{(B12)}$$

$$y_s = R\sin\theta_s + s_{\min}\cos\theta_s = -R\sin\frac{\pi}{8} - s_{\min}\cos\frac{\pi}{8} = -3.478R \qquad \text{(B13)}$$

Evidently, we have $|y_s| > (R + H)$. Therefore the particle can indeed reach its maximum height without striking the surface of the rod.

Part C

(j) Assume the weight is initially lower than O by h as shown in Fig. C1.

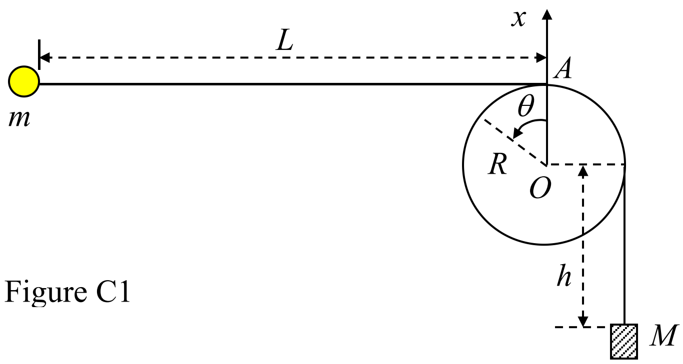


Figure C1

When the weight has fallen a distance D and stopped, the law of conservation of total mechanical energy as applied to the particle-weight pair as a system leads to

$$-Mgh = E' - Mg(h+D) \tag{C1}$$

where E' is the *total mechanical energy of the particle* when the weight has stopped. It follows

$$E' = MgD \tag{C2}$$

Let Λ be the total length of the string. Then, its value at $\theta = 0$ must be the same as at any other angular displacement θ. Thus we must have

$$\Lambda = L + \frac{\pi}{2}R + h = s + R(\theta + \frac{\pi}{2}) + (h+D) \tag{C3}$$

Noting that $D = \alpha L$ and introducing $\ell = L - D$, we may write

$$\ell = L - D = (1-\alpha)L \tag{C4}$$

From the last two equations, we obtain

$$s = L - D - R\theta = \ell - R\theta \tag{C5}$$

After the weight has stopped, the total mechanical energy of the particle must be conserved. According to Eq. (C2), we now have, instead of Eq. (B1), the following equation:

$$E' = MgD = \frac{1}{2}mv^2 - mg[R(1-\cos\theta) + s\sin\theta] \tag{C6}$$

The square of the particle's speed is accordingly given by

$$v^2 = (s\dot{\theta})^2 = \frac{2MgD}{m} + 2gR[(1-\cos\theta) + \frac{s}{R}\sin\theta] \quad . \tag{C7}$$

Since Eq. (B3) stills applies, the tension T of the string is given by

$$-T + mg\sin\theta = m(-s\dot{\theta}^2) \tag{C8}$$

From the last two equations, it follows

$$\begin{aligned} T &= m(s\dot{\theta}^2 + g\sin\theta) \\ &= \frac{mg}{s}[\frac{2M}{m}D + 2R(1-\cos\theta) + 3s\sin\theta] \\ &= \frac{2mgR}{s}[\frac{MD}{mR} + (1-\cos\theta) + \frac{3}{2}(\frac{\ell}{R} - \theta)\sin\theta] \end{aligned} \tag{C9}$$

where Eq. (C5) has been used to obtain the last equality.

We now introduce the function

$$f(\theta) = 1 - \cos\theta + \frac{3}{2}(\frac{\ell}{R} - \theta)\sin\theta \tag{C10}$$

From the fact $\ell = (L-D) >> R$, we may write

$$f(\theta) \approx 1 + \frac{3}{2}\frac{\ell}{R}\sin\theta - \cos\theta = 1 + A\sin(\theta - \phi) \tag{C11}$$

where we have introduced

$$A = \sqrt{1 + (\frac{3}{2}\frac{\ell}{R})^2}\ , \qquad \phi = \tan^{-1}\frac{\frac{3\ell}{2R}}{\sqrt{1 + (\frac{3\ell}{2R})^2}} \tag{C12}$$

From Eq. (C11), the minimum value of $f(\theta)$ is seen to be given by

$$f_{\min} = 1 - A = 1 - \sqrt{1 + (\frac{3}{2}\frac{\ell}{R})^2} \tag{C13}$$

Since the tension T remains nonnegative as the particle swings around the rod, we have from Eq. (C9) the inequality

$$\frac{MD}{mR} + f_{\min} = \frac{M(L-\ell)}{mR} + 1 - \sqrt{1 + (\frac{3\ell}{2R})^2} \geq 0 \tag{C14}$$

or

$$(\frac{ML}{mR}) + 1 \ \geq \ (\frac{M\ell}{mR}) + \sqrt{1 + (\frac{3\ell}{2R})^2} \approx (\frac{M\ell}{mR}) + (\frac{3\ell}{2R}) \tag{C15}$$

From Eq. (C4), Eq. (C15) may be written as

$$(\frac{ML}{mR}) + 1 \ \geq \ [(\frac{ML}{mR}) + (\frac{3L}{2R})](1 - \alpha) \tag{C16}$$

Neglecting terms of the order (R/L) or higher, the last inequality leads to

$$\alpha \ \geq \ 1 - \frac{(\frac{ML}{mR}) + 1}{(\frac{ML}{mR}) + (\frac{3L}{2R})} = \frac{(\frac{3L}{2R}) - 1}{(\frac{ML}{mR}) + (\frac{3L}{2R})} = \frac{1 - \frac{2R}{3L}}{\frac{2M}{3m} + 1} \approx \frac{1}{1 + \frac{2M}{3m}} \tag{C17}$$

The critical value for the ratio D/L is therefore

$$\alpha_c = \frac{1}{(1 + \frac{2M}{3m})} \qquad \text{(C18)*}$$

Solution- Theoretical Question 2
A Piezoelectric Crystal Resonator under an Alternating Voltage

Part A

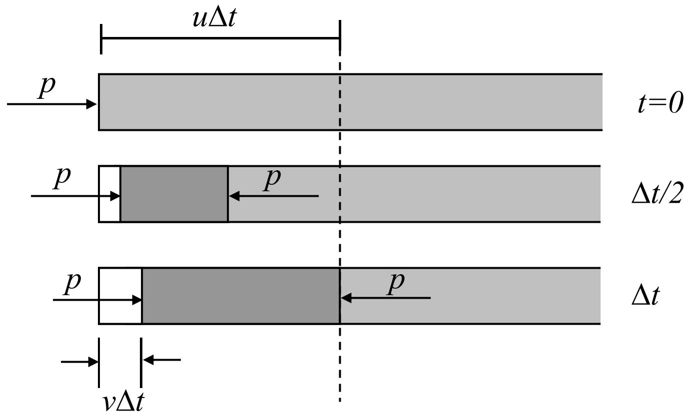
(a) Refer to Figure A1. The left face of the rod moves a distance $v\Delta t$ while the pressure wave travels a distance $u\Delta t$ with $u = \sqrt{Y/\rho}$. The strain at the left face is

$$S = \frac{\Delta \ell}{\ell} = \frac{-v\Delta t}{u\Delta t} = \frac{-v}{u} \qquad \text{(A1a)*[1]}$$

From Hooke's law, the pressure at the left face is

$$p = -YS = Y\frac{v}{u} = \rho uv \qquad \text{(A1b)*}$$

Figure A1



(b) The velocity v is related to the displacement ξ as in a simple harmonic motion (or a uniform circular motion, as shown in Figure A2) of angular frequency $\omega = ku$. Therefore, if $\xi(x,t) = \xi_0 \sin k(x-ut)$, then

$$v(x,t) = -ku\xi_0 \cos k(x-ut) . \qquad \text{(A2)*}$$

The strain and pressure are related to velocity as in Problem (a). Hence,

$$S(x,t) = -v(x,t)/u = k\xi_0 \cos k(x-ut) \qquad \text{(A3)*}$$

$$\begin{aligned} p(x,t) &= \rho uv(x,t) = -k\rho u^2 \xi_0 \cos k(x-ut) \\ &= -YS(x,t) = -kY\xi_0 \cos k(x-ut) \end{aligned} \qquad \text{(A4)*}$$

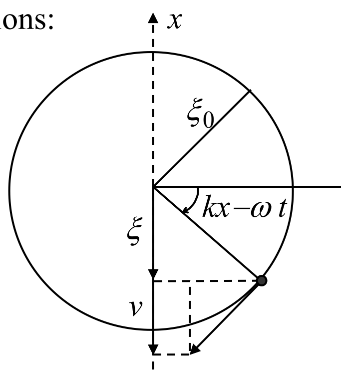
Alternatively, the answers may be obtained by differentiations:

$$v(x,t) = \frac{\Delta\xi}{\Delta t} = -ku\xi_0 \cos k(x-ut) ,$$

$$S(x,t) = \frac{\Delta\xi}{\Delta x} = k\xi_0 \cos k(x-ut) ,$$

$$p(x,t) = -Y\frac{\Delta\xi}{\Delta x} = -kY\xi_0 \cos k(x-ut) .$$

Figure A2



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[1] An equations marked with an asterisk contains answer to the problem.

Part B

(c) Since the angular frequency ω and speed of propagation u are given, the wavelength is given by $\lambda = 2\pi / k$ with $k = \omega / u$. The spatial variation of the displacement ξ is therefore described by

$$g(x) = B_1 \sin k(x - \frac{b}{2}) + B_2 \cos k(x - \frac{b}{2}) \qquad \text{(B1)}$$

Since the centers of the electrodes are assumed to be stationary, $g(b/2) = 0$. This leads to $B_2 = 0$. Given that the maximum of $g(x)$ is 1, we have $A = \pm 1$ and

$$g(x) = \pm \sin \frac{\omega}{u}(x - \frac{b}{2}) \qquad \text{(B2)*}$$

Thus, the displacement is

$$\xi(x,t) = \pm 2\xi_0 \sin \frac{\omega}{u}(x - \frac{b}{2}) \cos \omega t \qquad \text{(B3)}$$

(d) Since the pressure p (or stress T) must vanish at the end faces of the quartz slab (i.e., $x = 0$ and $x = b$), the answer to this problem can be obtained, by analogy, from the resonant frequencies of sound waves in an open pipe of length b. However, given that the centers of the electrodes are stationary, all even harmonics of the fundamental tone must be excluded because they have antinodes, rather than nodes, of displacement at the bisection plane of the slab.

Since the fundamental tone has a wavelength $\lambda = 2b$, the fundamental frequency is given by $f_1 = u/(2b)$. The speed of propagation u is given by

$$u = \sqrt{\frac{Y}{\rho}} = \sqrt{\frac{7.87 \times 10^{10}}{2.65 \times 10^3}} = 5.45 \times 10^3 \text{ m/s} \qquad \text{(B4)}$$

and, given that $b = 1.00 \times 10^{-2}$ m, the two lowest standing wave frequencies are

$$f_1 = \frac{u}{2b} = 273 \text{ (kHz)}, \quad f_3 = 3f_1 = \frac{3u}{2b} = 818 \text{ (kHz)} \qquad \text{(B5)*}$$

[*Alternative solution to Problems (c) and (d)*]:

A longitudinal standing wave in the quartz slab has a displacement node at $x = b/2$. It may be regarded as consisting of two waves traveling in opposite directions. Thus, its displacement and velocity must have the following form

$$\begin{aligned} \xi(x,t) &= 2\xi_m \sin k(x - \frac{b}{2}) \cos \omega t \\ &= \xi_m [\sin k(x - \frac{b}{2} - ut) + \sin k(x - \frac{b}{2} + ut)] \end{aligned} \qquad \text{(B6)}$$

$$\begin{aligned} v(x,t) &= -ku\xi_m [\cos k(x - \frac{b}{2} - ut) - \cos k(x - \frac{b}{2} + ut)] \\ &= -2\omega \xi_m \sin k(x - \frac{b}{2}) \sin \omega t \end{aligned} \qquad \text{(B7)}$$

where $\omega = ku$ and the first and second factors in the square brackets represent waves traveling along the $+x$ and $-x$ directions, respectively. Note that Eq. (B6) is identical to Eq. (B3) if we set $\xi_m = \pm \xi_0$.

For a wave traveling along the $-x$ direction, the velocity v must be replaced by $-v$ in Eqs. (A1a) and (A1b) so that we have

$$S = \frac{-v}{u} \quad \text{and} \quad p = \rho u v \qquad \text{(waves traveling along } +x) \tag{B8}$$

$$S = \frac{v}{u} \quad \text{and} \quad p = -\rho u v \qquad \text{(waves traveling along } -x) \tag{B9}$$

As in Problem (b), the strain and pressure are therefore given by

$$\begin{aligned} S(x,t) &= -k\xi_m[-\cos k(x-\frac{b}{2}-ut) - \cos k(x-\frac{b}{2}+ut)] \\ &= 2k\xi_m \cos k(x-\frac{b}{2})\cos\omega t \end{aligned} \tag{B10}$$

$$\begin{aligned} p(x,t) &= -\rho u\omega\xi_m[\cos k(x-\frac{b}{2}-ut) + \cos k(x-\frac{b}{2}+ut)] \\ &= -2\rho u\omega\xi_m \cos k(x-\frac{b}{2})\cos\omega t \end{aligned} \tag{B11}$$

Note that v, S, and p may also be obtained by differentiating ξ as in Problem (b).

The stress T or pressure p must be zero at both ends ($x = 0$ and $x = b$) of the slab at all times because they are free. From Eq. (B11), this is possible only if $\cos(kb/2) = 0$ or

$$kb = \frac{\omega}{u}b = \frac{2\pi f}{\lambda f}b = n\pi, \qquad n = 1, 3, 5, \cdots \tag{B12}$$

In terms of wavelength λ, Eq. (B12) may be written as

$$\lambda = \frac{2b}{n}, \qquad n = 1, 3, 5, \cdots. \tag{B13}$$

The frequency is given by

$$f = \frac{u}{\lambda} = \frac{nu}{2b} = \frac{n}{2b}\sqrt{\frac{Y}{\rho}}, \qquad n = 1, 3, 5, \cdots. \tag{B14}$$

This is identical with the results given in Eqs. (B4) and (B5).

(e) From Eqs. (5a) and (5b) in the Question, the piezoelectric effect leads to the equations

$$T = Y(S - d_p E) \tag{B15}$$

$$\sigma = Yd_p S + \varepsilon_T(1 - Y\frac{d_p^2}{\varepsilon_T})E \tag{B16}$$

Because $x = b/2$ must be a node of displacement for any longitudinal standing wave in the slab, the displacement ξ and strain S must have the form given in Eqs. (B6) and (B10), i.e., with $\omega = ku$,

$$\xi(x,t) = \xi_m \sin k(x-\frac{b}{2})\cos(\omega t + \phi) \tag{B17}$$

$$S(x,t) = k\xi_m \cos k(x-\frac{b}{2})\cos(\omega t + \phi) \tag{B18}$$

where a phase constant ϕ is now included in the time-dependent factors.

By assumption, the electric field E between the electrodes is uniform and depends only on time:

$$E(x,t) = \frac{V(t)}{h} = \frac{V_m \cos\omega t}{h}. \tag{B19}$$

Substituting Eqs. (B18) and (B19) into Eq. (B15), we have

$$T = Y[k\xi_m \cos k(x - \frac{b}{2})\cos(\omega t + \phi) - \frac{d_p}{h}V_m \cos\omega t] \tag{B20}$$

The stress T must be zero at both ends ($x = 0$ and $x = b$) of the slab at all times because they are free. This is possible only if $\phi = 0$ and

$$k\xi_m \cos\frac{kb}{2} = d_p \frac{V_m}{h} \tag{B21}$$

Since $\phi = 0$, Eqs. (B16), (B18), and (B19) imply that the surface charge density must have the same dependence on time t and may be expressed as

$$\sigma(x,t) = \sigma(x)\cos\omega t \tag{B22}$$

with the dependence on x given by

$$\begin{aligned}\sigma(x) &= Yd_p k\xi_m \cos k(x - \frac{b}{2}) + \varepsilon_T(1 - Y\frac{d_p^2}{\varepsilon_T})\frac{V_m}{h} \\ &= [Y\frac{d_p^2}{\cos\frac{kb}{2}}\cos k(x - \frac{b}{2}) + \varepsilon_T(1 - Y\frac{d_p^2}{\varepsilon_T})]\frac{V_m}{h}\end{aligned} \tag{B23)*}$$

(f) At time t, the total surface charge $Q(t)$ on the lower electrode is obtained by integrating $\sigma(x,t)$ in Eq. (B22) over the surface of the electrode. The result is

$$\begin{aligned}\frac{Q(t)}{V(t)} &= \frac{1}{V(t)}\int_0^b \sigma(x,t)\,w dx = \frac{1}{V_m}\int_0^b \sigma(x)\,w dx \\ &= \frac{w}{h}\int_0^b [Y\frac{d_p^2}{\cos\frac{kb}{2}}\cos k(x - \frac{b}{2}) + \varepsilon_T(1 - Y\frac{d_p^2}{\varepsilon_T})]dx \\ &= (\varepsilon_T\frac{bw}{h})[Y\frac{d_p^2}{\varepsilon_T}(\frac{2}{kb}\tan\frac{kb}{2}) + (1 - Y\frac{d_p^2}{\varepsilon_T})] \\ &= C_0[\alpha^2(\frac{2}{kb}\tan\frac{kb}{2}) + (1 - \alpha^2)]\end{aligned} \tag{B24}$$

where

$$C_0 = \varepsilon_T\frac{bw}{h}, \qquad \alpha^2 = Y\frac{d_p^2}{\varepsilon_T} = \frac{(2.25)^2 \times 10^{-2}}{1.27 \times 4.06} = 9.82 \times 10^{-3} \tag{B25)*}$$

(The constant α is called the *electromechanical coupling coefficient*.)

Note: The result $C_0 = \varepsilon_T bw/h$ can readily be seen by considering the static limit $k = 0$ of Eq. (5) in the Question. Since $\tan x \approx x$ when $x << 1$, we have

$$\lim_{k\to 0} Q(t)/V(t) \approx C_0[\alpha^2 + (1 - \alpha^2)] = C_0 \tag{B26}$$

Evidently, the constant C_0 is the capacitance of the parallel-plate capacitor formed by the electrodes (of area bw) with the quartz slab (of thickness h and permittivity

ε_T) serving as the dielectric medium. It is therefore given by $\varepsilon_T bw/h$.

(B47)*

Solution- Theoretical Question 3

Part A

Neutrino Mass and Neutron Decay

(a) Let $(c^2 E_e, c\vec{q}_e)$, $(c^2 E_p, c\vec{q}_p)$, and $(c^2 E_\nu, c\vec{q}_\nu)$ be the energy-momentum 4-vectors of the electron, the proton, and the anti-neutrino, respectively, in the rest frame of the neutron. Notice that $E_e, E_p, E_\nu, \vec{q}_e, \vec{q}_p, \vec{q}_\nu$ are all in units of mass.

The proton and the anti-neutrino may be considered as forming a system of total rest mass M_c, total energy $c^2 E_c$, and total momentum $c\vec{q}_c$. Thus, we have

$$E_c = E_p + E_\nu, \qquad \vec{q}_c = \vec{q}_p + \vec{q}_\nu, \qquad M_c^2 = E_c^2 - q_c^2 \tag{A1}$$

Note that the magnitude of the vector $\vec{q}_c$ is denoted as q_c. The same convention also applies to all other vectors.

Since energy and momentum are conserved in the neutron decay, we have

$$E_c + E_e = m_n \tag{A2}$$

$$\vec{q}_c = -\vec{q}_e \tag{A3}$$

When squared, the last equation leads to the following equality

$$q_c^2 = q_e^2 = E_e^2 - m_e^2 \tag{A4}$$

From Eq. (A4) and the third equality of Eq. (A1), we obtain

$$E_c^2 - M_c^2 = E_e^2 - m_e^2 \tag{A5}$$

With its second and third terms moved to the other side of the equality, Eq. (A5) may be divided by Eq. (A2) to give

$$E_c - E_e = \frac{1}{m_n}(M_c^2 - m_e^2) \tag{A6}$$

As a system of coupled linear equations, Eqs. (A2) and (A6) may be solved to give

$$E_c = \frac{1}{2m_n}(m_n^2 - m_e^2 + M_c^2) \tag{A7}$$

$$E_e = \frac{1}{2m_n}(m_n^2 + m_e^2 - M_c^2) \tag{A8}$$

Using Eq. (A8), the last equality in Eq. (A4) may be rewritten as

$$\begin{aligned} q_e &= \frac{1}{2m_n}\sqrt{(m_n^2 + m_e^2 - M_c^2)^2 - (2m_n m_e)^2} \\ &= \frac{1}{2m_n}\sqrt{(m_n + m_e + M_c)(m_n + m_e - M_c)(m_n - m_e + M_c)(m_n - m_e - M_c)} \end{aligned} \tag{A9}$$

Eq. (A8) shows that a maximum of E_e corresponds to a minimum of M_c^2. Now the rest mass M_c is the total energy of the proton and anti-neutrino pair in their center of mass (or momentum) frame so that it achieves the minimum

$$M = m_p + m_v \tag{A10}$$

when the proton and the anti-neutrino are both at rest in the center of mass frame. Hence, from Eqs. (A8) and (A10), the maximum energy of the electron $E = c^2 E_e$ is

$$E_{\max} = \frac{c^2}{2m_n}\left[m_n^2 + m_e^2 - (m_p + m_v)^2\right] \approx 1.292569 \text{ MeV} \approx 1.29 \text{MeV} \quad \text{(A11)*}$$ [1]

When Eq. (A10) holds, the proton and the anti-neutrino move with the same velocity v_m of the center of mass and we have

$$\frac{v_m}{c} = \left(\frac{q_v}{E_v}\right)\Big|_{E=E_{\max}} = \left(\frac{q_p}{E_p}\right)\Big|_{E=E_{\max}} = \left(\frac{q_c}{E_c}\right)\Big|_{E=E_{\max}} = \left(\frac{q_e}{E_c}\right)\Big|_{M_c = m_p + m_v} \tag{A12}$$

where the last equality follows from Eq. (A3). By Eqs. (A7) and (A9), the last expression in Eq. (A12) may be used to obtain the speed of the anti-neutrino when $E = E_{\max}$. Thus, with $M = m_p + m_v$, we have

$$\frac{v_m}{c} = \frac{\sqrt{(m_n + m_e + M)(m_n + m_e - M)(m_n - m_e + M)(m_n - m_e - M)}}{m_n^2 - m_e^2 + M^2} \quad \text{(A13)*}$$

$$\approx 0.00126538 \approx 0.00127$$

[Alternative Solution]

Assume that, in the rest frame of the neutron, the electron comes out with momentum $c\vec{q}_e$ and energy $c^2 E_e$, the proton with $c\vec{q}_p$ and $c^2 E_p$, and the anti-neutrino with $c\vec{q}_v$ and $c^2 E_v$. With the magnitude of vector $\vec{q}_\alpha$ denoted by the symbol q_α, we have

$$E_p^2 = m_p^2 + q_p^2, \qquad E_v^2 = m_v^2 + q_v^2, \qquad E_e^2 = m_e^2 + q_e^2 \tag{1A}$$

Conservation of energy and momentum in the neutron decay leads to

$$E_p + E_v = m_n - E_e \tag{2A}$$

$$\vec{q}_p + \vec{q}_v = -\vec{q}_e \tag{3A}$$

When squared, the last two equations lead to

$$E_p^2 + E_v^2 + 2E_p E_v = (m_n - E_e)^2 \tag{4A}$$

$$q_p^2 + q_v^2 + 2\vec{q}_p \cdot \vec{q}_v = q_e^2 = E_e^2 - m_e^2 \tag{5A}$$

Subtracting Eq. (5A) from Eq. (4A) and making use of Eq. (1A) then gives

$$m_p^2 + m_v^2 + 2(E_p E_v - \vec{q}_p \cdot \vec{q}_v) = m_n^2 + m_e^2 - 2m_n E_e \tag{6A}$$

[1] An equation marked with an asterisk contains answer to the problem.

or, equivalently,

$$2m_n E_e = m_n^2 + m_e^2 - m_p^2 - m_\nu^2 - 2(E_p E_\nu - \vec{q}_p \cdot \vec{q}_\nu) \tag{7A}$$

If θ is the angle between $\vec{q}_p$ and $\vec{q}_\nu$, we have $\vec{q}_p \cdot \vec{q}_\nu = q_p q_\nu \cos\theta \le q_p q_\nu$ so that Eq. (7A) leads to the relation

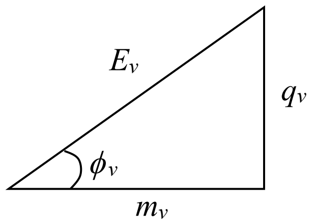
$$2m_n E_e \le m_n^2 + m_e^2 - m_p^2 - m_\nu^2 - 2(E_p E_\nu - q_p q_\nu) \tag{8A}$$

Note that the equality in Eq. (8A) holds only if $\theta = 0$, i.e., the energy of the electron $c^2 E_e$ takes on its maximum value only when the anti-neutrino and the proton *move in the same direction*.

Let the speeds of the proton and the anti-neutrino in the rest frame of the neutron be $c\beta_p$ and $c\beta_\nu$, respectively. We then have $q_p = \beta_p E_p$ and $q_\nu = \beta_\nu E_\nu$. As shown in Fig. A1, we introduce the angle ϕ_ν ($0 \le \phi_\nu < \pi/2$) for the antineutrino by

$$q_\nu = m_\nu \tan\phi_\nu, \qquad E_\nu = \sqrt{m_\nu^2 + q_\nu^2} = m_\nu \sec\phi_\nu, \qquad \beta_\nu = q_\nu / E_\nu = \sin\phi_\nu \tag{9A}$$

Figure A1



Similarly, for the proton, we write, with $0 \le \phi_p < \pi/2$,

$$q_p = m_p \tan\phi_p, \quad E_p = \sqrt{m_p^2 + q_p^2} = m_p \sec\phi_p, \quad \beta_p = q_p / E_p = \sin\phi_p \tag{10A}$$

Eq. (8A) may then be expressed as

$$2m_n E_e \le m_n^2 + m_e^2 - m_p^2 - m_\nu^2 - 2m_p m_\nu \left(\frac{1 - \sin\phi_p \sin\phi_\nu}{\cos\phi_p \cos\phi_\nu}\right) \tag{11A}$$

The factor in parentheses at the end of the last equation may be expressed as

$$\frac{1 - \sin\phi_p \sin\phi_\nu}{\cos\phi_p \cos\phi_\nu} = \frac{1 - \sin\phi_p \sin\phi_\nu - \cos\phi_p \cos\phi_\nu}{\cos\phi_p \cos\phi_\nu} + 1 = \frac{1 - \cos(\phi_p - \phi_\nu)}{\cos\phi_p \cos\phi_\nu} + 1 \ge 1 \tag{12A}$$

and clearly assumes its minimum possible value of 1 when $\phi_p = \phi_\nu$, i.e., when the anti-neutrino and the proton *move with the same velocity* so that $\beta_p = \beta_\nu$. Thus, it follows from Eq. (11A) that the maximum value of E_e is

$$\begin{aligned}(E_e)_{\max} &= \frac{1}{2m_n}(m_n^2 + m_e^2 - m_p^2 - m_\nu^2 - 2m_p m_\nu) \\ &= \frac{1}{2m_n}[m_n^2 + m_e^2 - (m_p + m_\nu)^2]\end{aligned} \tag{13A*}$$

and the maximum energy of the electron $E = c^2 E_e$ is

$$E_{\max} = c^2 (E_e)_{\max} \approx 1.292569 \text{ MeV} \approx 1.29\,\text{MeV} \tag{14A}^*$$

When the anti-neutrino and the proton move with the same velocity, we have, from Eqs. (9A), (10A), (2A) ,(3A), and (1A), the result

$$\beta_\nu = \beta_p = \frac{q_p}{E_p} = \frac{q_\nu}{E_\nu} = \frac{q_p + q_\nu}{E_p + E_\nu} = \frac{q_e}{m_n - E_e} = \frac{\sqrt{E_e^2 - m_e^2}}{m_n - E_e} \tag{15A}$$

Substituting the result of Eq. (13A) into the last equation, the speed v_m of the anti-neutrino when the electron attains its maximum value $E_{\max}$ is, with $M = m_p + m_\nu$, given by

$$\frac{v_m}{c} = (\beta_\nu)_{\max E_e} = \frac{\sqrt{(E_e)^2_{\max} - m_e^2}}{m_n - (E_e)_{\max}} = \frac{\sqrt{(m_n^2 + m_e^2 - M^2)^2 - 4m_n^2 m_e^2}}{2m_n^2 - (m_n^2 + m_e^2 - M^2)}$$

$$= \frac{\sqrt{(m_n + m_e + M)(m_n + m_e - M)(m_n - m_e + M)(m_n - m_e - M)}}{m_n^2 - m_e^2 + M^2} \tag{16A}^*$$

$$\approx 0.00126538 \approx 0.00127$$

Part B

Light Levitation

(b) Refer to Fig. B1. Refraction of light at the spherical surface obeys Snell's law and leads to

$$n \sin\theta_i = \sin\theta_t \tag{B1}$$

Neglecting terms of the order $(\delta/R)^3$or higher in sine functions, Eq. (B1) becomes

$$n\theta_i \approx \theta_t \tag{B2}$$

For the triangle ΔFAC in Fig. B1, we have

$$\beta = \theta_t - \theta_i \approx n\theta_i - \theta_i = (n-1)\theta_i \tag{B3}$$

Let f_0 be the frequency of the incident light. If n_p is the number of photons incident on the plane surface per unit area per unit time, then the total number of photons incident on the plane surface per unit time is $n_p \pi\delta^2$. The total power P of photons incident on the plane surface is $(n_p \pi\delta^2)(hf_0)$, with h being Planck's constant. Hence,

$$n_p = \frac{P}{\pi\delta^2 hf_0} \tag{B4}$$

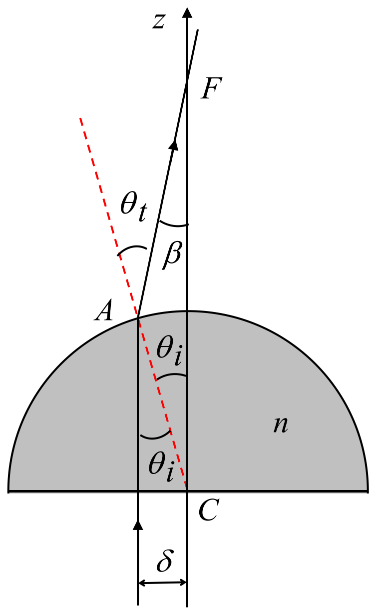


Fig. B1

The number of photons incident on an annular disk of inner radius r and outer radius $r+dr$ on the plane surface per unit time is $n_p(2\pi r dr)$, where $r = R\tan\theta_i \approx R\theta_i$. Therefore,

$$n_p(2\pi r dr) \approx n_p(2\pi R^2)\theta_i d\theta_i \tag{B5}$$

The z-component of the momentum carried away per unit time by these photons when refracted at the spherical surface is

$$\begin{aligned} dF_z &= n_p \frac{hf_o}{c}(2\pi r dr)\cos\beta \approx n_p \frac{hf_0}{c}(2\pi R^2)(1-\frac{\beta^2}{2})\theta_i d\theta_i \\ &\approx n_p \frac{hf_0}{c}(2\pi R^2)[\theta_i - \frac{(n-1)^2}{2}\theta_i^3]d\theta_i \end{aligned} \tag{B6}$$

so that the z-component of the total momentum carried away per unit time is

$$\begin{aligned} F_z &= 2\pi R^2 n_p (\frac{hf_0}{c})\int_0^{\theta_{im}}[\theta_i - \frac{(n-1)^2}{2}\theta_i^3]d\theta_i \\ &= \pi R^2 n_p (\frac{hf_0}{c})\theta_{im}^2[1-\frac{(n-1)^2}{4}\theta_{im}^2] \end{aligned} \tag{B7}$$

where $\tan\theta_{im} = \frac{\delta}{R} \approx \theta_{im}$. Therefore, by the result of Eq. (B5), we have

$$F_z = \frac{\pi R^2 P}{\pi\delta^2 hf_0}(\frac{hf_0}{c})\frac{\delta^2}{R^2}[1-\frac{(n-1)^2\delta^2}{4R^2}] = \frac{P}{c}[1-\frac{(n-1)^2\delta^2}{4R^2}] \tag{B8}$$

The force of optical levitation is equal to the sum of the z-components of the forces exerted by the incident and refracted lights on the glass hemisphere and is given by

$$\frac{P}{c} + (-F_z) = \frac{P}{c} - \frac{P}{c}[1-\frac{(n-1)^2\delta^2}{4R^2}] = \frac{(n-1)^2\delta^2}{4R^2}\frac{P}{c} \tag{B9}$$

Equating this to the weight mg of the glass hemisphere, we obtain the minimum laser power required to levitate the hemisphere as

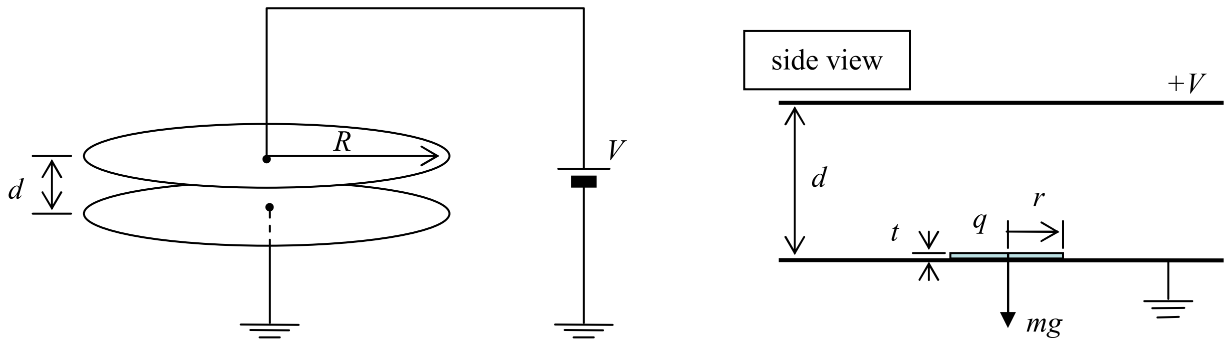
$$P = \frac{4mgcR^2}{(n-1)^2\delta^2} \tag{B10*}$$

Theoretical Question 1:

"Ping-Pong" Resistor

A capacitor consists of two circular parallel plates both with radius R separated by distance d, where $d \ll R$, as shown in Fig. 1.1(a). The top plate is connected to a *constant* voltage source at a potential V while the bottom plate is grounded. Then a thin and small disk of mass m with radius r ($\ll R, d$) and thickness t ($\ll r$) is placed on the center of the bottom plate, as shown in Fig. 1.1(b).

Let us assume that the space between the plates is in vacuum with the dielectric constant ε_0; the plates and the disk are made of perfect conductors; and all the electrostatic edge effects may be neglected. The inductance of the whole circuit and the relativistic effects can be safely disregarded. The image charge effect can also be neglected.



(a) (b)

Figure 1.1 Schematic drawings of (a) a *parallel plate* capacitor connected to a constant voltage source and (b) a side view of the parallel *plates* with a small *disk* inserted inside the capacitor. (See text for details.)

(a) [1.2 points] Calculate the electrostatic force F_{p} between *the plates* separated by d before inserting the disk in-between as shown in Fig. 1.1(a).

(b) [0.8 points] When the disk is placed on the bottom plate, a charge q on *the disk* of Fig. 1.1(b) is related to the voltage V by $q = \chi V$. Find χ in terms of r, d, and ε_0.

(c) [0.5 points] The parallel plates lie perpendicular to a uniform gravitational field g. To lift up the disk at rest initially, we need to increase the applied voltage beyond a threshold voltage V_{th}. Obtain V_{th} in terms of m, g, d, and χ.

(d) [2.3 points] When $V > V_{\text{th}}$, the disk makes an up-and-down motion between the plates. (Assume that the disk moves only vertically *without any wobbling.*) The collisions between the disk and the plates are inelastic with the restitution coefficient $\eta \equiv (\mathrm{v}_{\text{after}} / \mathrm{v}_{\text{before}})$, where $\mathrm{v}_{\text{before}}$ and $\mathrm{v}_{\text{after}}$ are the speeds of the disk just before and after the collision respectively. The plates are stationarily fixed in position. The speed of the disk *just after* the collision at the bottom plate approaches a "steady-state speed" v_{s}, which depends on V as follows:

$$\mathrm{v}_{\text{s}} = \sqrt{\alpha V^2 + \beta}\,. \tag{1.1}$$

Obtain the coefficients α and β in terms of m, g, χ, d, and η. Assume that the whole surface of the disk touches the plate evenly and simultaneously so that the complete charge exchange happens instantaneously at every collision.

(e) [2.2 points] After reaching its steady state, the time-averaged current I through the capacitor plates can be approximated by $I = \gamma V^2$ when $qV >> mgd$. Express the coefficient γ in terms of m, χ, d, and η.

(f) [3 points] When the applied voltage V is decreased (extremely slowly), there exists a critical voltage V_{c} below which the charge will cease to flow. Find V_{c} and the corresponding current I_{c} in terms of m, g, χ, d, and η. By comparing V_{c} with the lift-up threshold V_{th} discussed in (c), make a rough sketch of the $I - V$ characteristics when V is increased and decreased in the range from $V = 0$ to $3\,V_{\text{th}}$.

Country Code	Student Code	Question Number
		1

Answer Form

Theoretical Question 1:

(a) $F_{\mathrm{p}} =$

(b) $\chi =$

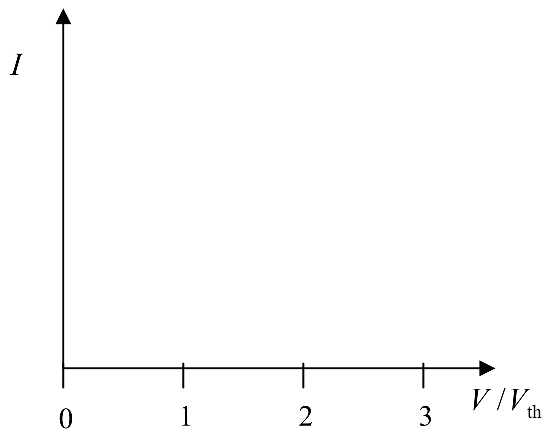
(c) $V_{\mathrm{th}} =$

(d) $\alpha =$ $\beta =$

(e) $\gamma =$

(f) $I_c =$

$V_c =$



Theoretical Question 1: *Ping-Pong Resistor*

1. Answers

(a) $F_{\text{R}} = -\dfrac{1}{2}\pi R^2 \varepsilon_0 \dfrac{V^2}{d^2}$

(b) $\chi = -\varepsilon_0 \dfrac{\pi r^2}{d}$

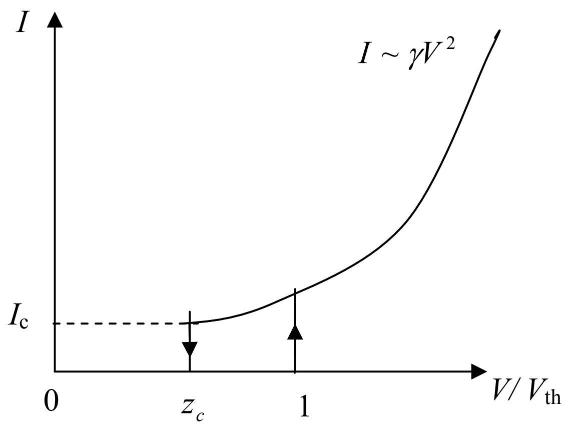
(c) $V_{\text{th}} = \sqrt{\dfrac{2mgd}{\chi}}$

(d) $\text{v}_{\text{s}} = \sqrt{\alpha V^2 + \beta}$

$$\alpha = \left(\frac{\eta^2}{1-\eta^2}\right)\left(\frac{2\chi}{m}\right), \quad \beta = \left(\frac{\eta^2}{1+\eta^2}\right)(2gd)$$

(e) $\gamma = \sqrt{\dfrac{1+\eta}{1-\eta}}\sqrt{\dfrac{\chi^3}{2md^2}}$

(f) $V_c = \sqrt{\dfrac{1-\eta^2}{1+\eta^2}}\sqrt{\dfrac{mgd}{\chi}}$, $I_c = \dfrac{2\eta\sqrt{1-\eta^2}}{(1+\eta)(1+\eta^2)} g\sqrt{m\chi}$



2. Solutions

(a) [1.2 points]

The charge Q induced by the external bias voltage V can be obtained by applying the Gauss law:

$$\varepsilon_0 \oint \vec{E} \cdot d\vec{s} = Q \tag{a1}$$

$$Q = \varepsilon_0 E \cdot (\pi R^2) = \varepsilon_0 \left(\frac{V}{d} \right) \cdot (\pi R^2), \tag{a2}$$

where $V = Ed$.

The energy stored in the capacitor:

$$U = \int_0^V Q(V')dV' = \int_0^V \varepsilon_0 \pi R^2 \left(\frac{V'}{d} \right) dV' = \frac{1}{2} \varepsilon_0 \pi R^2 \frac{V^2}{d} . \tag{a3}$$

The force acting on the plate, when the bias voltage V is kept constant:

$$\therefore F_{\mathrm{R}} = + \frac{\partial U}{\partial d} = - \frac{1}{2} \varepsilon_0 \pi R^2 \frac{V^2}{d^2} . \tag{a4}$$

[An alternative solution:]

Since the electric field E' acting on one plate should be generated by the other plate and its magnitude is

$$E' = \frac{1}{2} E = \frac{V}{2d}, \tag{a5}$$

the force acting on the plate can be obtained by

$$F_{\mathrm{R}} = QE' . \tag{a6}$$

(b) [0.8 points]

The charge q on the small disk can also be calculated by applying the Gauss law:

$$\varepsilon_0 \oint \vec{E} \cdot d\vec{s} = q . \tag{b1}$$

Since one side of the small disk is in contact with the plate,

$$q = -\varepsilon_0 E \cdot (\pi r^2) = -\varepsilon_0 \frac{\pi r^2}{d} V = \chi V . \tag{b2}$$

Alternatively, one may use the area ratio for $q = -\left(\frac{\pi r^2}{\pi R^2}\right)Q$.

$$\therefore \chi = -\varepsilon_0 \frac{\pi r^2}{d}. \tag{b3}$$

(c) [0.5 points]

The net force, F_{net}, acting on the small disk should be a sum of the gravitational and electrostatic forces:

$$F_{\text{net}} = F_{\text{g}} + F_{\text{e}}. \tag{c1}$$

The gravitational force: $F_{\text{g}} = -mg$.

The electrostatic force can be derived from the result of (a) above:

$$F_{\text{e}} = \frac{1}{2}\varepsilon_0 \frac{\pi r^2}{d^2} V^2 = \frac{\chi}{2d} V^2. \tag{c2}$$

In order for the disk to be lifted, one requires $F_{\text{net}} > 0$:

$$\frac{\chi}{2d} V^2 - mg > 0. \tag{c3}$$

$$\therefore V_{\text{th}} = \sqrt{\frac{2mgd}{\chi}}. \tag{c4}$$

(d) [2.3 points]

Let v_{s} be the steady velocity of the small disk just after its collision with the bottom plate. Then the *steady-state* kinetic energy K_{s} of the disk just above the bottom plate is given by

$$K_{\text{s}} = \frac{1}{2} m \text{v}_{\text{s}}^2. \tag{d1}$$

For each round trip, the disk gains electrostatic energy by

$$\Delta U = 2qV. \tag{d2}$$

For each inelastic collision, the disk lose its kinetic energy by

$$\Delta K_{\text{loss}} = K_{\text{before}} - K_{\text{after}} = (1-\eta^2) K_{\text{before}} = \left(\frac{1}{\eta^2} - 1\right) K_{\text{after}}. \tag{d3}$$

Since K_{s} is the energy after the collision at the bottom plate and $(K_{\text{s}} + qV - mgd)$ is the energy before the collision at the top plate, the total energy loss during the round trip can be written in terms of K_s:

$$\Delta K_{tot} = \left(\frac{1}{\eta^2} - 1\right) K_s + (1-\eta^2)(K_s + qV - mgd). \tag{d4}$$

In its steady state, ΔU should be compensated by ΔK_{tot}.

$$2qV = \left(\frac{1}{\eta^2} - 1\right) K_s + (1-\eta^2)(K_s + qV - mgd). \tag{d5}$$

Rearranging Eq. (d5), we have

$$\begin{aligned} K_s &= \frac{\eta^2}{1-\eta^4}\left[(1+\eta^2)qV + (1-\eta^2)mgd\right] \\ &= \left(\frac{\eta^2}{1-\eta^2}\right) qV + \left(\frac{\eta^2}{1+\eta^2}\right) mgd \\ &= \frac{1}{2} m \mathrm{v}_s^2. \end{aligned} \tag{d6}$$

Therefore,

$$\mathrm{v}_s = \sqrt{\left(\frac{\eta^2}{1-\eta^2}\right)\left(\frac{2\chi V^2}{m}\right) + \left(\frac{\eta^2}{1+\eta^2}\right)(2gd)}. \tag{d7}$$

Comparing with the form:

$$\mathrm{v}_s = \sqrt{\alpha V^2 + \beta}, \tag{d8}$$

$$\alpha = \left(\frac{\eta^2}{1-\eta^2}\right)\left(\frac{2\chi}{m}\right), \quad \beta = \left(\frac{\eta^2}{1+\eta^2}\right)(2gd). \tag{d9}$$

[An alternative solution:]

Let v_n be the velocity of the small disk just after n-th collision with the bottom plate. Then the kinetic energy of the disk just above the bottom plate is given by

$$K_n = \frac{1}{2} m \mathrm{v}_n^2. \tag{d10}$$

When it reaches the top plate, the disk gains energy by the increase of potential energy:

$$\Delta U_{up} = qV - mgd. \tag{d11}$$

Thus, the kinetic energy just before its collision with the top plate becomes

$$K_{n-up} = \frac{1}{2} m \mathrm{v}_{up}^2 = K_n + \Delta U_{up}. \tag{d12}$$

Since $\eta = \mathrm{v}_{\text{after}} / \mathrm{v}_{\text{before}}$, the kinetic energy after the collision with the top plate becomes scaled down by a factor of η^2:

$$K'_{n-\text{up}} = \eta^2 \cdot K_{n-\text{up}} . \tag{d13}$$

Now the potential energy gain by the downward motion is:

$$\Delta U_{\text{down}} = qV + mgd \tag{d14}$$

so that the kinetic energy just before it collides with the bottom plate becomes:

$$K_{n-\text{down}} = K'_{n-\text{up}} + \Delta U_{\text{down}} . \tag{d15}$$

Again, due to the loss of energy by the collision with the bottom plate, the kinetic energy after its $(n+1)$-th collision can be obtained by

$$\begin{aligned} K_{n+1} &= \eta^2 \cdot K_{n-\text{down}} \\ &= \eta^2 (K'_{n-\text{up}} + \Delta U_{\text{down}}) \\ &= \eta^2 (\eta^2 (K_n + \Delta U_{\text{up}}) + \Delta U_{\text{down}}) \\ &= \eta^2 (\eta^2 (K_n + qV - mgd) + qV + mgd) \\ &= \eta^4 K_n + \eta^2 (1+\eta^2) qV + \eta^2 (1-\eta^2) mgd. \end{aligned} \tag{d16}$$

As $n \to \infty$, we expect the velocity $\mathrm{v}_n \to \mathrm{v}_\mathrm{s}$, that is, $K_n \to K_\mathrm{s} = \frac{1}{2} m \mathrm{v}_\mathrm{s}^2$:

$$\begin{aligned} K_\mathrm{s} &= \frac{1}{1-\eta^4} \left[\eta^2 (1+\eta^2) qV + \eta^2 (1-\eta^2) mgd \right] \\ &= \left(\frac{\eta^2}{1-\eta^2} \right) qV + \left(\frac{\eta^2}{1+\eta^2} \right) mgd \\ &= \frac{1}{2} m \mathrm{v}_\mathrm{s}^2 \end{aligned} \tag{d17}$$

(e) [2.2 points]

The amount of charge carried by the disk during its round trip between the plates is $\Delta Q = 2q$, and the time interval $\Delta t = t_+ + t_-$, where t_+ (t_-) is the time spent during the up- (down-) ward motion respectively.

Here t_+ (t_-) can be determined by

$$\begin{aligned} \mathrm{v}_{0+} t_+ + \frac{1}{2} a_+ t_+^2 &= d \\ \mathrm{v}_{0-} t_- + \frac{1}{2} a_- t_-^2 &= d \end{aligned} \tag{e1}$$

where v_{0+} (v_{0-}) is the initial velocity at the bottom (top) plate and a_+ (a_-) is the up-

(down-) ward acceleration respectively.

Since the force acting on the disk is given by

$$F = ma_{\pm} = qE \mp mg = \frac{qV}{d} \mp mg \, , \tag{e2}$$

in the limit of $mgd << qV$, $a_{\pm}$ can be approximated by

$$a_0 = a_+ = a_- \approx \frac{qV}{md} \, , \tag{e3}$$

which implies that the upward and down-ward motion should be symmetric. Thus, Eq.(e1) can be described by a single equation with $t_0 = t_+ = t_-$, $\mathrm{v_s} = \mathrm{v}_{0+} = \mathrm{v}_{0-}$, and $a_0 = a_+ = a_-$. Moreover, since the speed of the disk just after the collision should be the same for the top- and bottom-plates, one can deduce the relation:

$$\mathrm{v_s} = \eta(\mathrm{v_s} + a_0 t_0) \, , \tag{e4}$$

from which we obtain the time interval $\Delta t = 2t_0$,

$$\Delta t = 2t_0 = 2\left(\frac{1-\eta}{\eta}\right)\frac{\mathrm{v_s}}{a_0} \, . \tag{e5}$$

From Eq. (d6), in the limit of $mgd << qV$, we have

$$K_\mathrm{s} = \frac{1}{2}m\mathrm{v_s^2} \approx \left(\frac{\eta^2}{1-\eta^2}\right) qV \, . \tag{e6}$$

By substituting the results of Eqs. (e3) and (e6), we get

$$\Delta t = 2\left(\frac{1-\eta}{\eta}\right)\sqrt{\frac{2\eta^2}{1-\eta^2}}\sqrt{\frac{md^2}{qV}} = 2\sqrt{\frac{1-\eta}{1+\eta}}\sqrt{\frac{2md^2}{\chi V^2}} \, . \tag{e7}$$

Therefore, from $I = \dfrac{\Delta Q}{\Delta t} = \dfrac{2q}{\Delta t}$,

$$I = \frac{2q}{\Delta t} = \chi V\sqrt{\frac{1+\eta}{1-\eta}}\sqrt{\frac{\chi V^2}{2md^2}} = \sqrt{\frac{1+\eta}{1-\eta}}\sqrt{\frac{\chi^3}{2md^2}}V^2 \, . \tag{e8}$$

$$\therefore \gamma = \sqrt{\frac{1+\eta}{1-\eta}}\sqrt{\frac{\chi^3}{2md^2}} \tag{e9}$$

<u>[Alternative solution #1:]</u>

Starting from Eq. (e3), we can solve the quadratic equation of Eq. (e1) so that

$$t_{\pm} = \frac{\mathrm{v}_{0\pm}}{a_0}\left(\sqrt{1+\frac{2da_0}{\mathrm{v}_{0\pm}^2}}-1\right). \tag{e10}$$

When it reaches the steady state, the initial velocities $\mathrm{v}_{0\pm}$ are given by

$$\mathrm{v}_{0+} = \mathrm{v}_{\mathrm{s}} \tag{e11}$$

$$\mathrm{v}_{0-} = \eta\cdot(\mathrm{v}_{\mathrm{s}} + a_0 t_+) = \eta \mathrm{v}_{\mathrm{s}}\sqrt{1+\frac{2da_0}{\mathrm{v}_{\mathrm{s}}^2}}\ , \tag{e12}$$

where v_{s} can be rewritten by using the result of Eq. (e6),

$$\mathrm{v}_{\mathrm{s}}^2 \approx \alpha V = \left(\frac{\eta^2}{1-\eta^2}\right)\frac{2qV}{m} = \left(\frac{\eta^2}{1-\eta^2}\right)2a_0 d\ . \tag{e13}$$

As a result, we get $\mathrm{v}_{0-} \cong \eta \mathrm{v}_{\mathrm{s}}\cdot\frac{1}{\eta} = \mathrm{v}_{\mathrm{s}}$ and consequently $t_{\pm} = \frac{\mathrm{v}_{\mathrm{s}}}{a_0}\left(\frac{1}{\eta}-1\right)$, which is equivalent to Eq. (e4).

[Alternative solution #2:]

The current I can be obtained from

$$I = \frac{2q}{\Delta t} = \frac{2q\bar{\mathrm{v}}}{d}, \tag{e14}$$

where $\bar{\mathrm{v}}$ is an average velocity. Since the up and down motions are symmetric with the same constant acceleration in the limit of $mgd << qV$,

$$\bar{\mathrm{v}} = \frac{1}{2}\left(\mathrm{v}_{\mathrm{s}} + \frac{\mathrm{v}_{\mathrm{s}}}{\eta}\right). \tag{e15}$$

Thus, we have

$$I = \frac{q}{2d}\left(1+\frac{1}{\eta}\right)\mathrm{v}_{\mathrm{s}}\ . \tag{e16}$$

Inserting the expression (Eq. (e15)) of v_{s} into Eq. (e16), one obtains an expression identical to Eq. (e8).

(f) [3 points]

The disk will lose its kinetic energy and eventually cease to move when the disk can not reach the top plate. In other words, the threshold voltage V_{c} can be determined from the condition that the velocity $\mathrm{v}_{0\text{-}}$ of the disk at the top plate is zero, i.e., $\mathrm{v}_{0\text{-}} = 0$.

In order for the disk to have $\mathrm{v}_{0\text{-}} = 0$ at the top plate, the kinetic energy $\overline{K}_{\mathrm{s}}$ at the top plate should satisfy the relation:

$$\overline{K}_{\mathrm{s}} = K_{\mathrm{s}} + qV_c - mgd = 0, \tag{f1}$$

where K_{s} is the *steady-state* kinetic energy at the bottom plate after the collision. Therefore, we have

$$\left(\frac{\eta^2}{1-\eta^2}\right)qV_c + \left(\frac{\eta^2}{1+\eta^2}\right)mgd + qV_c - mgd = 0, \tag{f2}$$

or equivalently,

$$(1+\eta^2)qV_c - (1-\eta^2)mgd = 0. \tag{f3}$$

$$\therefore \quad qV_{\mathrm{c}} = \frac{1-\eta^2}{1+\eta^2}mgd \tag{f4}$$

From the relation $q = \chi V_{\mathrm{c}}$,

$$\therefore V_{\mathrm{c}} = \sqrt{\frac{1-\eta^2}{1+\eta^2}}\sqrt{\frac{mgd}{\chi}}. \tag{f5}$$

In comparison with the threshold voltage V_{th} of Eq. (c4), we can rewrite Eq. (f5) by

$$V_c = z_c V_{\mathrm{th}} \tag{f6}$$

where z_c should be used in the plot of I vs. (V/V_{th}) and

$$z_c = \sqrt{\frac{1-\eta^2}{2(1+\eta^2)}}. \tag{f7}$$

[Note that an alternative derivation of Eq. (f1) is possible if one applies the energy compensation condition of Eq. (d5) or the recursion relation of Eq. (d17) at the top plate instead of the bottom plate.]

Now we can setup equations to determine the time interval $\Delta t = t_- + t_+$:

$$\mathrm{v}_{0-}t_- + \frac{1}{2}a_- t_-^2 = d \tag{f8}$$

$$\mathrm{v}_{0+}t_+ + \frac{1}{2}a_+ t_+^2 = d \tag{f9}$$

where the accelerations are given by

$$a_+ = \frac{qV_{\mathrm{c}}}{md} - g = \left[\frac{1-\eta^2}{1+\eta^2} - 1\right]g = \left(\frac{-2\eta^2}{1+\eta^2}\right)g \tag{f10}$$

$$a_- = \frac{qV_c}{md} + g = \left[\frac{1-\eta^2}{1+\eta^2} + 1\right] g = \left(\frac{2}{1+\eta^2}\right) g \tag{f11}$$

$$\frac{a_+}{a_-} = -\eta^2 \tag{f12}$$

Since $v_{0-} = 0$, we have $v_{0+} = \eta(a_- t_-)$ and $t_-^2 = 2d / a_-$.

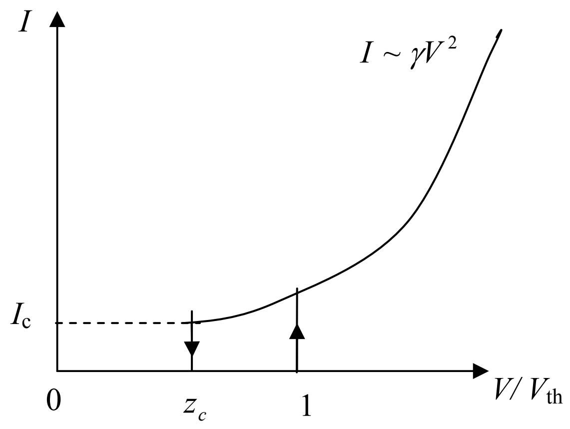
$$t_- = \sqrt{\frac{2d}{a_-}} = \sqrt{(1+\eta^2)\left(\frac{d}{g}\right)}, \tag{f13}$$

By using $v_{0+}^2 = \eta^2(2da_-) = -2da_+$, we can solve the quadratic equation of Eq. (f9):

$$t_+ = \frac{v_{0+}}{a_+}\left(\sqrt{1 + \frac{2da_+}{v_{0+}^2}} - 1\right) = -\frac{v_{0+}}{a_+} = \sqrt{\frac{2d}{|a_+|}} = \sqrt{\left(\frac{1+\eta^2}{\eta^2}\right)\left(\frac{d}{g}\right)} = \frac{t_-}{\eta}. \tag{f14}$$

$$\therefore \Delta t = t_- + t_+ = \left(1 + \frac{1}{\eta}\right)\sqrt{\left(1+\eta^2\right)\left(\frac{d}{g}\right)} \tag{f15}$$

$$I_c = \frac{\Delta Q_c}{\Delta t} = \frac{2q}{\Delta t} = \frac{2\chi V_c}{\Delta t} = \frac{2\eta\sqrt{1-\eta^2}}{(1+\eta)(1+\eta^2)} g\sqrt{m\chi}. \tag{f16}$$



[A more elaborate Solution:]

One may find a general solution for an arbitrary value of V. By solving the quadratic equations of Eqs. (f8) and (f9), we have

$$t_{\pm} = \frac{\mathrm{v}_{0\pm}}{a_{\pm}}\left[-1+\sqrt{1+\frac{2da_{\pm}}{\mathrm{v}_{0\pm}^2}}\right]. \tag{f17}$$

(It is noted that one has to keep the smaller positive root.)

To simplify the notation, we introduce a few variables:

(i) $y = \dfrac{V}{V_{\text{th}}}$ where $V_{\text{th}} = \sqrt{\dfrac{2mgd}{\chi}}$,

(ii) $z_c = \sqrt{\dfrac{1-\eta^2}{2(1+\eta^2)}}$, which is defined in Eq. (f7),

(iii) $w_0 = 2\eta\sqrt{\dfrac{gd}{1-\eta^2}}$ and $w_1 = 2\sqrt{\dfrac{d}{(1-\eta^2)g}}$,

In terms of y, w, and z_c,

$$a_+ = \frac{qV}{md} - g = g(2y^2-1) \tag{f18}$$

$$a_- = \frac{qV}{md} + g = g(2y^2+1) \tag{f19}$$

$$\mathrm{v}_{0+} = \mathrm{v}_{\mathrm{s}} = w_0\sqrt{y^2+z_c^2} \tag{f20}$$

$$\mathrm{v}_{0-} = \eta(\mathrm{v}_{\mathrm{s}} + a_+t_+) = w_0\sqrt{y^2-z_c^2} \tag{f21}$$

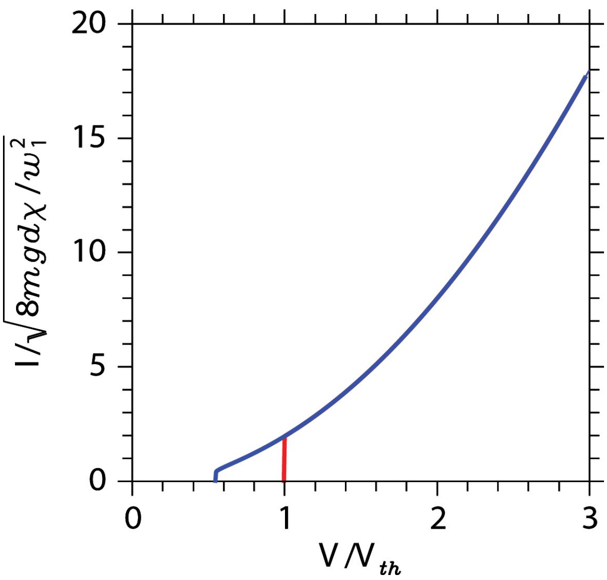
$$t_+ = w_1\frac{\sqrt{y^2-z_c^2}-\eta\sqrt{y^2+z_c^2}}{2y^2-1} \tag{f22}$$

$$t_- = w_1\frac{\sqrt{y^2+z_c^2}-\eta\sqrt{y^2-z_c^2}}{2y^2+1} \tag{f21}$$

$$I = \frac{\Delta Q}{\Delta t} = \frac{2q}{t_+ + t_-} = \left(2\chi V_{\text{th}}\right)\frac{y}{\Delta t} = \frac{\sqrt{8mgd\chi}}{w_1}F(y) \tag{f22}$$

where

$$F(y) = y\left\{\frac{\sqrt{y^2 - z_c^2} - \eta\sqrt{y^2 + z_c^2}}{2y^2 - 1} + \frac{\sqrt{y^2 + z_c^2} - \eta\sqrt{y^2 - z_c^2}}{2y^2 + 1}\right\}^{-1} \tag{f23}$$



3. Mark Distribution

No.	Total Pt.	Partial Pt.	Contents	
(a)	1.2	0.3	Gauss law, or a formula for the capacitance of a parallel plate	
		0.5	Total energy of a capacitor at V	E'= electrical field by the other plate
		0.4	Force from the energy expression	$F = QE'$
(b)	0.8	0.3	Gauss law	Use of area ratio and result of (a)
		0.5	Correct answer	
(c)	0.5	0.1	Correct lift-up condition with force balance	
		0.2	Use of area ratio and result of (a)	
		0.2	Correct answer	
(d)	2.3	0.5	Energy conservation and the work done by the field	
		0.5	Loss of energy due to collisions	
		0.8	Condition for the steady state: energy balance equation (loss = gain)	Condition for the steady state: recursion relation
		0.5	Correct answer	
(e)	2.2	0.2	$\Delta Q = 2q$ per trip	
		0.5	Acceleration $a_{\pm}$ in the limit of $qV >> mgd$; $a_{+} = a_{-}$ by symmetry	
		0.3	Kinetic equations for d , v , a , and t , solutions for t_{+}	By using the symmetry, derive the relation (e4)
		0.4	Expression of $\mathrm{v}_{0\pm}$ and $t_{\pm}$ in its steady state	
		0.4	Solutions of $t_{\pm}$ in approximation	
		0.4	Correct answer	
(f)	3.0	0.5	Condition for V_c ; $K_{\mathrm{up}} = 0$ or $\mathrm{v}_{\mathrm{s,up}} = 0$	Using (d8), Recursion relations
		0.3	energy balance equation	
		0.3	Correct answer of V_c	
		0.7	Kinetic equations for Δt	
		0.3	Correct answer of I_c	
		0.9	Distinction between V_{th} and V_c , the asymptotic behavior $I = \gamma V^2$ in plots	
Total	10			

Theoretical Question 2

Rising Balloon

A rubber balloon filled with helium gas goes up high into the sky where the pressure and temperature decrease with height. In the following questions, assume that the shape of the balloon remains spherical regardless of the payload, and neglect the payload volume. Also assume that the temperature of the helium gas inside of the balloon is always the same as that of the ambient air, and treat all gases as ideal gases. The universal gas constant is R=8.31 J/mol·K and the molar masses of helium and air are $M_H = 4.00\times10^{-3}$ kg/mol and $M_A = 28.9\times10^{-3}$ kg/mol, respectively. The gravitational acceleration is $g = 9.8$ m/s^2.

[Part A]

(a) [1.5 points] Let the pressure of the ambient air be P and the temperature be T. The pressure inside of the balloon is higher than that of outside due to the surface tension of the balloon. The balloon contains n moles of helium gas and the pressure inside is $P+\Delta P$. Find the buoyant force F_B acting on the balloon as a function of P and ΔP.

(b) [2 points] On a particular summer day in Korea, the air temperature T at the height z from the sea level was found to be $T(z) = T_0(1 - z/z_0)$ in the range of $0 < z < 15$ km with z_0=49 km and T_0 =303 K. The pressure and density at the sea level were P_0 = 1.0 atm = 1.01×10^5 Pa and ρ_0 = 1.16 kg/m^3, respectively. For this height range, the pressure takes the form

$$P(z) = P_0(1 - z/z_0)^\eta \ . \tag{2.1}$$

Express η in terms of z_0, ρ_0, P_0, and g, and find its numerical value to the *two* significant digits. Treat the gravitational acceleration as a constant, independent of height.

[Part B]

When a rubber balloon of spherical shape with un-stretched radius r_0 is inflated to a sphere of radius r $(\geq r_0)$, the balloon surface contains extra elastic energy due to the stretching. In a simplistic theory, the elastic energy at constant temperature T can be expressed by

$$U = 4\pi r_0^2 \kappa RT (2\lambda^2 + \frac{1}{\lambda^4} - 3) \tag{2.2}$$

where $\lambda \equiv r/r_0$ (≥ 1) is the size-inflation ratio and κ is a constant in units of mol/m^2.

(c) [2 points] Express ΔP in terms of parameters given in Eq. (2.2), and sketch ΔP as a function of $\lambda = r/r_0$.

(d) [1.5 points] The constant κ can be determined from the amount of the gas needed to inflate the balloon. At T_0=303 K and P_0=1.0 atm = 1.01×10^5 Pa, an un-stretched balloon ($\lambda = 1$) contains n_0=12.5 moles of helium. It takes n=3.6n_0=45 moles in total to inflate the balloon to λ=1.5 at the same T_0 and P_0. Express the balloon parameter a, defined as $a = \kappa/\kappa_0$, in terms of n, n_0, and λ, where $\kappa_0 \equiv \frac{r_0 P_0}{4RT_0}$. Evaluate a to the two significant digits.

[Part C]

A balloon is prepared as in (d) at the sea level (inflated to $\lambda = 1.5$ with $n = 3.6n_0 = 45$ moles of helium gas at T_0=303 K and P_0=1 atm=1.01×10^5 Pa). The total mass including gas, balloon itself, and other payloads is $M_{\mathrm{T}} = 1.12$ kg. Now let the balloon rise from the sea level.

(e) [3 points] Suppose that the balloon eventually stops at the height z_f where the buoyant force balances the total weight. Find z_f and the inflation ratio λ_f at that height. Give the answers in two significant digits. Assume there are no drift effect and no gas leakage during the upward flight.

Country Code	Student Code	Question Number
		2

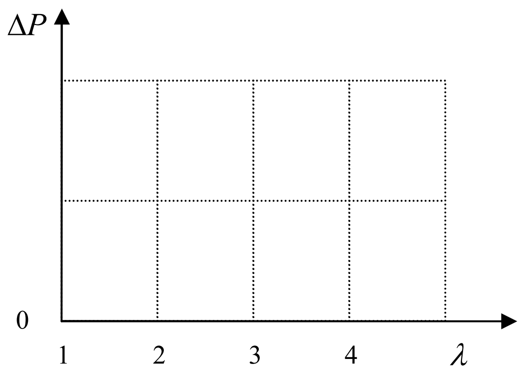
Answer Form

Theoretical Question 2:

(a) F_B =

(b) $\eta =$ Numerical value of $\eta =$

(c) $\Delta P =$



(d) a =   Numerical value of $a =$

(e) z_f =   km λ_f =

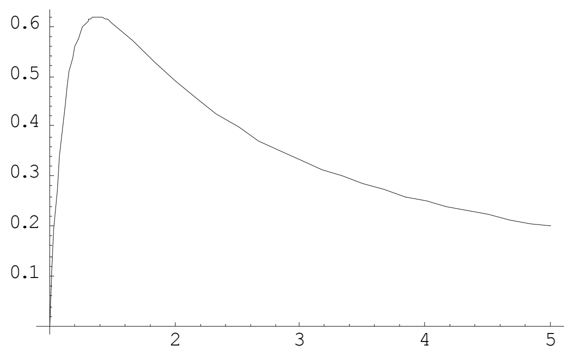
Theoretical Question 2: *Rising Balloon*

1. Answers

(a) $F_B = M_A ng \dfrac{P}{P + \Delta P}$

(b) $\gamma = \dfrac{\rho_0 z_0 g}{P_0} = 5.5$

(c) $\Delta P = \dfrac{4\kappa RT}{r_0}\left(\dfrac{1}{\lambda} - \dfrac{1}{\lambda^7}\right)$



(d) a =0.110

(e) z_f =11 km, λ_f =2.1.

2. Solutions

[Part A]

(a) [1.5 points]

Using the ideal gas equation of state, the volume of the helium gas of n moles at pressure $P+\Delta P$ and temperature T is

$$V = nRT/(P+\Delta P) \tag{a1}$$

while the volume of n' moles of air gas at pressure P and temperature T is

$$V = n'RT/P. \tag{a2}$$

Thus the balloon displaces $n' = n\dfrac{P}{P+\Delta P}$ moles of air whose weight is $M_A n' g$.

This displaced air weight is the buoyant force, i.e.,

$$F_B = M_A n g \frac{P}{P+\Delta P}. \tag{a3}$$

(Partial credits for subtracting the gas weight.)

(b) [2 points]

The pressure difference arising from a height difference of z is $-\rho g z$ when the air density ρ is a constant. When it varies as a function of the height, we have

$$\frac{dP}{dz} = -\rho g = -\frac{\rho_0 T_0}{P_0}\frac{P}{T} g \tag{b1}$$

where the ideal gas law $\rho T/P =$ constant is used. Inserting Eq. (2.1) and $T/T_0 = 1 - z/z_0$ on both sides of Eq. (b1), and comparing the two, one gets

$$\gamma = \frac{\rho_0 z_0 g}{P_0} = \frac{1.16\times 4.9\times 10^4 \times 9.8}{1.01\times 10^5} = 5.52. \tag{b2}$$

The required numerical value is 5.5.

[Part B]

(c) [2 points]

The work needed to increase the radius from r to $r+dr$ under the pressure difference ΔP is

$$dW = 4\pi r^2 \Delta P dr, \tag{c1}$$

while the increase of the elastic energy for the same change of r is

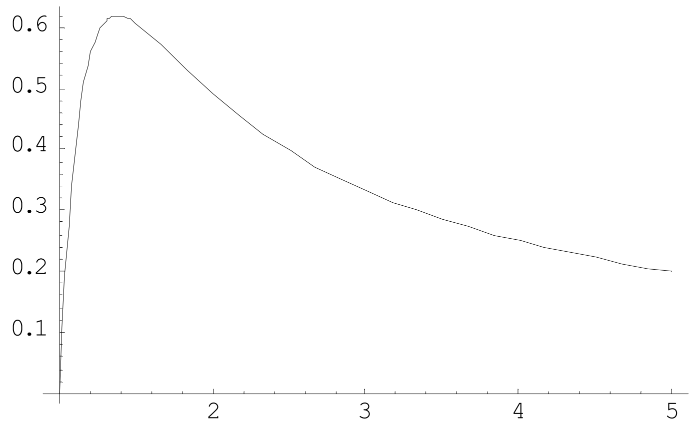
$$dW = \left(\frac{dU}{dr}\right)dr = 4\pi\kappa RT(4r - 4\frac{r_0^6}{r^5})dr \,. \tag{c2}$$

Equating the two expressions of dW, one gets

$$\Delta P = 4\kappa RT(\frac{1}{r} - \frac{r_0^6}{r^7}) \;=\; \frac{4\kappa RT}{r_0}\left(\frac{1}{\lambda} - \frac{1}{\lambda^7}\right). \tag{c3}$$

This is the required answer.

The graph as a function of λ (>1) increases sharply initially, has a maximum at $\lambda = 7^{1/6}$ =1.38, and decreases as λ^{-1} for large λ. The plot of $\Delta P/(4\kappa RT/r_0)$ is given below.



(d) [1.5 points]

From the ideal gas law,

$$P_0V_0 = n_0RT_0 \tag{d1}$$

where V_0 is the unstretched volume.

At volume $V = \lambda^3 V_0$ containing n moles, the ideal gas law applied to the gas inside at $T = T_0$ gives the inside pressure P_{in} as

$$P_{\text{in}} = nRT_0 / V = \frac{n}{n_0\lambda^3}P_0 \quad . \tag{d2}$$

On the other hand, the result of (c) at $T = T_0$ gives

$$P_{\text{in}} = P_0 + \Delta P = P_0 + \frac{4\kappa RT_0}{r_0}(\frac{1}{\lambda} - \frac{1}{\lambda^7}) = (1 + a(\frac{1}{\lambda} - \frac{1}{\lambda^7}))P_0 \,. \tag{d3}$$

Equating (d2) and (d3) to solve for a,

$$a = \frac{n/(n_0\lambda^3)-1}{\lambda^{-1}-\lambda^{-7}} . \tag{d5}$$

Inserting n/n_0=3.6 and λ=1.5 here, a=0.110.

[Part C]

(e) [3 points]

The buoyant force derived in problem (a) should balance the total mass of M_T=1.12 kg. Thus, from Eq. (a3), at the weight balance,

$$\frac{P}{P+\Delta P} = \frac{M_\mathrm{T}}{M_A n} . \tag{e1}$$

On the other hand, applying again the ideal gas law to the helium gas inside of volume $V = \frac{4}{3}\pi r^3 = \lambda^3 \frac{4}{3}\pi r_0^{\ 3} = \lambda^3 V_0$, for arbitrary ambient P and T, one has

$$(P+\Delta P)\lambda^3 = \frac{nRT}{V_0} = P_0 \frac{T}{T_0}\frac{n}{n_0} \tag{e2}$$

for n moles of helium. Eqs. (c3), (e1), and (e2) determine the three unknowns P, ΔP, and λ as a function of T and other parameters. Using Eq. (e2) in Eq. (e1), one has an alternative condition for the weight balance as

$$\frac{P}{P_0}\frac{T_0}{T}\lambda^3 = \frac{M_\mathrm{T}}{M_A n_0} . \tag{e3}$$

Next using (c3) for ΔP in (e2), one has

$$P\lambda^3 + \frac{4\kappa RT}{r_0}\lambda^2(1-\lambda^{-6}) = P_0 \frac{T}{T_0}\frac{n}{n_0}$$

or, rearranging it,

$$\frac{P}{P_0}\frac{T_0}{T}\lambda^3 = \frac{n}{n_0} - a\lambda^2(1-\lambda^{-6}) , \tag{e4}$$

where the definition of a has been used again.

Equating the right hand sides of Eqs. (e3) and (e4), one has the equation for λ as

$$\lambda^2(1-\lambda^{-6}) = \frac{1}{an_0}\left(n - \frac{M_\mathrm{T}}{M_A}\right) = 4.54. \tag{e5}$$

The solution for λ can be obtained by

$$\lambda^2 \approx 4.54/(1-4.54^{-3}) \approx 4.54 : \ \lambda_f \cong 2.13. \tag{e6}$$

To find the height, replace $(P/P_0)/(T/T_0)$ on the left hand side of Eq. (e3) as a function of the height given in (b) as

$$\frac{P}{P_0}\frac{T_0}{T}\lambda^3 = (1 - z_f / z_0)^{\gamma-1}\lambda_f^3 = \frac{M_{\mathrm{T}}}{M_A n_0} = 3.10 \ . \tag{e7}$$

Solution of Eq. (e7) for z_f with λ_f=2.13 and $\gamma - 1$=4.5 is

$$z_f = 49 \times \left(1 - (3.10 / 2.13^3)^{1/4.5}\right) = 10.9 \text{ (km)}. \tag{e8}$$

The required answers are $\lambda_f = 2.1$, and $z_f = 11$ km.

3. Mark Distribution

No.	Total Pt.	Partial Pt.	Contents
(a)	1.5	0.5	Archimedes' principle
		0.5	Ideal gas law applied correctly
		0.5	Correct answer (partial credits 0.3 for subtracting He weight)
(b)	2.0	0.8	Relation of pressure difference to air density
		0.5	Application of ideal gas law to convert the density into pressure
		0.5	Correct formula for γ
		0.2	Correct number in answer
(c)	2.0	0.7	Relation of mechanical work to elastic energy change
		0.3	Relation of pressure to force
		0.5	Correct answer in formula
		0.5	Correct sketch of the curve
(d)	1.5	0.3	Use of ideal gas law for the increased pressure inside
		0.4	Expression of inside pressure in terms of a at the given conditions
		0.5	Formula or correct expression for a
		0.3	Correct answer
(e)	3.0	0.3	Use of force balance as one condition to determine unknowns
		0.3	Ideal gas law applied to the gas as an independent condition to determine unknowns
		0.5	The condition to determine λ_f numerically
		0.7	Correct answer for λ_f
		0.5	The relation of z_f versus λ_f
		0.7	Correct answer for z_f
Total	10		

Theoretical Question 3

Atomic Probe Microscope

Atomic probe microscopes (APMs) are powerful tools in the field of nano-science. The motion of a cantilever in APM can be detected by a photo-detector monitoring the reflected laser beam, as shown in Fig. 3.1. The cantilever can move only in the vertical direction and its displacement z as a function of time t can be described by the equation

$$m\frac{d^2z}{dt^2}+b\frac{dz}{dt}+kz=F\,, \tag{3.1}$$

where m is the cantilever mass, $k=m\omega_0^2$ is the spring constant of the cantilever, b is a small damping coefficient satisfying $\omega_0 >> (b/m) > 0$, and finally F is an external driving force of the piezoelectric tube.

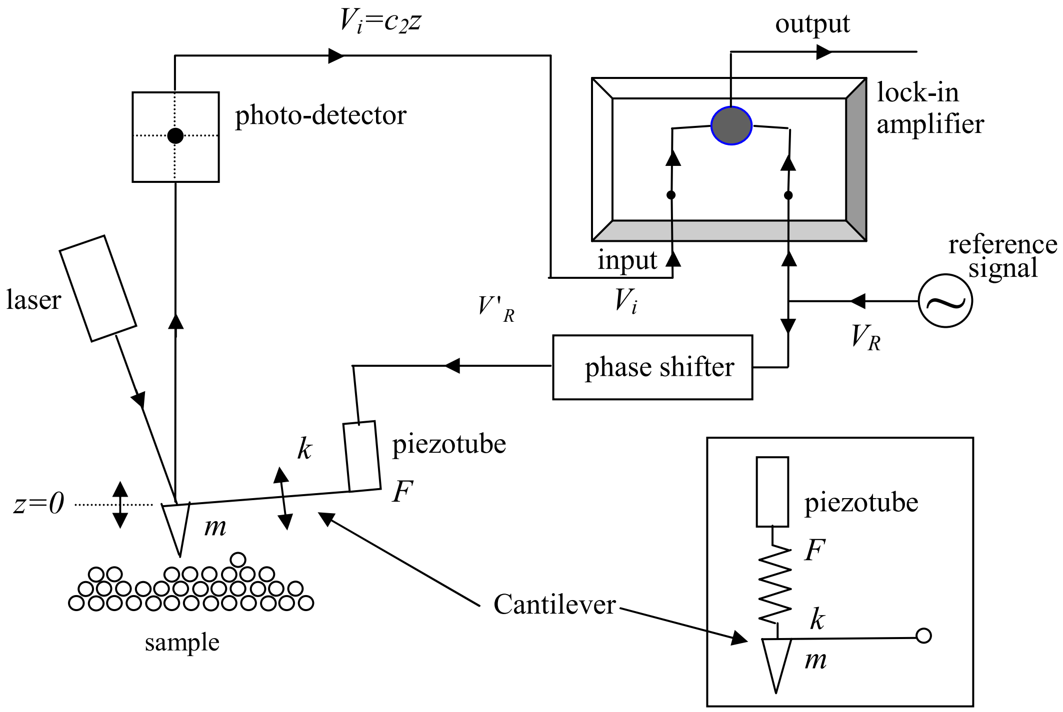


Figure 3.1 A schematic diagram for a scanning probe microscope (SPM). The inset in the lower right corner represents a simplified mechanical model to describe the coupling of the piezotube with the cantilever.

[Part A]

(a) [1.5 points] When $F=F_0\sin\omega t$, $z(t)$ satisfying Eq. (3.1) can be written as $z(t)=A\sin(\omega t-\phi)$, where $A>0$ and $0\le\phi\le\pi$. Find the expression of the amplitude A and $\tan\phi$ in terms of F_0, m, ω, ω_0, and b. Obtain A and the phase ϕ at the resonance frequency $\omega = \omega_0$.

(b) [1 point] A lock-in amplifier shown in Fig.3.1 multiplies an input signal by the lock-in reference signal, $V_R = V_{R0}\sin\omega t$, and then passes *only* the dc (direct current) component of the multiplied signal. Assume that the input signal is given by $V_i = V_{i0}\ \sin(\omega_i t - \phi_i)$. Here V_{R0}, V_{i0}, ω_i, and ϕ_i are all positive given constants. Find the condition on ω (>0) for a non-vanishing output signal. What is the expression for the magnitude of the non-vanishing *dc output signal* at this frequency?

(c) [1.5 points] Passing through the phase shifter, the lock-in reference voltage $V_R = V_{R0}\sin\omega t$ changes to $V'_R = V_{R0}\sin(\omega t + \pi/2)$. V'_R, applied to the piezoelectric tube, drives the cantilever with a force $F = c_1 V'_R$. Then, the photo-detector converts the displacement of the cantilever, z, into a voltage $V_i = c_2 z$. Here c_1 and c_2 are constants. Find the expression for the magnitude of the *dc output signal* at $\omega = \omega_0$.

(d) [2 points] The small change Δm of the cantilever mass shifts the resonance frequency by $\Delta\omega_0$. As a result, the phase ϕ at the original resonance frequency ω_0 shifts by $\Delta\phi$. Find the mass change Δm corresponding to the phase shift $\Delta\phi = \pi/1800$, which is a typical resolution in phase measurements. The physical parameters of the cantilever are given by $m = 1.0\times10^{-12}$ kg, $k = 1.0$ N/m, and $(b/m) = 1.0\times10^{3}$ s^{-1}. Use the approximations $(1+x)^a \approx 1 + ax$ and $\tan(\pi/2 + x) \approx -1/x$ when $|x| << 1$.

[Part B]

From now on let us consider the situation that some forces, besides the driving force discussed in Part A, act on the cantilever due to the sample as shown in Fig.3.1.

(e) [1.5 points] Assuming that the additional force $f(h)$ depends only on the distance h between the cantilever and the sample surface, one can find a new equilibrium position h_0. Near $h = h_0$, we can write $f(h) \approx f(h_0) + c_3(h - h_0)$, where c_3 is a constant in h. Find the new resonance frequency ω'_0 in terms of ω_0, m, and c_3.

(f) [2.5 points] While scanning the surface by moving the sample horizontally, the tip of the cantilever charged with $Q = 6e$ encounters an electron of charge $q = e$ trapped

(localized in space) at some distance below the surface. During the scanning around the electron, the maximum shift of the resonance frequency $\Delta\omega_0 (= \omega'_0 - \omega_0)$ is observed to be much smaller than ω_0. Express the distance d_0 from the cantilever to the trapped electron at the maximum shift in terms of m, q, Q, ω_0, $\Delta\omega_0$, and the Coulomb constant k_e. Evaluate d_0 in nm (1 nm = 1×10^{-9} m) for $\Delta\omega_0 = 20$ s^{-1}.
The physical parameters of the cantilever are $m = 1.0\times10^{-12}$ kg and $k = 1.0$ N/m. Disregard any polarization effect in both the cantilever tip and the surface. Note that $k_e = 1/4\pi\varepsilon_0 = 9.0\times10^9$ N·m^2/C^2 and $e = -1.6\times10^{-19}$ C.

Country Code	Student Code	Question Number
		3

Answer Form

Theoretical Question 3:

(a) $A =$ [ ] and $\tan\phi =$ [ ]

At $\omega = \omega_0$, $A =$ [ ] and $\phi =$ [ ]

(b) The condition on ω for a non-vanishing output signal : [ ]

The magnitude of the dc signal = [ ]

(c) The magnitude of the signal = [ ]

(d) $\Delta m =$ [ ] kg

Country Code	Student Code	Question Number
		3

(e) $\omega'_0 =$

(f) $d_0 =$; Evaluated $d_0 =$ nm.

Theoretical Question 3: *Scanning Probe Microscope*

1. Answers

(a) $A = \dfrac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + b^2\omega^2}}$ and $\tan\phi = \dfrac{b\omega_0}{m(\omega_0^2 - \omega^2)}$. At $\omega = \omega_0$, $A = \dfrac{F_0}{b\omega_0}$

and $\phi = \dfrac{\pi}{2}$.

(b) A non-vanishing dc component exists only when $\omega = \omega_i$.

In this case the amplitude of the dc signal will be $\dfrac{1}{2}V_{i0}V_{R0}\cos\phi_i$.

(c) $\dfrac{c_1 c_2}{2}\dfrac{V_{R0}^2}{b\omega_0}$ at the resonance frequency ω_0.

(d) $\Delta m = 1.7 \times 10^{-18}$ kg.

(e) $\omega'_0 = \omega_0\left(1 - \dfrac{c_3}{m\omega_0^2}\right)^{1/2}$.

(f) $d_0 = \left(k_e \dfrac{qQ}{m\omega_0 \Delta\omega_0}\right)^{1/3}$

$d_0 = 41$ nm.

2. Solutions

(a) [1.5 points]

Substituting $z(t) = A\sin(\omega t - \phi)$ in the equation $m\dfrac{d^2z}{dt^2} + b\dfrac{dz}{dt} + m\omega_0^2 z = F_0 \sin\omega t$

yields,

$$-m\omega^2\sin(\omega t - \phi) + b\omega\cos(\omega t - \phi) + m\omega_0{}^2\sin(\omega t - \phi) = \frac{F_0}{A}\sin\omega t\,. \tag{a1}$$

Collecting terms proportional to $\sin\omega t$ and $\cos\omega t$, one obtains

$$\left\{m(\omega_0^2 - \omega^2)\cos\phi + b\omega\sin\phi - \frac{F_0}{A}\right\}\sin\omega t + \left\{-m(\omega_0^2 - \omega^2)\sin\phi + b\omega\cos\phi\right\}\cos\omega t = 0 \tag{a2}$$

Zeroing the each curly square bracket produces

$$\tan\phi = \frac{b\omega}{m(\omega_0^2 - \omega^2)}, \tag{a3}$$

$$A = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + b^2\omega^2}}\,. \tag{a4}$$

At $\omega = \omega_0$,

$$A = \frac{F_0}{b\omega_0} \text{ and } \phi = \frac{\pi}{2}. \tag{a5}$$

(b) [1 point]

The multiplied signal is

$$\begin{aligned} &V_{i0}\sin(\omega_i t - \phi_i)V_{R0}\sin(\omega t) \\ &= \frac{1}{2}V_{i0}V_{R0}[\cos\{(\omega_i - \omega)t - \phi_i\} - \cos\{(\omega_i + \omega)t - \phi_i\}] \end{aligned} \tag{b1}$$

A non-vanishing dc component exists only when $\omega = \omega_i$. In this case the amplitude of the dc signal will be

$$\frac{1}{2}V_{i0}V_{R0}\cos\phi_i\,. \tag{b2}$$

(c) [1.5 points]

Since the lock-in amplifier measures the ac signal of the same frequency with its reference signal, the frequency of the piezoelectric tube oscillation, the frequency of the cantilever, and the frequency of the photodiode detector should be same. The magnitude of the input signal at the resonance is

$$V_{i0} = c_2 \frac{F_0}{b\omega_0} = \frac{c_1 c_2 V_{R0}}{b\omega_0}. \tag{c1}$$

Then, since the phase of the input signal is $-\frac{\pi}{2}+\frac{\pi}{2}=0$ at the resonance, $\phi_i = 0$ and the lock-in amplifier signal is

$$\frac{1}{2}V_{i0}V_{R0}\cos 0 = \frac{c_1 c_2}{2}\frac{V_{R0}^2}{b\omega_0}. \tag{c2}$$

(d) [2 points]

The original resonance frequency $\omega_0 = \sqrt{\frac{k}{m}}$ is shifted to

$$\sqrt{\frac{k}{m+\Delta m}} = \sqrt{\frac{k}{m}}\left(1+\frac{\Delta m}{m}\right)^{-\frac{1}{2}} \cong \sqrt{\frac{k}{m}}\left(1-\frac{1}{2}\frac{\Delta m}{m}\right) = \omega_0\left(1-\frac{1}{2}\frac{\Delta m}{m}\right). \tag{d1}$$

Thus

$$\Delta\omega_0 = -\frac{1}{2}\omega_0\frac{\Delta m}{m}. \tag{d2}$$

Near the resonance, by substituting $\phi \to \frac{\pi}{2}+\Delta\phi$ and $\omega_0 \to \omega_0 + \Delta\omega_0$ in Eq. (a3), the change of the phase due to the small change of ω_0 (not the change of ω) is

$$\tan\left(\frac{\pi}{2}+\Delta\phi\right) = -\frac{1}{\tan\Delta\phi} = \frac{b}{2m\Delta\omega_0}. \tag{d3}$$

Therefore,

$$\Delta\phi \approx \tan\Delta\phi = -\frac{2m\Delta\omega_0}{b}. \tag{d4}$$

From Eqs. (d2) and (d4),

$$\Delta m = \frac{b}{\omega_0}\Delta\phi = \frac{10^3\cdot 10^{-12}}{10^6}\frac{\pi}{1800} = \frac{\pi}{1.8}10^{-18} = 1.7\times 10^{-18}\ \text{kg}. \tag{d5}$$

(e) [1.5 points]

In the presence of interaction, the equation of motion near the new equilibrium position h_0 becomes

$$m\frac{d^2z}{dt^2}+b\frac{dz}{dt}+m\omega_0^2 z-c_3 z=F_0\sin\omega t \qquad \text{(e1)}$$

where we used $f(h)\approx f(h_0)+c_3 z$ with $z=h-h_0$ being the displacement from the new equilibrium position h_0. Note that the constant term $f(h_0)$ is cancelled at the new equilibrium position.

Thus the original resonance frequency $\omega_0=\sqrt{\frac{k}{m}}$ will be shifted to

$$\omega'_0=\sqrt{\frac{k-c_3}{m}}=\sqrt{\frac{m\omega_0^2-c_3}{m}}=\omega_0\sqrt{1-\frac{c_3}{m\omega_0^2}}\,. \qquad \text{(e3)}$$

Hence the resonance frequency shift is given by

$$\Delta\omega_0=\omega_0\left[\sqrt{1-\frac{c_3}{m\omega_0^2}}-1\right]. \qquad \text{(e4)}$$

(f) [2.5 points]

The maximum shift occurs when the cantilever is on top of the charge, where the interacting force is given by

$$f(h)=k_e\frac{qQ}{h^2}\,. \qquad \text{(f1)}$$

From this,

$$c_3=\left.\frac{df}{dh}\right|_{h=d_0}=-2k_e\frac{qQ}{d_0^3}\,. \qquad \text{(f2)}$$

Since $\Delta\omega_0 << \omega_0$, we can approximate Eq. (e4) as

$$\Delta\omega_0\approx-\frac{c_3}{2m\omega_0}\,. \qquad \text{(f3)}$$

From Eqs. (f2) and (f3), we have

$$\Delta\omega_0=-\frac{1}{2m\omega_0}\left(-2k_e\frac{qQ}{d_0^3}\right)=k_e\frac{qQ}{m\omega_0 d_0^3}\,. \qquad \text{(f4)}$$

Here $q=e=-1.6\times10^{-19}$ Coulomb and $Q=6e=-9.6\times10^{-19}$ Coulomb. Using the values provided,

$$d_0 = \left(k_e \frac{qQ}{m\omega_0 \Delta\omega_0} \right)^{1/3} = 4.1 \times 10^{-8} \text{ m} = 41 \text{ nm}. \qquad \text{(f5)}$$

Thus the trapped electron is 41 nm from the cantilever.

3. Mark Distribution

No.	Total Pt.	Partial Pt.	Contents
(a)	1.5	0.7	Equations for A and ϕ (substitution and manipulation)
		0.4	Correct answers for A and ϕ
		0.4	A and ϕ at ω_0
(b)	1.0	0.4	Equation for the multiplied signal
		0.3	Condition for the non-vanishing dc output
		0.3	Correct answer for the dc output
(c)	1.5	0.6	Relation between V_i and V_R
		0.4	Condition for the maximum dc output
		0.5	Correct answer for the magnitude of dc output
(d)	2.0	0.5	Relation between Δm and $\Delta\omega_0$
		1.0	Relations between $\Delta\omega_0$ (or Δm) and $\Delta\phi$
		0.5	Correct answer (Partial credit of 0.2 for the wrong sign.)
(e)	1.5	1.0	Modification of the equation with $f(h)$ and use of a proper approximation for the equation
		0.5	Correct answer
(f)	2.5	0.5	Use of a correct formula of Coulomb force
		0.3	Evaluation of c_3
		0.6	Use of the result in (e) for either $\Delta\omega_0$ or $\omega'^2_0-\omega_0^2$
		0.6	Expression for d_0
		0.5	Correct answer
Total	10		

Th 1 AN ILL FATED SATELLITE

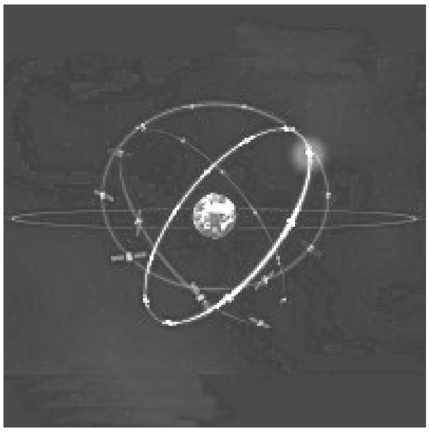


Image: ESA

The most frequent orbital manoeuvres performed by spacecraft consist of velocity variations along the direction of flight, namely accelerations to reach higher orbits or brakings done to initiate re-entering in the atmosphere. In this problem we will study the orbital variations when the engine thrust is applied in a radial direction.

To obtain numerical values use: Earth radius $R_T = 6.37\cdot10^6$ m, Earth surface gravity $g = 9.81\,\text{m/s}^2$, and take the length of the sidereal day to be $T_0 = 24.0$ h.

We consider a geosynchronous[1] communications satellite of mass m placed in an equatorial circular orbit of radius r_0. These satellites have an "apogee engine" which provides the tangential thrusts needed to reach the final orbit.

Marks are indicated at the beginning of each subquestion, in parenthesis.

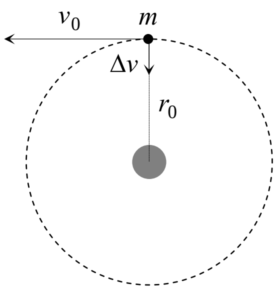
Question 1

1.1 (0.3) Compute the numerical value of r_0.

1.2 (0.3+0.1) Give the analytical expression of the velocity v_0 of the satellite as a function of g, R_T, and r_0, and calculate its numerical value.

1.3 (0.4+0.4) Obtain the expressions of its angular momentum L_0 and mechanical energy E_0, as functions of v_0, m, g and R_T.

Once this geosynchronous circular orbit has been reached (see Figure F-1), the satellite has been stabilised in the desired location, and is being readied to do its work, an error by the ground controllers causes the apogee engine to be fired again. The thrust happens to be directed towards the Earth and, despite the quick reaction of the ground crew to shut the engine off, an unwanted velocity variation Δv is imparted on the satellite. We characterize this boost by the parameter $\beta = \Delta v / v_0$. The duration of the engine burn is always negligible with respect to any other orbital times, so that it can be considered as instantaneous.



F-1

Question 2

Suppose $\beta < 1$.

2.1 (0.4+0.5) Determine the parameters of the new orbit[2], s*emi-latus-rectum* l and *eccentricity* ε, in terms of r_0 and β.

2.2 (1.0) Calculate the angle α between the major axis of the new orbit and the position vector at the accidental misfire.

2.3 (1.0+0.2) Give the analytical expressions of the perigee r_{min} and apogee r_{max} distances to the Earth centre, as functions of r_0 and β, and calculate their numerical values for $\beta = 1/4$.

2.4 (0.5+0.2) Determine the period of the new orbit, T, as a function of T_0 and β, and calculate its numerical value for $\beta = 1/4$.

---

[1] Its revolution period is T_0.

[2] See the "hint".

Question 3

3.1 (0.5) Calculate the minimum boost parameter, β_{esc}, needed for the satellite to escape Earth gravity.

3.2 (1.0) Determine in this case the closest approach of the satellite to the Earth centre in the new trajectory, r'_{min}, as a function of r_0.

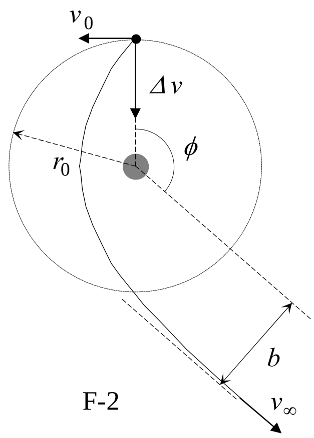
Question 4

Suppose $\beta > \beta_{esc}$.

4.1 (1.0) Determine the residual velocity at the infinity, v_∞, as a function of v_0 and β.

4.2 (1.0) Obtain the "impact parameter" b of the asymptotic escape direction in terms of r_0 and β. (See Figure F-2).

4.3 (1.0+0.2) Determine the angle ϕ of the asymptotic escape direction in terms of β. Calculate its numerical value for $\beta = \frac{3}{2}\beta_{esc}$.

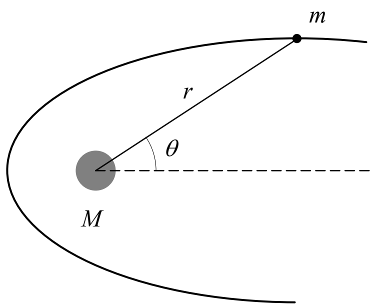


F-2

HINT

Under the action of central forces obeying the inverse-square law, bodies follow trajectories described by ellipses, parabolas or hyperbolas. In the approximation $m << M$ the gravitating mass M is at one of the focuses. Taking the origin at this focus, the general polar equation of these curves can be written as (see Figure F-3)

$$r(\theta) = \frac{l}{1-\varepsilon\ \cos\theta}$$



F-3

where l is a positive constant named the *semi-latus-rectum* and ε is the *eccentricity* of the curve. In terms of constants of motion:

$$l = \frac{L^2}{GMm^2} \qquad \text{and} \qquad \varepsilon = \left(1 + \frac{2EL^2}{G^2M^2m^3}\right)^{1/2}$$

where G is the Newton constant, L is the modulus of the angular momentum of the orbiting mass, with respect to the origin, and E is its mechanical energy, with zero potential energy at infinity.

We may have the following cases:

i) If $0 \le \varepsilon < 1$, the curve is an ellipse (circumference for $\varepsilon = 0$).

ii) If $\varepsilon = 1$, the curve is a parabola.

iii) If $\varepsilon > 1$, the curve is a hyperbola.

COUNTRY CODE	STUDENT CODE	PAGE NUMBER	TOTAL No OF PAGES

Th 1 ANSWER SHEET

Question	Basic formulas and ideas used	Analytical results	Numerical results	Marking guideline
1.1			$r_0 =$	0.3
1.2		$v_0 =$	$v_0 =$	0.4
1.3		$L_0 =$ $E_0 =$		0.4 0.4
2.1		$l =$ $\varepsilon =$		0.4 0.5
2.2			$\alpha =$	1.0
2.3		$r_{max} =$ $r_{min} =$	$r_{max} =$ $r_{min} =$	1.2
2.4		$T =$	$T =$	0.7
3.1			$\beta_{esc} =$	0.5
3.2		$r'_{min} =$		1.0
4.1		$v_\infty =$		1.0
4.2		$b =$		1.0
4.3		$\phi =$	$\phi =$	1.2

Th1 AN ILL FATED SATELLITE
SOLUTION

1.1 and 1.2

$$\left.\begin{array}{l} G\dfrac{M_T m}{r_0^2} = m\dfrac{v_0^2}{r_0} \\ v_0 = \dfrac{2\pi r_0}{T_0} \\ g = \dfrac{GM_T}{R_T^2} \end{array}\right\} \Rightarrow \left\{\begin{array}{ll} r_0 = \left(\dfrac{g\,R_T^2\,T_0^2}{4\pi^2}\right)^{1/3} & \Rightarrow \boxed{r_0 = 4.22\cdot 10^7 \text{ m}} \\ \boxed{v_0 = R_T\sqrt{\dfrac{g}{r_0}}} & \Rightarrow \boxed{v_0 = 3.07\cdot 10^3 \text{ m/s}} \end{array}\right.$$

1.3

$$L_0 = r_0\, m v_0 = \frac{g\,R_T^2}{v_0^2} m v_0 \quad\Rightarrow\quad \boxed{L_0 = \frac{m\,g\,R_T^2}{v_0}}$$

$$E_0 = \frac{1}{2}mv_0^2 - G\frac{M_T m}{r_0} = \frac{1}{2}mv_0^2 - \frac{g\,R_T^2\,m}{r_0} = \frac{1}{2}mv_0^2 - mv_0^2 \quad\Rightarrow\quad \boxed{E_0 = -\frac{1}{2}mv_0^2}$$

2.1

The value of the *semi-latus-rectum* l is obtained taking into account that the orbital angular momentum is the same in both orbits. That is

$$l = \frac{L_0^2}{G M_T\, m^2} = \frac{m^2 g^2 R_T^4}{v_0^2}\frac{1}{g\,R_T^2\, m^2} = \frac{g\,R_T^2}{v_0^2} = r_0 \qquad\Rightarrow\qquad \boxed{l = r_0}$$

The eccentricity value is

$$\varepsilon^2 = 1 + \frac{2\,E\,L_0^2}{G^2 M_T^2\, m^3}$$

where E is the new satellite mechanical energy

$$E = \frac{1}{2}m\left(v_0^2 + \Delta v^2\right) - G\frac{M_T m}{r_0} = \frac{1}{2}m\Delta v^2 + E_0 = \frac{1}{2}m\Delta v^2 - \frac{1}{2}mv_0^2$$

that is

$$E = \frac{1}{2}mv_0^2\left(\frac{\Delta v^2}{v_0^2} - 1\right) = \frac{1}{2}mv_0^2\left(\beta^2 - 1\right)$$

Combining both, one gets $\boxed{\varepsilon = \beta}$

This is an elliptical trajectory because $\varepsilon = \beta < 1$.

2.2

The initial and final orbits cross at P, where the satellite engine fired instantaneously (see Figure 4). At this point

$$r(\theta = \alpha) = r_0 = \frac{r_0}{1-\beta\cos\alpha} \quad \Rightarrow \quad \boxed{\alpha = \frac{\pi}{2}}$$

2.3

From the trajectory expression one immediately obtains that the maximum and minimum values of r correspond to $\theta = 0$ and $\theta = \pi$ respectively (see Figure 4). Hence, they are given by

$$r_{max} = \frac{l}{1-\varepsilon} \qquad r_{min} = \frac{l}{1+\varepsilon}$$

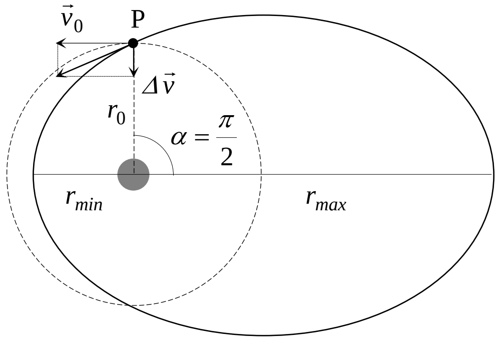


Figure 4

that is

$$\boxed{r_{\max} = \frac{r_0}{1-\beta}} \quad \text{and} \quad \boxed{r_{\min} = \frac{r_0}{1+\beta}}$$

For $\beta = 1/4$, one gets

$$\boxed{r_{max} = 5.63\cdot 10^7 \text{ m}; \quad r_{min} = 3.38\cdot 10^7 \text{ m}}$$

The distances r_{max} and r_{min} can also be obtained from mechanical energy and angular momentum conservation, taking into account that $\vec{r}$ and $\vec{v}$ are orthogonal at apogee and at perigee

$$E = \frac{1}{2}mv_0^2\left(\beta^2 - 1\right) = \frac{1}{2}mv^2 - \frac{gR_T^2 m}{r}$$

$$L_0 = \frac{mgR_T^2}{v_0} = m\,v\,r$$

What remains of them, after eliminating v, is a second-degree equation whose solutions are r_{max} and r_{min}.

2.4

By the Third Kepler Law, the period T in the new orbit satisfies that

$$\frac{T^2}{a^3} = \frac{T_0^2}{r_0^3}$$

where a, the semi-major axis of the ellipse, is given by

$$a = \frac{r_{max} + r_{min}}{2} = \frac{r_0}{1-\beta^2}$$

Therefore

$$\boxed{T = T_0\left(1-\beta^2\right)^{-3/2}}$$

For $\beta = 1/4$ $\qquad \boxed{T = T_0\left(\frac{15}{16}\right)^{-3/2} = 26.4\,\text{h}}$

3.1

Only if the satellite follows an open trajectory it can escape from the Earth gravity attraction. Then, the orbit eccentricity has to be equal or larger than one. The minimum boost corresponds to a parabolic trajectory, with $\varepsilon = 1$

$$\varepsilon = \beta \qquad \Rightarrow \qquad \boxed{\beta_{esc} = 1}$$

This can also be obtained by using that the total satellite energy has to be zero to reach infinity (E_p = 0) without residual velocity (E_k = 0)

$$E = \frac{1}{2} m v_0^2 \left(\beta_{esc}^2 - 1\right) = 0 \quad \Rightarrow \qquad \beta_{esc} = 1$$

This also arises from $T = \infty$ or from $r_{max} = \infty$.

3.2

Due to $\varepsilon = \beta_{esc} = 1$, the polar parabola equation is

$$r = \frac{l}{1 - \cos\theta}$$

where the semi-latus-rectum continues to be $l = r_0$. The minimum Earth - satellite distance corresponds to $\theta = \pi$, where

$$\boxed{r'_{min} = \frac{r_0}{2}}$$

This also arises from energy conservation (for E = 0) and from the equality between the angular momenta (L_0) at the initial point P and at maximum approximation, where $\vec{r}$ and $\vec{v}$ are orthogonal.

4.1

If the satellite escapes to infinity with residual velocity v_∞ , by energy conservation

$$E = \frac{1}{2} m v_0^2 \left(\beta^2 - 1\right) = \frac{1}{2} m v_\infty^2 \qquad \Rightarrow$$

$$\boxed{v_\infty = v_0 \left(\beta^2 - 1\right)^{1/2}}$$

4.2

As $\varepsilon = \beta > \beta_{esc} = 1$ the satellite trajectory will be a hyperbola.

The satellite angular momentum is the same at P than at the point where its residual velocity is v_∞ (Figure 5), thus

$$m\, v_0 r_0 = m\, v_\infty b$$

So

$$b = r_0 \frac{v_0}{v_\infty} \quad \Rightarrow \quad \boxed{b = r_0 \left(\beta^2 - 1\right)^{-1/2}}$$

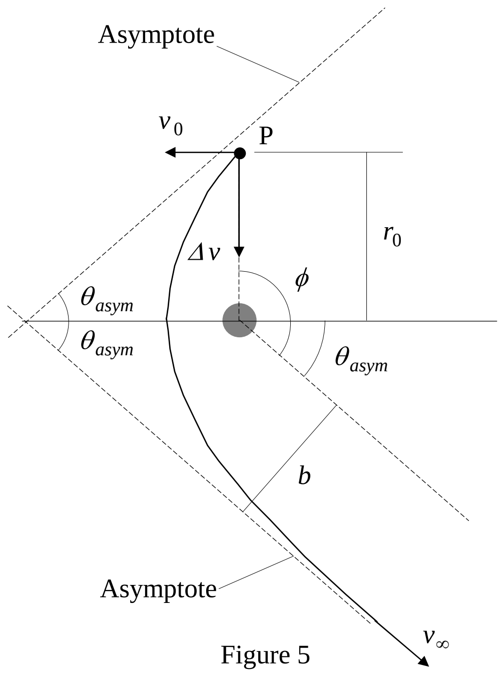


Figure 5

4.3

The angle between each asymptote and the hyperbola axis is that appearing in its polar equation in the limit $r \to \infty$. This is the angle for which the equation denominator vanishes

$$1 - \beta \cos\theta_{asym} = 0 \quad \Rightarrow \quad \theta_{asym} = \cos^{-1}\left(\frac{1}{\beta}\right)$$

According to Figure 5

$$\phi = \frac{\pi}{2} + \theta_{asym} \quad \Rightarrow \quad \boxed{\phi = \frac{\pi}{2} + \cos^{-1}\left(\frac{1}{\beta}\right)}$$

For $\beta = \frac{3}{2}\beta_{esc} = \frac{3}{2}$, one gets $\boxed{\phi = 138^o = 2.41\,\text{rad}}$

Th 1 ANSWER SHEET

Question	Basic formulas and ideas used	Analytical results	Numerical results	Marking guideline
1.1	$G\frac{M_T m}{r_0^2} = m\frac{v_0^2}{r_0}$		$r_0 = 4.22\cdot 10^7$ m	0.3
1.2	$v_0 = \frac{2\pi r_0}{T_0}$ $g = \frac{GM_T}{R_T^2}$	$v_0 = R_T\sqrt{\frac{g}{r_0}}$	$v_0 = 3.07\cdot 10^3$ m/s	0.3 + 0.1
1.3	$\vec{L} = m\,\vec{r}\times\vec{v}$ $E = \frac{1}{2}mv^2 - G\frac{Mm}{r}$	$L_0 = \frac{mgR_T^2}{v_0}$ $E_0 = -\frac{1}{2}mv_0^2$		0.4 0.4
2.1	Hint on the conical curves	$l = r_0$ $\varepsilon = \beta$		0.4 0.5
2.2			$\alpha = \frac{\pi}{2}$	1.0
2.3	Results of 2.1, or conservation of E and L	$r_{max} = \frac{r_0}{1-\beta}$ $r_{min} = \frac{r_0}{1+\beta}$	$r_{\max} = 5.63\cdot 10^7$ m $r_{\min} = 3.38\cdot 10^7$ m	1.0 + 0.2
2.4	Third Kepler's Law	$T = T_0\left(1-\beta^2\right)^{-3/2}$	$T = 26.4$ h	0.5 + 0.2
3.1	$\varepsilon = 1$, $E = 0$, $T = \infty$ or $r_{max} = \infty$		$\beta_{esc} = 1$	0.5
3.2	$\varepsilon = 1$ and results of 2.1	$r'_{min} = \frac{r_0}{2}$		1.0
4.1	Conservation of E	$v_\infty = v_0\left(\beta^2 - 1\right)^{1/2}$		1.0
4.2	Conservation of L	$b = r_0\left(\beta^2 - 1\right)^{-1/2}$		1.0
4.3	Hint on the conical curves	$\phi = \frac{\pi}{2} + \cos^{-1}\left(\frac{1}{\beta}\right)$	$\phi = 138^\circ = 2.41\,\text{rad}$	1.0 + 0.2

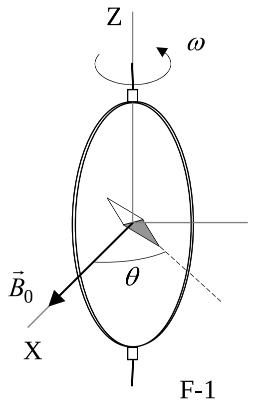
Th 2 ABSOLUTE MEASUREMENTS OF ELECTRICAL QUANTITIES

The technological and scientific transformations underwent during the XIX century produced a compelling need of universally accepted standards for the electrical quantities. It was thought the new absolute units should only rely on the standards of length, mass and time established after the French Revolution. An intensive experimental work to settle the values of these units was developed from 1861 until 1912. We propose here three case studies.

Marks are indicated at the beginning of each subquestion, in parenthesis.

Determination of the ohm (Kelvin)

A closed circular coil of N turns, radius a and total resistance R is rotated with uniform angular velocity ω about a vertical diameter in a horizontal magnetic field $\vec{B}_0 = B_0\,\vec{i}$.



F-1

1. (0.5+1.0) Compute the electromotive force ε induced in the coil, and also the mean power[1] $\langle P \rangle$ required for maintaining the coil in motion. Neglect the coil self inductance.

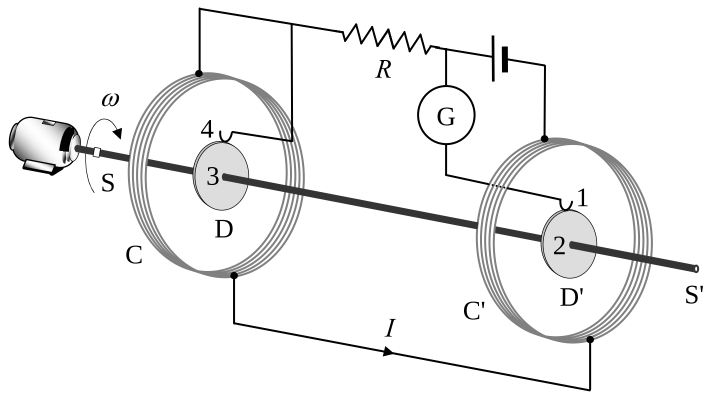
A small magnetic needle is placed at the center of the coil, as shown in Figure F-1. It is free to turn slowly around the Z axis in a horizontal plane, but it cannot follow the rapid rotation of the coil.

2. (2.0) Once the stationary regime is reached, the needle will set at a direction making a small angle θ with $\vec{B}_0$. Compute the resistance R of the coil in terms of this angle and the other parameters of the system.

Lord Kelvin used this method in the 1860s to set the absolute standard for the ohm. To avoid the rotating coil, Lorenz devised an alternative method used by Lord Rayleigh and Ms. Sidgwick, that we analyze in the next paragraphs.

Determination of the ohm (Rayleigh, Sidgwick).

The experimental setup is shown in Figure F-2. It consists of two identical metal disks D and D' of radius b mounted on the conducting shaft SS'. A motor rotates the set at an angular velocity ω, which can be adjusted for measuring R. Two identical coils C and C' (of radius a and with $\boldsymbol{N}$ turns each) surround the disks. They are connected in such a form that the current I flows through them in opposite directions. The whole apparatus serves to measure the resistance R.



F-2

[1] The mean value $\langle X \rangle$ of a quantity $X(t)$ in a periodic system of period T is $\langle X \rangle = \frac{1}{T}\int_0^T X(t)\,dt$

You may need one or more of these integrals:

$$\int_0^{2\pi} \sin x\,dx = \int_0^{2\pi} \cos x\,dx = \int_0^{2\pi} \sin x \cos x\,dx = 0\,, \quad \int_0^{2\pi} \sin^2 x\,dx = \int_0^{2\pi} \cos^2 x\,dx = \pi\,, \quad \text{and later} \quad \int x^n\,dx = \frac{1}{n+1}x^{n+1}$$

3. (2.0) Assume that the current I flowing through the coils C and C' creates a uniform magnetic field B around D and D', equal to the one at the centre of the coil. Compute[1] the electromotive force $\mathcal{E}$ induced between the rims 1 and 4, assuming that the distance between the coils is much larger than the radius of the coils and that $a >> b$.

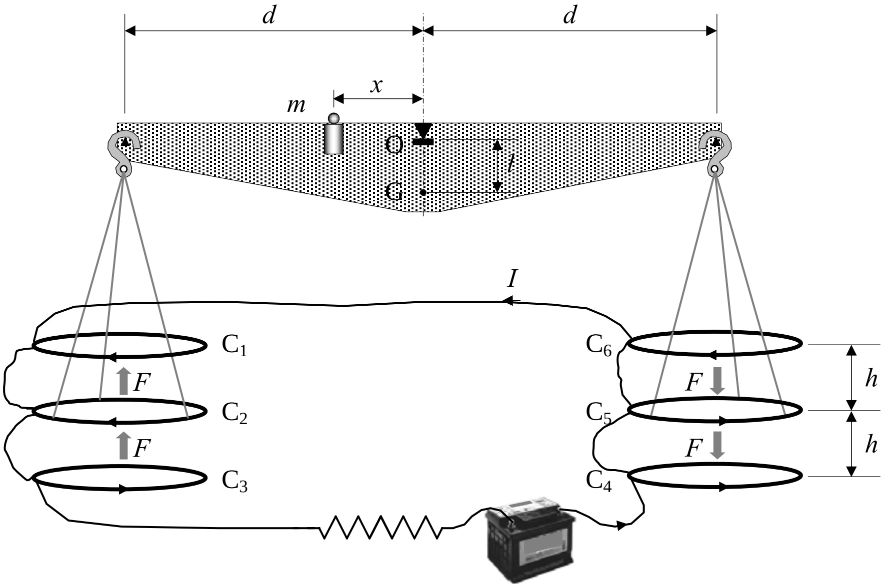
The disks are connected to the circuit by brush contacts at their rims 1 and 4. The galvanometer G detects the flow of current through the circuit 1-2-3-4.

4. (0.5) The resistance R is measured when G reads zero. Give R in terms of the physical parameters of the system.

Determination of the ampere

Passing a current through two conductors and measuring the force between them provides an absolute determination of the current itself. The "Current Balance" designed by Lord Kelvin in 1882 exploits this method. It consists of six identical single turn coils $C_1 \dots C_6$ of radius a, connected in series. As shown in Figure F-3, the fixed coils C_1, C_3, C_4, and C_6 are on two horizontal planes separated by a small distance $2h$. The coils C_2 and C_5 are carried on balance arms of length d, and they are, in equilibrium, equidistant from both planes.

The current I flows through the various coils in such a direction that the magnetic force on C_2 is upwards while that on C_5 is downwards. A mass m at a distance x from the fulcrum O is required to restore the balance to the equilibrium position described above when the current flows through the circuit.



F-3

5. (1.0) Compute the force F on C_2 due to the magnetic interaction with C_1. For simplicity assume that the force per unit length is the one corresponding to two long, straight wires carrying parallel currents.

6. (1.0) The current I is measured when the balance is in equilibrium. Give the value of I in terms of the physical parameters of the system. The dimensions of the apparatus are such that we can neglect the mutual effects of the coils on the left and on the right.

Let M be the mass of the balance (except for m and the hanging parts), G its centre of mass and l the distance $\overline{\mathrm{OG}}$.

7. (2.0) The balance equilibrium is stable against deviations producing small changes δz in the height of C_2 and $-\delta z$ in C_5. Compute[2] the maximum value $\delta z_{\max}$ so that the balance still returns towards the equilibrium position when it is released.

[2] Consider that the coils centres remain approximately aligned.

Use the approximations $\dfrac{1}{1 \pm \beta} \approx 1 \mp \beta + \beta^2$ or $\dfrac{1}{1 \pm \beta^2} \approx 1 \mp \beta^2$ for $\beta << 1$, and $\sin\theta \approx \tan\theta$ for small θ.

COUNTRY CODE	STUDENT CODE	PAGE NUMBER	TOTAL No OF PAGES

Th 2 ANSWER SHEET

Question	Basic formulas used	Analytical results	Marking guideline
1		$\varepsilon =$ $\langle P \rangle =$	1.5
2		$R =$	2.0
3		$\varepsilon =$	2.0
4		$R =$	0,5
5		$F =$	1.0
6		$I =$	1.0
7		$\delta z_{\max} =$	2.0

Th 2 ABSOLUTE MEASUREMENTS OF ELECTRICAL QUANTITIES

SOLUTION

1. After some time *t*, the normal to the coil plane makes an angle ωt with the magnetic field $\vec{B}_0 = B_0 \vec{i}$. Then, the magnetic flux through the coil is

$$\phi = N \; \vec{B}_0 \cdot \vec{S}$$

where the vector surface $\vec{S}$ is given by $\vec{S} = \pi a^2 \left(\cos\omega t \, \vec{i} + \sin\omega t \, \vec{j} \right)$

Therefore $\phi = N \pi a^2 B_0 \cos\omega t$

The induced electromotive force is

$$\varepsilon = -\frac{d\phi}{dt} \qquad \Rightarrow \qquad \boxed{\varepsilon = N \pi a^2 B_0 \omega \sin\omega t}$$

The instantaneous power is $P = \varepsilon^2 / \mathrm{R}$, therefore

$$\boxed{\langle P \rangle = \frac{\left(N \pi a^2 B_0 \omega\right)^2}{2R}}$$

where we used $< \sin^2\omega t >= \frac{1}{T}\int_0^T \sin^2 \omega t \, dt = \frac{1}{2}$

2. The total field at the center the coil at the instant *t* is

$$\vec{B}_t = \vec{B}_0 + \vec{B}_i$$

where $\vec{B}_i$ is the magnetic field due to the induced current $\vec{B}_i = B_i \left(\cos\omega t \, \vec{i} + \sin\omega t \, \vec{j} \right)$

with $\quad B_i = \frac{\mu_0 N I}{2a} \quad$ and $\quad I = \varepsilon / R$

Therefore $\quad B_i = \frac{\mu_0 N^2 \pi a B_0 \omega}{2R} \sin\omega t$

The mean values of its components are

$$\langle B_{ix} \rangle = \frac{\mu_0 N^2 \pi a B_0 \omega}{2R} \langle \sin\omega t \cos\omega t \rangle = 0$$

$$\langle B_{iy} \rangle = \frac{\mu_0 N^2 \pi a B_0 \omega}{2R} \langle \sin^2 \omega t \rangle = \frac{\mu_0 N^2 \pi a B_0 \omega}{4R}$$

And the mean value of the total magnetic field is

$$\langle \vec{B}_t \rangle = B_0 \, \vec{i} + \frac{\mu_0 N^2 \pi a B_0 \omega}{4R} \vec{j}$$

The needle orients along the mean field, therefore

$$\tan\theta = \frac{\mu_0 N^2 \pi a \omega}{4R}$$

Finally, the resistance of the coil measured by this procedure, in terms of θ, is

$$\boxed{R = \frac{\mu_0 N^2 \pi a \omega}{4 \tan\theta}}$$

3. The force on a unit positive charge in a disk is radial and its modulus is

$$\left|\vec{v} \times \vec{B}\right| = v B = \omega r B$$

where B is the magnetic field at the center of the coil

$$B = N \frac{\mu_0 I}{2a}$$

Then, the electromotive force (e.m.f.) induced on each disk by the magnetic field B is

$$\varepsilon_D = \varepsilon_{D'} = B\omega \int_0^b r\, dr = \frac{1}{2} B \omega b^2$$

Finally, the induced e.m.f. between 1 and 4 is $\varepsilon = \varepsilon_D + \varepsilon_{D'}$

$$\boxed{\varepsilon = N \frac{\mu_0 b^2 \omega I}{2a}}$$

4. When the reading of G vanishes, $I_G = 0$ and Kirchoff laws give an immediate answer. Then we have

$$\varepsilon = I R \qquad \Rightarrow \qquad \boxed{R = N \frac{\mu_0 b^2 \omega}{2a}}$$

5. The force per unit length f between two indefinite parallel straight wires separated by a distance h is.

$$f = \frac{\mu_0}{2\pi} \frac{I_1 I_2}{h}$$

for $I_1 = I_2 = I$ and length $2\pi a$, the force F induced on C_2 by the neighbor coils C_1 is

$$\boxed{F = \frac{\mu_0 a}{h} I^2}$$

6. In equilibrium

$$m g x = 4 F d$$

Then

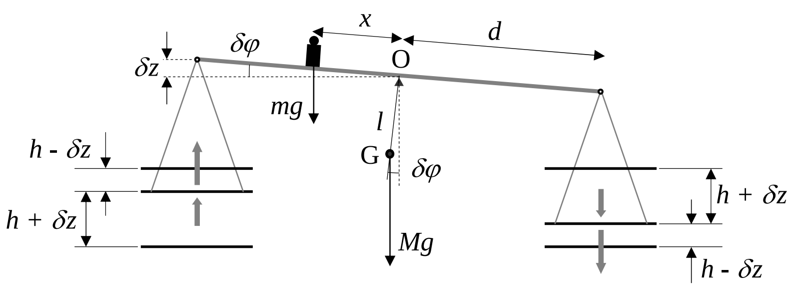
$$m g x = \frac{4 \mu_0 a d}{h} I^2 \qquad (1)$$

so that

$$\boxed{I = \left(\frac{m g h x}{4 \mu_0 a d} \right)^{1/2}}$$

7. The balance comes back towards the equilibrium position for a little angular deviation $\delta\varphi$ if the gravity torques with respect to the fulcrum O are greater than the magnetic torques.

$$Mgl\sin\delta\varphi + mgx\cos\delta\varphi > 2\mu_0 aI^2\left(\frac{1}{h-\delta z}+\frac{1}{h+\delta z}\right)d\cos\delta\varphi$$



Therefore, using the suggested approximation

$$Mgl\sin\delta\varphi + mgx\cos\delta\varphi > \frac{4\mu_0 adI^2}{h}\left(1+\frac{\delta z^2}{h^2}\right)\cos\delta\varphi$$

Taking into account the equilibrium condition (1), one obtains

$$M\,g\,l\,sin\,\delta\varphi > m\,g\,x\frac{\delta z^2}{h^2}cos\,\delta\varphi$$

Finally, for $\tan\delta\varphi \approx \sin\delta\varphi = \dfrac{\delta z}{d}$

$$\delta z < \frac{Mlh^2}{mxd} \qquad \Rightarrow \qquad \boxed{\delta z_{\max} = \frac{Mlh^2}{mxd}}$$

Th 2 ANSWER SHEET

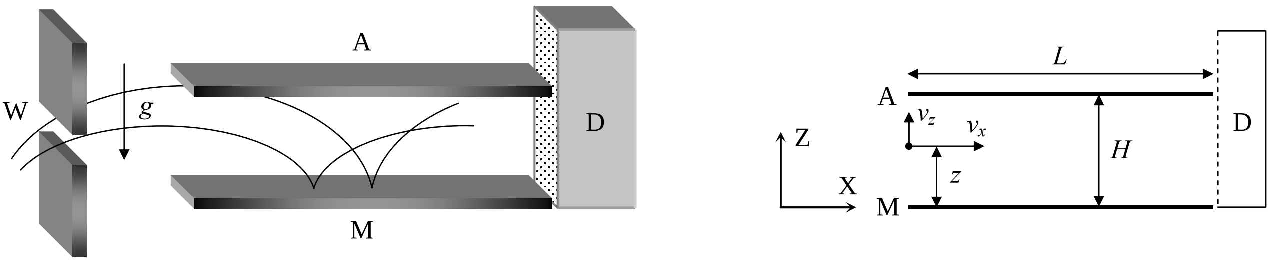
Question	Basic formulas and ideas used	Analytical results	Marking guideline
1	$\Phi = N\,\vec{B}_0 \cdot \vec{S}$ $\mathcal{E} = -\dfrac{d\Phi}{dt}$ $P = \dfrac{\mathcal{E}^2}{R}$	$\mathcal{E} = N\pi a^2 B_0 \omega \sin\omega t$ $\langle P \rangle = \dfrac{(N\pi a^2 B_0 \omega)^2}{2R}$	0.5 1.0
2	$\vec{B} = \vec{B}_0 + \vec{B}_i$ $B_i = \dfrac{\mu_0 N}{2a} I$ $\tan\theta = \dfrac{\langle B_y \rangle}{\langle B_x \rangle}$	$R = \dfrac{\mu_0 N^2 \pi a \omega}{4\tan\theta}$	2.0
3	$\vec{E} = \vec{v} \times \vec{B}$ $v = \omega r$ $B = N\dfrac{\mu_0 I}{2a}$ $\mathcal{E} = \int_0^b \vec{E}\, d\vec{r}$	$\mathcal{E} = N\dfrac{\mu_0 b^2 \omega I}{2a}$	2.0
4	$\mathcal{E} = RI$	$R = N\dfrac{\mu_0 b^2 \omega}{2a}$	0,5
5	$f = \dfrac{\mu_0}{2\pi}\dfrac{I\,I'}{h}$	$F = \dfrac{\mu_0\, a}{h} I^2$	1.0
6	$mgx = 4Fd$	$I = \left(\dfrac{mghx}{4\mu_0\, ad}\right)^{1/2}$	1.0
7	$\Gamma_{grav} > \Gamma_{mag}$	$\delta z_{\max} = \dfrac{Mlh^2}{mxd}$	2.0

Th 3 NEUTRONS IN A GRAVITATIONAL FIELD

In the familiar classical world, an elastic bouncing ball on the Earth's surface is an ideal example for perpetual motion. The ball is trapped: it can not go below the surface or above its turning point. It will remain bounded in this state, turning down and bouncing up once and again, forever. Only air drag or inelastic bounces could stop the process and will be ignored in the following.

A group of physicists from the Institute Laue - Langevin in Grenoble reported[1] in 2002 experimental evidence on the behaviour of neutrons in the gravitational field of the Earth. In the experiment, neutrons moving to the right were allowed to fall towards a horizontal crystal surface acting as a neutron mirror, where they bounced back elastically up to the initial height once and again.

The setup of the experiment is sketched in Figure F-1. It consists of the opening W, the neutron mirror M (at height z = 0), the neutron absorber A (at height $z = H$ and with length L) and the neutron detector D. The beam of neutrons flies with constant horizontal velocity component v_x from W to D through the cavity between A and M. All the neutrons that reach the surface of A are absorbed and disappear from the experiment. Those that reach the surface of M are reflected elastically. The detector D counts the transmission rate $N(H)$, that is, the total number of neutrons that reach D per unit time.



F-1

Marks are indicated at the beginning of each subquestion, in parenthesis.

The neutrons enter the cavity with a wide range of positive and negative vertical velocities, v_z. Once in the cavity, they fly between the mirror below and the absorber above.

1. (1.5) Compute classically the range of vertical velocities $v_z(z)$ of the neutrons that, entering at a height z, can arrive at the detector D. Assume that L is much larger than any other length in the problem.

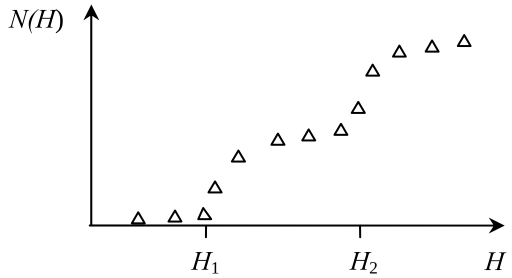
2. (1.5) Calculate classically the minimum length L_c of the cavity to ensure that all neutrons outside the previous velocity range, regardless of the values of z, are absorbed by A. Use v_x = 10 m s^{-1} and H = 50 μm.

The neutron transmission rate $N(H)$ is measured at D. We expect that it increases monotonically with H.

3. (2.5) Compute the classical rate $N_c(H)$ assuming that neutrons arrive at the cavity with vertical velocity v_z and at height z, being all the values of v_z and z equally probable. Give the answer in terms of ρ, the constant number of neutrons per unit time, per unit vertical velocity, per unit height, that enter the cavity with vertical velocity v_z and at height z.

[1] V. V. Nesvizhevsky et al. "Quantum states of neutrons in the Earth's gravitational field." Nature, **415** (2002) 297. Phys Rev D 67, 102002 (2003).

The experimental results obtained by the Grenoble group disagree with the above classical predictions, showing instead that the value of $N(H)$ experiences sharp increases when H crosses some critical heights H_1, H_2 ... (Figure F-2 shows a sketch). In other words, the experiment showed that the vertical motion of neutrons bouncing on the mirror is quantized. In the language that Bohr and Sommerfeld used to obtain the energy levels of the hydrogen atom, this can be written as: "The action S of these neutrons along the vertical direction is an integer multiple of the Planck action constant h". Here S is given by



F-2

$$S = \int p_z(z)\, dz = n\, h, \quad n = 1, 2, 3 \ldots \qquad \text{(Bohr-Sommerfeld quantization rule)}$$

where p_z is the vertical component of the classical momentum, and the integral covers a whole bouncing cycle. Only neutrons with these values of S are allowed in the cavity.

4. (2.5) Compute the turning heights H_n and energy levels E_n (associated to the vertical motion) using the Bohr-Sommerfeld quantization condition. Give the numerical result for H_1 in μm and for E_1 in eV.

The uniform initial distribution ρ of neutrons at the entrance changes, during the flight through a long cavity, into the step-like distribution detected at D (see Figure F-2). From now on, we consider for simplicity the case of a long cavity with $H < H_2$. Classically, all neutrons with energies in the range considered in question 1 were allowed through it, while quantum mechanically only neutrons in the energy level E_1 are permitted. According to the time-energy Heisenberg uncertainty principle, this reshuffling requires a minimum time of flight. The uncertainty of the vertical motion energy will be significant if the cavity length is small. This phenomenon will give rise to the widening of the energy levels.

5. (2.0) Estimate the minimum time of flight t_q and the minimum length L_q of the cavity needed to observe the first sharp increase in the number of neutrons at D. Use v_x = 10 m s^{-1}.

Data:

Planck action constant	$h = 6.63 \cdot 10^{-34}$ J s
Speed of light in vacuum	$c = 3.00 \cdot 10^{8}$ m s^{-1}
Elementary charge	$e = 1.60 \cdot 10^{-19}$ C
Neutron mass	$M = 1.67 \cdot 10^{-27}$ kg
Acceleration of gravity on Earth	g = 9.81 m s^{-2}
If necessary, use the expression:	$\int (1-x)^{1/2} dx = -\frac{2(1-x)^{3/2}}{3}$

COUNTRY CODE	STUDENT CODE	PAGE NUMBER	TOTAL No OF PAGES

Th 3 ANSWER SHEET

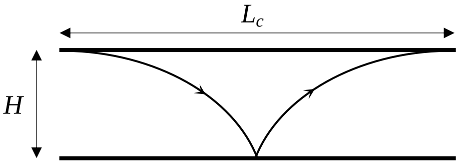
Question	Basic formulas used	Analytical results	Numerical results	Marking guideline
1		$\leq v_z(z) \leq$		1.5
2		$L_c =$	$L_c =$	1.5
3		$N_c(H)=$		2.5
4		$H_n =$ $E_n =$	$H_1=$ μm $E_1 =$ eV	2.5
5		$t_q =$ $L_q =$	$t_q =$ $L_q =$	2.0

Th3 QUANTUM EFFECTS OF GRAVITY
SOLUTION

1. The only neutrons that will survive absorption at A are those that cannot cross H. Their turning points will be below H. So that, for a neutron entering to the cavity at height z with vertical velocity v_z, conservation of energy implies

$$\frac{1}{2}M\,v_z^2 + M\,g\,z \le M\,g\,H \qquad \Rightarrow \qquad \boxed{-\sqrt{2g(H-z)} \le v_z(z) \le \sqrt{2g(H-z)}}$$

2. The cavity should be long enough to ensure the absorption of all neutrons with velocities outside the allowed range. Therefore, neutrons have to reach its maximum height at least once within the cavity. The longest required length corresponds to neutrons that enter at $z = H$ with $v_z = 0$ (see the figure). Calling t_f to their time of fall



$$\left.\begin{aligned} L_c &= v_x\,2t_f \\ H &= \frac{1}{2}g\,t_f^2 \end{aligned}\right\} \quad \Rightarrow \quad \boxed{L_c = 2v_x\sqrt{\frac{2H}{g}}} \qquad \boxed{L_c = 6.4 \text{ cm}}$$

3. The rate of transmitted neutrons entering at a given height z, per unit height, is proportional to the range of allowed velocities at that height, ρ being the proportionality constant

$$\frac{dN_c(z)}{dz} = \rho\left[v_{z,\max}(z) - v_{z,\min}(z)\right] = 2\rho\sqrt{2g(H-z)}$$

The total number of transmitted neutrons is obtained by adding the neutrons entering at all possible heights. Calling $y = z/H$

$$N_c(H) = \int_0^H dN_c(z) = \int_0^H 2\rho\sqrt{2g(H-z)}dz = 2\rho\sqrt{2g}\,H^{3/2}\int_0^1 (1-y)^{1/2}dy = 2\rho\sqrt{2g}\,H^{3/2}\left[-\frac{2}{3}(1-y)^{3/2}\right]_0^1$$

$$\Rightarrow \qquad \boxed{N_c(H) = \frac{4}{3}\rho\sqrt{2g}\,H^{3/2}}$$

4. For a neutron falling from a height H, the action over a bouncing cycle is twice the action during the fall or the ascent

$$S = 2\int_0^H p_z dz = 2M\sqrt{2g}\,H^{3/2}\int_0^1 (1-y)^{1/2}dy = \frac{4}{3}M\sqrt{2g}\,H^{3/2}$$

Using the BS quantization condition

$$S = \frac{4}{3}M\sqrt{2g}\,H^{3/2} = n\,h \qquad \Rightarrow \qquad \boxed{H_n = \left(\frac{9\,h^2}{32\,M^2 g}\right)^{1/3} n^{2/3}}$$

The corresponding energy levels (associated to the vertical motion) are

$$E_n = M\,g\,H_n \qquad \Rightarrow \qquad \boxed{E_n = \left(\frac{9M\,g^2 h^2}{32}\right)^{1/3} n^{2/3}}$$

Numerical values for the first level:

$$H_1 = \left(\frac{9 h^2}{32 M^2 g} \right)^{1/3} = 1.65 \times 10^{-5} \text{ m} \qquad \boxed{H_1 = 16.5\ \mu\text{m}}$$

$$E_1 = M g H_1 = 2.71 \times 10^{-31} \text{ J} = 1.69 \times 10^{-12} \text{ eV} \qquad \boxed{E_1 = 1.69 \text{ peV}}$$

Note that H_1 is of the same order than the given cavity height, H = 50 μm. This opens up the possibility for observing the spatial quantization when varying H.

5. The uncertainty principle says that the minimum time Δt and the minimum energy ΔE satisfy the relation $\Delta E\, \Delta t \geq \hbar$. During this time, the neutrons move to the right a distance

$$\Delta x = v_x \Delta t \geq v_x \frac{\hbar}{\Delta E}$$

Now, the minimum neutron energy allowed in the cavity is E_1, so that $\Delta E \approx E_1$. Therefore, an estimation of the minimum time and the minimum length required is

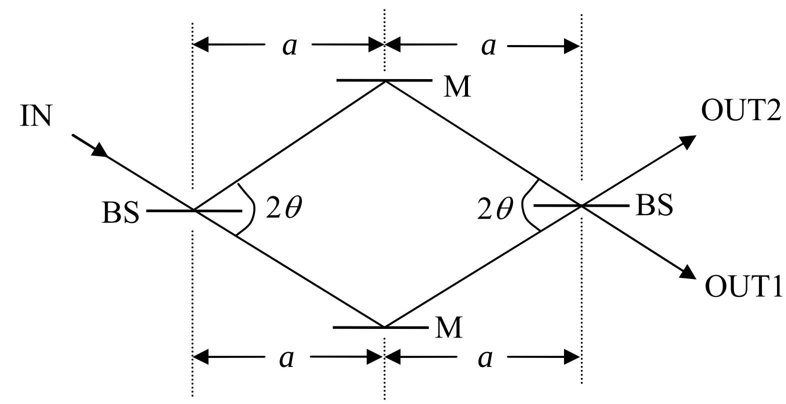
$$\boxed{t_q \approx \frac{\hbar}{E_1} = 0.4 \cdot 10^{-3} \text{ s} = 0.4 \text{ ms}} \qquad \boxed{L_q \approx v_x \frac{\hbar}{E_1} = 4 \cdot 10^{-3} \text{ m} = 4 \text{ mm}}$$

Th 3 ANSWER SHEET

Question	Basic formulas used	Analytical results	Numerical results	Marking guideline
1	$\frac{1}{2}M v_z^2 + M g z \le M g H$	$-\sqrt{2g(H-z)} \le v_z(z) \le \sqrt{2g(H-z)}$		1.5
2	$L_c = v_x 2t_f$ $H = \frac{1}{2} g t_f^2$	$L_c = 2v_x\sqrt{\frac{2H}{g}}$	$L_c = 6.4$ cm	1.3 + 0.2
3	$\frac{dN_c}{dz} = \rho\left[v_{z,\max} - v_{z,\min}\right]$ $N_c(H) = \int_0^H dN_c(z)$	$N_c(H) = \frac{4}{3}\rho\sqrt{2g}\, H^{3/2}$		2.5
4	$S = 2\int_0^H p_z dz = nh$	$H_n = \left(\frac{9h^2}{32M^2 g}\right)^{1/3} n^{2/3}$ $E_n = \left(\frac{9M g^2 h^2}{32}\right)^{1/3} n^{2/3}$	$H_1 = 16.5$ µm $E_1 = 1.69$ peV	1.6 + 0.2 0.5 + 0.2
5	$\Delta E\, \Delta t \ge \hbar$ $\Delta E \approx E_1$ $\Delta x = v_x \Delta t$	$t_q \approx \frac{\hbar}{E_1}$ $L_q \approx v_x \frac{\hbar}{E_1}$	$t_q \approx 0.4$ ms $L_q \approx 4$ mm	1.3 + 0.2 0.3 + 0.2

Theory Question 1: Gravity in a Neutron Interferometer

Enter all your answers into the **Answer Script**.



BS - Beam Splitters M - Mirror

Figure 1a

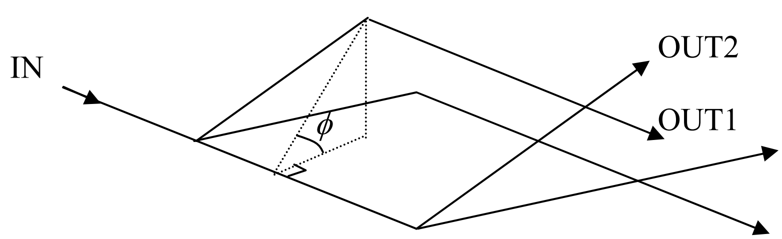


Figure 1b

Physical situation We consider the situation of the famous neutron-interferometer experiment by Collela, Overhauser and Werner, but idealize the set-up inasmuch as we shall assume perfect beam splitters and mirrors within the interferometer. The experiment studies the effect of the gravitational pull on the de Broglie waves of neutrons.

The symbolic representation of this interferometer in analogy to an optical interferometer is shown in Figure 1a. The neutrons enter the interferometer through the IN port and follow the two paths shown. The neutrons are detected at either one of the two output ports, OUT1 or OUT2. The two paths enclose a diamond-shaped area, which is typically a few cm^2 in size.

The neutron de Broglie waves (of typical wavelength of 10^{-10} m) interfere such that all neutrons emerge from the output port OUT1 if the interferometer plane is horizontal. But when the interferometer is tilted around the axis of the incoming neutron beam by angle ϕ (Figure 1b), one observes a ϕ dependent redistribution of the neutrons between the two output ports OUT1 and OUT2.

Geometry For $\phi = 0°$ the interferometer plane is horizontal; for $\phi = 90°$ the plane is vertical with the output ports above the tilt axis.

1.1 (1.0) How large is the diamond-shaped area A enclosed by the two paths of the interferometer?

1.2 (1.0) What is the height H of output port OUT1 above the horizontal plane of the tilt axis?

Express A and H in terms of a, θ, and ϕ.

Optical path length The optical path length N_{opt} (a number) is the ratio of the geometrical path length (a distance) and the wavelength λ. If λ changes along the path, N_{opt} is obtained by integrating λ^{-1} along the path.

1.3 (3.0) What is the difference ΔN_{opt} in the optical path lengths of the two paths when the interferometer has been tilted by angle ϕ? Express your answer in terms of a, θ, and ϕ as well as the neutron mass M, the de Broglie wavelength λ_0 of the incoming neutrons, the gravitational acceleration g, and Planck's constant h.

1.4 (1.0) Introduce the volume parameter

$$V = \frac{h^2}{gM^2}$$

and express ΔN_{opt} solely in terms of A, V, λ_0, and ϕ. State the value of V for $M = 1.675 \times 10^{-27}\,\text{kg}$, $g = 9.800\,\text{m s}^{-2}$, and $h = 6.626 \times 10^{-34}\,\text{J s}$.

1.5 (2.0) How many cycles — from high intensity to low intensity and back to high intensity — are completed by output port OUT1 when ϕ is increased from $\phi = -90°$ to $\phi = 90°$?

Experimental data The interferometer of an actual experiment was characterized by $a = 3.600\,\text{cm}$ and $\theta = 22.10°$, and 19.00 full cycles were observed.

1.6 (1.0) How large was λ_0 in this experiment?

1.7 (1.0) If one observed 30.00 full cycles in another experiment of the same kind that uses neutrons with $\lambda_0 = 0.2000\,\text{nm}$, how large would be the area A?

<u>Hint:</u> If $|\alpha x| << 1$, it is permissible to replace $(1+x)^\alpha$ by $1+\alpha x$.

Country Code	Student Code	Question Number
		1

Answer Script

Geometry

1.1 The area is

$A =$

1.2 The height is

$H =$

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1.0

1.0

Country Code	Student Code	Question Number
		1

Optical path length

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1.3 In terms of a, θ, ϕ, M, λ_0, g, and h : $\Delta N_{\text{opt}} =$	**3.0**
1.4 In terms of A, V, λ_0, and ϕ: $\Delta N_{\text{opt}} =$	**0.8**
The numerical value of V is $V =$	**0.2**
1.5 The number of cycles is # of cycles =	**2.0**

Country Code	Student Code	Question Number
		1

Experimental data

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1.6 The de Broglie wavelength was

$\lambda_0 =$

1.0

1.7 The area is

$A =$

1.0

SOLUTIONS to Theory Question 1

Geometry Each side of the diamond has length $L = \dfrac{a}{\cos\theta}$ and the distance between parallel sides is $D = \dfrac{a}{\cos\theta}\sin(2\theta) = 2a\sin\theta$. The area is the product thereof, $A = LD$, giving

1.1

$$\boxed{A = 2a^2\tan\theta\,.}$$

The height H by which a tilt of ϕ lifts OUT1 above IN is $H = D\sin\phi$ or

1.2

$$\boxed{H = 2a\sin\theta\,\sin\phi\,.}$$

Optical path length Only the two parallel lines for IN and OUT1 matter, each having length L. With the de Broglie wavelength λ_0 on the IN side and λ_1 on the OUT1 side, we have

$$\Delta N_{\text{opt}} = \frac{L}{\lambda_0} - \frac{L}{\lambda_1} = \frac{a}{\lambda_0\cos\theta}\left(1 - \frac{\lambda_0}{\lambda_1}\right).$$

The momentum is h/λ_0 or h/λ_1, respectively, and the statement of energy conservation reads

$$\frac{1}{2M}\left(\frac{h}{\lambda_0}\right)^2 = \frac{1}{2M}\left(\frac{h}{\lambda_1}\right)^2 + MgH\,,$$

which implies

$$\frac{\lambda_0}{\lambda_1} = \sqrt{1 - 2\frac{gM^2}{h^2}\lambda_0^2 H}\,.$$

Upon recognizing that $(gM^2/h^2)\lambda_0^2 H$ is of the order of 10^{-7}, this simplifies to

$$\frac{\lambda_0}{\lambda_1} = 1 - \frac{gM^2}{h^2}\lambda_0^2 H\,,$$

and we get

$$\Delta N_{\text{opt}} = \frac{a}{\lambda_0\cos\theta}\frac{gM^2}{h^2}\lambda_0^2 H$$

or

1.3

$$\Delta N_{\text{opt}} = 2\frac{gM^2}{h^2}a^2\lambda_0 \tan\theta \, \sin\phi \, .$$

A more compact way of writing this is

1.4

$$\Delta N_{\text{opt}} = \frac{\lambda_0 A}{V} \sin\phi \, ,$$

where

1.4

$$V = 0.1597 \times 10^{-13}\,\text{m}^3 = 0.1597\,\text{nm}\,\text{cm}^2$$

is the numerical value for the volume parameter V.

There is constructive interference (high intensity in OUT1) when the optical path lengths of the two paths differ by an integer, $\Delta N_{\text{opt}} = 0, \pm 1, \pm 2, \ldots$, and we have destructive interference (low intensity in OUT1) when they differ by an integer plus half, $\Delta N_{\text{opt}} = \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \ldots$. Changing ϕ from $\phi = -90^\circ$ to $\phi = 90^\circ$ gives

$$\Delta N_{\text{opt}}\Big|_{\phi=-90^\circ}^{\phi=90^\circ} = \frac{2\lambda_0 A}{V} \, ,$$

which tell us that

1.5

$$\sharp \text{ of cycles} = \frac{2\lambda_0 A}{V} \, .$$

Experimental data For $a = 3.6\,\text{cm}$ and $\theta = 22.1^\circ$ we have $A = 10.53\,\text{cm}^2$, so that

1.6

$$\lambda_0 = \frac{19 \times 0.1597}{2 \times 10.53}\,\text{nm} = 0.1441\,\text{nm} \, .$$

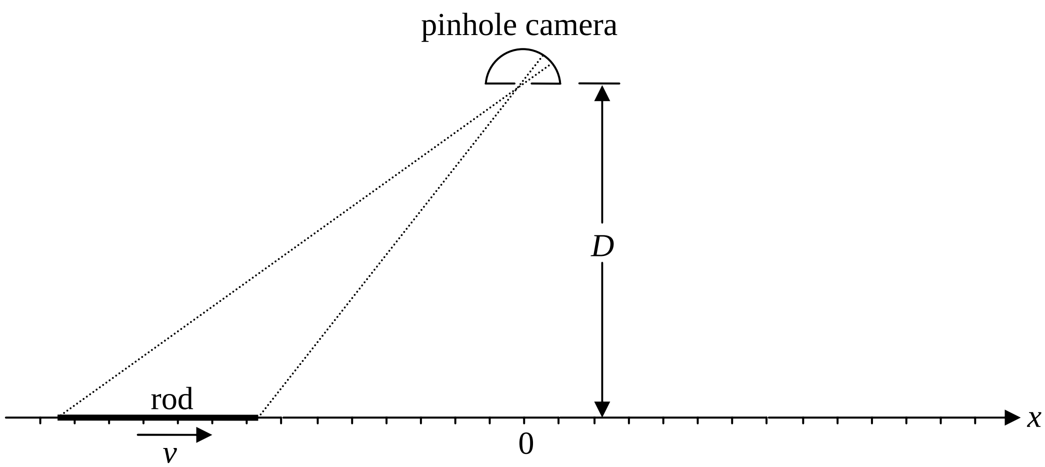
And 30 full cycles for $\lambda_0 = 0.2\,\text{nm}$ correspond to an area

1.7

$$A = \frac{30 \times 0.1597}{2 \times 0.2}\,\text{cm}^2 = 11.98\,\text{cm}^2 \, .$$

Theory Question 2: Watching a Rod in Motion

Enter all your answers into the **Answer Script**.



Physical situation A pinhole camera, with the pinhole at $x = 0$ and at distance D from the x axis, takes pictures of a rod, by opening the pinhole for a very short time. There are equidistant marks along the x axis by which the *apparent length* of the rod, as it is seen on the picture, can be determined from the pictures taken by the pinhole camera. On a picture of the rod *at rest*, its length is L. However, the rod is *not* at rest, but is moving with constant velocity υ along the x axis.

Basic relations A picture taken by the pinhole camera shows a tiny segment of the rod at position $\tilde{x}$.

2.1 (0.6) What is the *actual position* x of this segment at the time when the picture is taken? State your answer in terms of $\tilde{x}$, D, L, υ, and the speed of light $c = 3.00\times10^8\,\mathrm{m\,s^{-1}}$. Employ the quantities

$$\beta = \frac{\upsilon}{c} \text{ and } \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

if they help to simplify your result.

2.2 (0.9) Find also the corresponding inverse relation, that is: express $\tilde{x}$ in terms of x, D, L, υ, and c.

Note: The *actual position* is the position in the frame in which the camera is at rest

Apparent length of the rod The pinhole camera takes a picture at the instant when the actual position of the center of the rod is at some point x_0.

2.3 (1.5) In terms of the given variables, determine the apparent length of the rod on this picture.

2.4 (1.5) Check one of the boxes in the **Answer Script** to indicate how the apparent length changes with time.

Country Code	Student Code	Question Number
		1

Experimental data

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1.6 The de Broglie wavelength was

$\lambda_0 =$

1.0

1.7 The area is

$A =$

1.0

Country Code	Student Code	Question Number
		2

Answer Script

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Basic Relations

2.1 x value for given $\tilde{x}$ value: **0.6**

$x =$

2.2 $\tilde{x}$ value for given x value: **0.9**

$\tilde{x} =$

Apparent length of the rod

2.3 The apparent length is **1.5**

$\tilde{L}(x_0) =$

2.4 Check one: The apparent length **1.5**

- □ increases first, reaches a maximum value, then decreases.
- □ decreases first, reaches a minimum value, then increases.
- □ decreases all the time.
- □ increases all the time.

Country Code	Student Code	Question Number
		2

Symmetric picture

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2.5 The apparent length is

$\tilde{L} =$

0.8

2.6 The actual position of the middle of the rod is

$x_0 =$

1.0

2.7 The picture shows the middle of the rod at a distance

$l =$

from the image of the front end of the rod.

1.2

Country Code	Student Code	Question Number
		2

Very early and very late pictures

For Examiners Use Only

2.8 Check one: **0.5**

☐ The apparent length is 1 m on the early picture and 3 m on the late picture.

☐ The apparent length is 3 m on the early picture and 1 m on the late picture.

2.9 The velocity is **1.0**

$\upsilon =$

2.10 The rod has length **0.6**

$L =$

at rest.

2.11 The apparent length on the symmetric picture is **0.4**

$\tilde{L} =$

SOLUTIONS to Theory Question 2

Basic relations Position $\tilde{x}$ shows up on the picture if light was emitted from there at an instant that is earlier than the instant of the picture taking by the light travel time T that is given by

$$T = \sqrt{D^2 + \tilde{x}^2}\Big/c\,.$$

During the lapse of T the respective segment of the rod has moved the distance vT, so that its actual position x at the time of the picture taking is

2.1
$$\boxed{x = \tilde{x} + \beta\sqrt{D^2 + \tilde{x}^2}\,.}$$

Upon solving for $\tilde{x}$ we find

2.2
$$\boxed{\tilde{x} = \gamma^2 x - \beta\gamma\sqrt{D^2 + (\gamma x)^2}\,.}$$

Apparent length of the rod Owing to the Lorentz contraction, the actual length of the moving rod is L/γ, so that the actual positions of the two ends of the rod are

$$x_\pm = x_0 \pm \frac{L}{2\gamma} \text{ for the } \left\{\begin{matrix}\text{front end}\\ \text{rear end}\end{matrix}\right\} \text{ of the rod.}$$

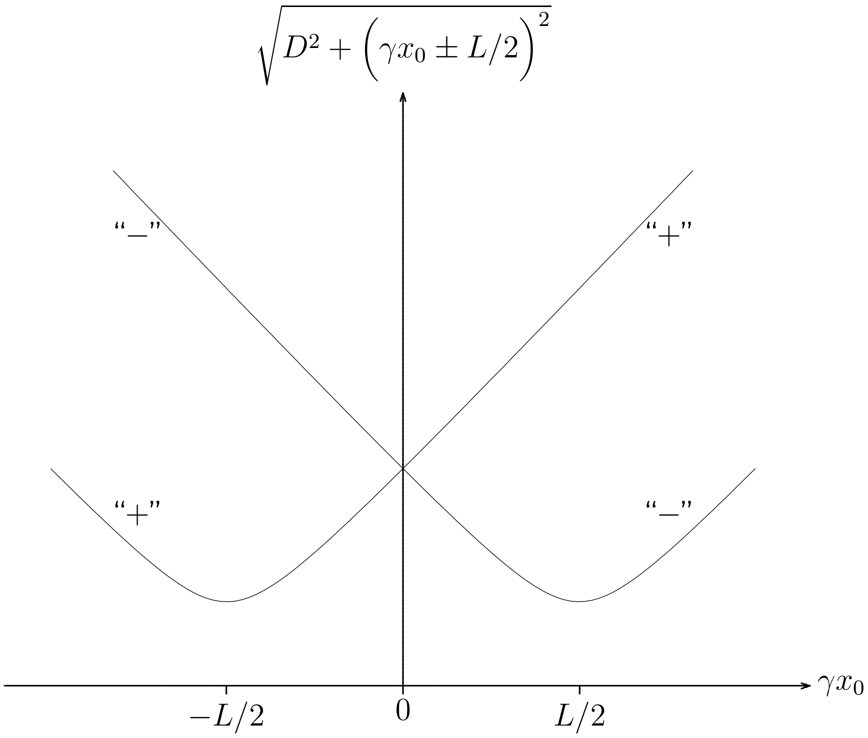
The picture taken by the pinhole camera shows the images of the rod ends at

$$\tilde{x}_\pm = \gamma\Big(\gamma x_0 \pm \frac{L}{2}\Big) - \beta\gamma\sqrt{D^2 + \Big(\gamma x_0 \pm \frac{L}{2}\Big)^2}\,.$$

The apparent length $\tilde{L}(x_0) = \tilde{x}_+ - \tilde{x}_-$ is therefore

2.3
$$\boxed{\tilde{L}(x_0) = \gamma L + \beta\gamma\sqrt{D^2 + \Big(\gamma x_0 - \frac{L}{2}\Big)^2} - \beta\gamma\sqrt{D^2 + \Big(\gamma x_0 + \frac{L}{2}\Big)^2}\,.}$$

Since the rod moves with the constant speed v, we have $\dfrac{\mathrm{d}x_0}{\mathrm{d}t} = v$ and therefore the question is whether $\tilde{L}(x_0)$ increases or decreases when x_0 increases. We sketch the two square root terms:



The difference of the square roots with "−" and "+" appears in the expression for $\tilde{L}(x_0)$, and this difference clearly decreases when x_0 increases.

2.4 The apparent length decreases all the time.

Symmetric picture For symmetry reasons, the apparent length on the symmetric picture is the actual length of the moving rod, because the light from the two ends was emitted simultaneously to reach the pinhole at the same time, that is:

2.5
$$\tilde{L} = \frac{L}{\gamma}\,.$$

The apparent endpoint positions are such that $\tilde{x}_- = -\tilde{x}_+$, or

$$0 = \tilde{x}_+ + \tilde{x}_- = 2\gamma^2 x_0 - \beta\gamma\sqrt{D^2 + \left(\gamma x_0 + \frac{L}{2}\right)^2} - \beta\gamma\sqrt{D^2 + \left(\gamma x_0 - \frac{L}{2}\right)^2}\,.$$

In conjunction with

$$\frac{L}{\gamma} = \tilde{x}_+ - \tilde{x}_- = \gamma L - \beta\gamma\sqrt{D^2 + \left(\gamma x_0 + \frac{L}{2}\right)^2} + \beta\gamma\sqrt{D^2 + \left(\gamma x_0 - \frac{L}{2}\right)^2}$$

this tells us that

$$\sqrt{D^2 + \left(\gamma x_0 \pm \frac{L}{2}\right)^2} = \frac{2\gamma^2 x_0 \pm (\gamma L - L/\gamma)}{2\beta\gamma} = \frac{\gamma x_0}{\beta} \pm \frac{\beta L}{2}\,.$$

As they should, both the version with the upper signs and the version with the lower signs give the same answer for x_0, namely

2.6

$$\boxed{x_0 = \beta\sqrt{D^2 + \left(\frac{L}{2\gamma}\right)^2}\,.}$$

The image of the middle of the rod on the symmetric picture is, therefore, located at

$$\begin{aligned}\tilde{x}_0 &= \gamma^2 x_0 - \beta\gamma\sqrt{D^2 + (\gamma x_0)^2} \\ &= \beta\gamma\left(\sqrt{(\gamma D)^2 + \left(\frac{L}{2}\right)^2} - \sqrt{(\gamma D)^2 + \left(\frac{\beta L}{2}\right)^2}\right),\end{aligned}$$

which is at a distance $\ell = \tilde{x}_+ - \tilde{x}_0 = \dfrac{L}{2\gamma} - \tilde{x}_0$ from the image of the front end, that is

2.7

$$\ell = \frac{L}{2\gamma} - \beta\gamma\sqrt{(\gamma D)^2 + \left(\frac{L}{2}\right)^2} + \beta\gamma\sqrt{(\gamma D)^2 + \left(\frac{\beta L}{2}\right)^2}$$

or

$$\ell = \frac{L}{2\gamma}\left[1 - \frac{\dfrac{\beta L}{2}}{\sqrt{(\gamma D)^2 + \left(\dfrac{L}{2}\right)^2} + \sqrt{(\gamma D)^2 + \left(\dfrac{\beta L}{2}\right)^2}}\right].$$

Very early and very late pictures At the very early time, we have a very large negative value for x_0, so that the apparent length on the very early picture is

$$\tilde{L}_{\text{early}} = \tilde{L}(x_0 \to -\infty) = (1+\beta)\gamma L = \sqrt{\frac{1+\beta}{1-\beta}}\,L\,.$$

Likewise, at the very late time, we have a very large positive value for x_0, so that the apparent length on the very late picture is

$$\tilde{L}_{\text{late}} = \tilde{L}(x_0 \to \infty) = (1-\beta)\gamma L = \sqrt{\frac{1-\beta}{1+\beta}}\, L\,.$$

It follows that $\tilde{L}_{\text{early}} > \tilde{L}_{\text{late}}$, that is:

2.8 The apparent length is 3 m on the early picture and 1 m on the late picture.

Further, we have

$$\beta = \frac{\tilde{L}_{\text{early}} - \tilde{L}_{\text{late}}}{\tilde{L}_{\text{early}} + \tilde{L}_{\text{late}}}\,,$$

so that $\beta = \dfrac{1}{2}$ and the velocity is

2.9

$$v = \frac{c}{2}\,.$$

It follows that $\gamma = \dfrac{\tilde{L}_{\text{early}} + \tilde{L}_{\text{late}}}{2\sqrt{\tilde{L}_{\text{early}}\tilde{L}_{\text{late}}}} = \dfrac{2}{\sqrt{3}} = 1.1547$. Combined with

2.10

$$L = \sqrt{\tilde{L}_{\text{early}}\tilde{L}_{\text{late}}} = 1.73\,\text{m}\,,$$

this gives the length on the symmetric picture as

2.11

$$\tilde{L} = \frac{2\tilde{L}_{\text{early}}\tilde{L}_{\text{late}}}{\tilde{L}_{\text{early}} + \tilde{L}_{\text{late}}} = 1.50\,\text{m}\,.$$

Theory Question 3

This question consists of five independent parts. Each of them asks for an estimate of an order of magnitude only, not for a precise answer. Enter all your answers into the **Answer Script**.

Digital Camera Consider a digital camera with a square CCD chip with linear dimension L = 35 mm having N_p = 5 Mpix (1 Mpix = 10^6 pixels). The lens of this camera has a focal length of f = 38 mm. The well known sequence of numbers (2, 2.8, 4, 5.6, 8, 11, 16, 22) that appear on the lens refer to the so called F-number, which is denoted by $F\#$ and defined as the ratio of the focal length and the diameter D of the aperture of the lens, $F\# = f / D$.

3.1 (1.0) Find the best possible spatial resolution $\Delta x_{\min}$, at the chip, of the camera as limited by the lens. Express your result in terms of the wavelength λ and the F-number $F\#$ and give the numerical value for λ = 500 nm.

3.2 (0.5) Find the necessary number N of Mpix that the CCD chip should possess in order to match this optimal resolution.

3.3 (0.5) Sometimes, photographers try to use a camera at the smallest practical aperture. Suppose that we now have a camera of N_0 = 16 Mpix, with the chip size and focal length as given above. Which value is to be chosen for $F\#$ such that the image quality is not limited by the optics?

3.4 (0.5) Knowing that the human eye has an approximate angular resolution of ϕ = 2 arcmin and that a typical photo printer will print a minimum of 300 dpi (dots per inch), at what minimal distance z should you hold the printed page from your eyes so that you do not see the individual dots?

<u>Data</u> 1 inch = 25.4 mm
1 arcmin = 2.91×10^{-4} rad

Hard-boiled egg An egg, taken directly from the fridge at temperature $T_0 = 4°C$, is dropped into a pot with water that is kept boiling at temperature T_1.

3.5 (0.5) How large is the amount of energy U that is needed to get the egg coagulated?

3.6 (0.5) How large is the heat flow J that is flowing into the egg?

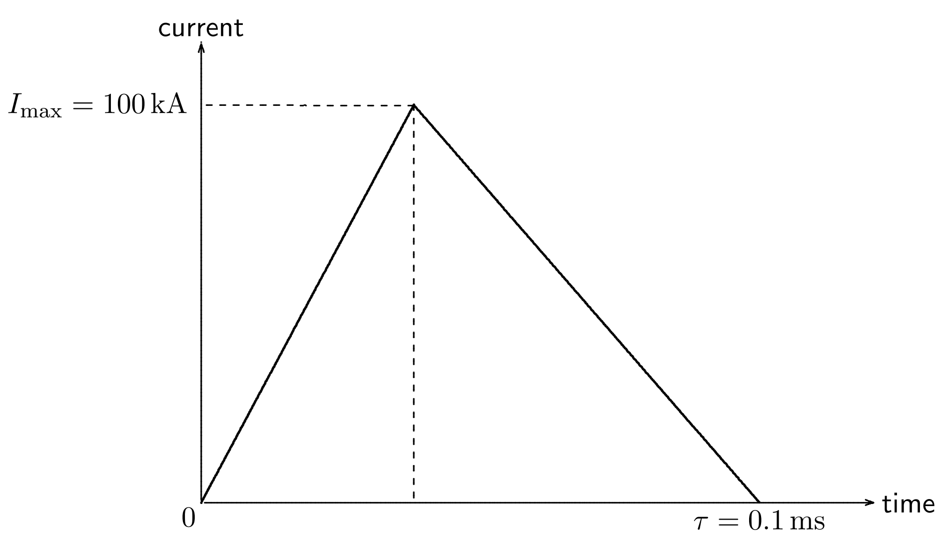
3.7 (0.5) How large is the heat power P transferred to the egg?

3.8 (0.5) For how long do you need to cook the egg so that it is hard-boiled?

Hint You may use the simplified form of Fourier's Law $J = \kappa \Delta T / \Delta r$, where ΔT is the temperature difference associated with Δr, the typical length scale of the problem. The heat flow J is in units of $\mathrm{W\,m^{-2}}$.

Data Mass density of the egg: $\mu = 10^3\,\mathrm{kg\,m^{-3}}$
Specific heat capacity of the egg: $C = 4.2\,\mathrm{J\,K^{-1}\,g^{-1}}$
Radius of the egg: $R = 2.5\,\mathrm{cm}$
Coagulation temperature of albumen (egg protein): $T_c = 65°C$
Heat transport coefficient: $\kappa = 0.64\,\mathrm{W\,K^{-1}\,m^{-1}}$ (assumed to be the same for liquid and solid albumen)

Lightning An oversimplified model of lightning is presented. Lightning is caused by the build-up of electrostatic charge in clouds. As a consequence, the bottom of the cloud usually gets positively charged and the top gets negatively charged, and the ground below the cloud gets negatively charged. When the corresponding electric field exceeds the breakdown strength value of air, a disruptive discharge occurs: this is lightning.



Idealized current pulse flowing between the cloud and the ground during lightning.

Answer the following questions with the aid of this simplified curve for the current as a function of time and these data:

Distance between the bottom of the cloud and the ground: $h = 1$ km;
Breakdown electric field of humid air: $E_0 = 300$ kV m^{-1};
Total number of lightning striking Earth per year: 32×10^6;
Total human population: 6.5×10^9 people.

3.9 (0.5) What is the total charge Q released by lightning?

3.10 (0.5) What is the average current I flowing between the bottom of the cloud and the ground during lightning?

3.11 (1.0) Imagine that the energy of all storms of one year is collected and equally shared among all people. For how long could you continuously light up a 100 W light bulb for your share?

Capillary Vessels Let us regard blood as an incompressible viscous fluid with mass density μ similar to that of water and dynamic viscosity $\eta = 4.5$ g m^{-1} s^{-1}. We model blood vessels as circular straight pipes with radius r and length L and describe the blood flow by Poiseuille's law,

$$\Delta p = RD\,,$$

the Fluid Dynamics analog of Ohm's law in Electricity. Here Δp is the pressure difference between the entrance and the exit of the blood vessel, $D = S\upsilon$ is the volume flow through the cross-sectional area S of the blood vessel and υ is the blood velocity. The hydraulic resistance R is given by

$$R = \frac{8\eta L}{\pi r^4}\,.$$

For the systemic blood circulation (the one flowing from the left ventricle to the right auricle of the heart), the blood flow is $D \approx 100$ cm^{3}s^{-1} for a man at rest. Answer the following questions under the assumption that all capillary vessels are connected in parallel and that each of them has radius $r = 4$ µm and length $L = 1$ mm and operates under a pressure difference $\Delta p = 1$ kPa.

3.12 (1.0) How many capillary vessels are in the human body?

3.13 (0.5) How large is the velocity υ with which blood is flowing through a capillary vessel?

Skyscraper At the bottom of a 1000 m high skyscraper, the outside temperature is T_{bot}= 30°C. The objective is to estimate the outside temperature T_{top} at the top. Consider a thin slab of air (ideal nitrogen gas with adiabatic coefficient γ = 7/5) rising slowly to height z where the pressure is lower, and assume that this slab expands adiabatically so that its temperature drops to the temperature of the surrounding air.

3.14 (0.5) How is the fractional change in temperature dT/T related to dp/p, the fractional change in pressure?

3.15 (0.5) Express the pressure difference dp in terms of dz, the change in height.

3.16 (1.0) What is the resulting temperature at the top of the building?

Data Boltzmann constant: $k = 1.38 \times 10^{-23}\,\mathrm{J\,K^{-1}}$
Mass of a nitrogen molecule: $m = 4.65 \times 10^{-26}$ kg
Gravitational acceleration: $g = 9.80\,\mathrm{m\,s^{-2}}$

Country Code	Student Code	Question Number
		3

Answer Script

Digital Camera

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3.1 The best spatial resolution is

(formula:) $\Delta x_{\text{min}} =$ **0.7**

which gives

(numerical:) $\Delta x_{\text{min}} =$ **0.3**

for $\lambda = 500\,\text{nm}$.

3.2 The number of Mpix is **0.5**

$N =$

3.3 The best F-number value is **0.5**

$F\# =$

3.4 The minimal distance is **0.5**

$z =$

Country Code	Student Code	Question Number
		3

Hard-boiled egg

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3.5 The required energy is **0.5**

$U =$

3.6 The heat flow is **0.5**

$J =$

3.7 The heat power transferred is **0.5**

$P =$

3.8 The time needed to hard-boil the egg is **0.5**

$\tau =$

Country Code	Student Code	Question Number
		3

Lightning

3.9 The total charge is

$Q =$

3.10 The average current is

$I =$

3.11 The light bulb would burn for the duration

$t =$

Capillary Vessels

3.12 There are

$N =$

capillary vessels in a human body.

3.13 The blood flows with velocity

$\upsilon =$

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0.5
0.5
1.0
1.0
0.5

Country Code	Student Code	Question Number
		3

Skyscraper

3.14 The fractional change in temperature is

$$\frac{dT}{T} =$$

3.15 The pressure difference is

$$dp =$$

3.16 The temperature at the top is

$$T_{\text{top}} =$$

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0.5

0.5

1.0

SOLUTIONS to Theory Question 3

Digital Camera Two factors limit the resolution of the camera as a photographic tool: the diffraction by the aperture and the pixel size. For diffraction, the inherent angular resolution θ_R is the ratio of the wavelength λ of the light and the aperture D of the camera,

$$\theta_R = 1.22\frac{\lambda}{D}\,,$$

where the standard factor of 1.22 reflects the circular shape of the aperture. When taking a picture, the object is generally sufficiently far away from the photographer for the image to form in the focal plane of the camera where the CCD chip should thus be placed. The Rayleigh diffraction criterion then states that two image points can be resolved if they are separated by more than

3.1

$$\Delta x = f\theta_R = 1.22\lambda\, F\sharp\,,$$

which gives

$$\Delta x = 1.22\ \mu\mathrm{m}$$

if we choose the largest possible aperture (or smallest value $F\sharp = 2$) and assume $\lambda = 500\,\mathrm{nm}$ for the typical wavelength of daylight

The digital resolution is given by the distance ℓ between the center of two neighboring pixels. For our 5 Mpix camera this distance is roughly

$$\ell = \frac{L}{\sqrt{N_p}} = 15.65\,\mu\mathrm{m}\,.$$

Ideally we should match the optical and the digital resolution so that neither aspect is overspecified. Taking the given optical resolution in the expression for the digital resolution, we obtain

3.2

$$N = \left(\frac{L}{\Delta x}\right)^2 \approx 823\ \mathrm{Mpix}\,.$$

Now looking for the unknown optimal aperture, we note that we should have $\ell \geq \Delta x$, that is: $F\sharp \leq F_0$ with

$$F_0 = \frac{L}{1.22\lambda\sqrt{N_0}} = 2\sqrt{\frac{N}{N_0}} = 14.34\,.$$

Since this $F\sharp$ value is not available, we choose the nearest value that has a higher optical resolution,

3.3
$$F_0 = 11\,.$$

When looking at a picture at distance z from the eye, the (small) subtended angle between two neighboring dots is $\phi = \ell/z$ where, as above, ℓ is the distance between neighboring dots. Accordingly,

3.4
$$z = \frac{\ell}{\phi} = \frac{2.54 \times 10^{-2}/300\,\mathrm{dpi}}{5.82 \times 10^{-4}\,\mathrm{rad}} = 14.55\,\mathrm{cm} \approx 15\,\mathrm{cm}\,.$$

Hard-boiled egg All of the egg has to reach coagulation temperature. This means that the increase in temperature is

$$\Delta T = T_\mathrm{c} - T_0 = 65\,^\circ\mathrm{C} - 4\,^\circ\mathrm{C} = 61\,^\circ\mathrm{C}\,.$$

Thus the minimum amount of energy that we need to get into the egg such that all of it has coagulated is given by $U = \mu V C \Delta T$ where $V = 4\pi R^3/3$ is the egg volume. We thus find

3.5
$$U = \mu \frac{4\pi R^3}{3} C(T_\mathrm{c} - T_0) = 16768\ \mathrm{J}\,.$$

The simplified equation for heat flow then allows us to calculate how much energy has flown into the egg through the surface per unit time. To get an approximate value for the time we assume that the center of the egg is at the initial temperature $T = 4\,^\circ\mathrm{C}$. The typical length scale is $\Delta r = R$, and the temperature difference associated with it is $\Delta T = T_1 - T_0$ where $T_1 = 100\,^\circ\mathrm{C}$ (boiling water). We thus get

3.6
$$J = \kappa(T_1 - T_0)/R = 2458\,\mathrm{W\,m^{-2}}\,.$$

Heat is transferred from the boiling water to the egg through the surface of the egg. This gives

3.7

$$P = 4\pi R^2 J = 4\pi\kappa R(T_1 - T_0) \approx 19.3\,\mathrm{W}$$

for the amount of energy transferred to the egg per unit time. From this we get an estimate for the time τ required for the necessary amount of heat to flow into the egg all the way to the center:

3.8

$$\tau = \frac{U}{P} = \frac{\mu C R^2}{3\kappa}\frac{T_\mathrm{c} - T_0}{T_1 - T_0} = \frac{16768}{19.3} = 869\ \mathrm{s} \approx 14.5\,\mathrm{min}\,.$$

Lightning The total charge Q is just the area under the curve of the figure. Because of the triangular shape, we immediately get

3.9

$$Q = \frac{I_0 \tau}{2} = 5\,\mathrm{C}\,.$$

The average current is

3.10

$$I = Q/\tau = \frac{I_0}{2} = 50\,\mathrm{kA}\,,$$

simply half the maximal value.

Since the bottom of the cloud gets negatively charged and the ground positively charged, the situation is essentially that of a giant parallel-plate capacitor. The amount of energy stored just before lightning occurs is $QE_0h/2$ where E_0h is the voltage difference between the bottom of the cloud and the ground, and lightning releases this energy. Altogether we thus get for one lightning the energy $QE_0h/2 = 7.5 \times 10^8$ J. It follows that you could light up the 100 W bulb for the duration

3.11

$$t = \frac{32 \times 10^6}{6.5 \times 10^9} \times \frac{7.5 \times 10^8\,\mathrm{J}}{100\,\mathrm{W}} \approx 10\,\mathrm{h}\,.$$

Capillary Vessels Considering *all* capillaries, one has

$$R_\mathrm{all} = \frac{\Delta p}{D} = 10^7\,\mathrm{Pa\,m^{-3}\,s}\,.$$

All capillaries are assumed to be connected in parallel. The analogy between Poiseuille's and Ohm's laws then gives the hydraulic resistance R of one capillary as

$$\frac{1}{R_{\text{all}}} = \frac{N}{R}\,.$$

We thus get

$$N = \frac{R}{R_{\text{all}}}$$

for the number of capillary vessels in the human body. Now calculate R using Poiseuille's law,

$$R = \frac{8\eta L}{\pi r^4} \approx 4.5 \times 10^{16}\,\text{kg}\,\text{m}^{-4}\,\text{s}^{-1}\,,$$

and arrive at

3.12
$$\boxed{N \approx \frac{4.5 \times 10^{16}}{10^7} = 4.5 \times 10^9\,.}$$

The volume flow is $D = S_{\text{all}}v$ where $S_{\text{all}} = N\pi r^2$ is the *total* cross-sectional area associated with all capillary vessels. We then get

3.13
$$\boxed{v = \frac{D}{N\pi r^2} = \frac{r^2 \Delta p}{8\eta L} = 0.44\,\text{mm}\,\text{s}^{-1}\,,}$$

where the second expression is found by alternatively considering one capillary vessel by itself.

Skyscraper When the slab is at height z above the ground, the air in the slab has pressure $p(z)$ and temperature $T(z)$ and the slab has volume $V(z) = Ah(z)$ where A is the cross-sectional area and $h(z)$ is the thickness of the slab. At any given height z, we combine the ideal gas law

$$pV = NkT \qquad (N \text{ is the number of molecules in the slab})$$

with the adiabatic law

$$pV^\gamma = \text{const} \quad \text{or} \quad (pV)^\gamma \propto p^{\gamma-1}$$

to conclude that $p^{\gamma-1} \propto T^\gamma$. Upon differentiation this gives $(\gamma-1)\dfrac{dp}{p} = \gamma\dfrac{dT}{T}$, so that

3.14
$$\boxed{\frac{dT}{T} = (1 - 1/\gamma)\frac{dp}{p}\,.}$$

Since the slab is not accelerated, the weight must be balanced by the force that results from the difference in pressure at the top and bottom of the slab. Taking downward forces as positive, we have the net force

$$0 = Nmg + A[p(z+h) - p(z)] = \frac{pV}{kT}mg + \frac{V}{h}\frac{dp}{dz}h\,,$$

so that $\dfrac{dp}{dz} = -\dfrac{mg}{k}\dfrac{p}{T}$ or

3.15
$$\boxed{dp = -\frac{mg}{k}\frac{p}{T}dz\,.}$$

Taken together, the two expressions say that

$$dT = -(1 - 1/\gamma)\frac{mg}{k}dz$$

and therefore we have

$$T_{\text{top}} = T_{\text{bot}} - (1 - 1/\gamma)\frac{mgH}{k}$$

for a building of height H, which gives

3.16
$$\boxed{T_{\text{top}} = 20.6\,^\circ\text{C}}$$

for $H = 1\,\text{km}$ and $T_{\text{bot}} = 30\,^\circ\text{C}$.

1.1) One may use any reasonable equation to obtain the dimension of the questioned quantities.

I) The Planck relation is $h\nu = E \quad \Rightarrow \quad [h][\nu] = [E] \quad \Rightarrow \quad [h] = [E][\nu]^{-1} = ML^2T^{-1}$
(0.2)

II) $[c] = LT^{-1}$ (0.2)

III) $F = \dfrac{Gmm}{r^2} \quad \Rightarrow \quad [G] = [F][r^2][m]^{-2} = M^{-1}L^3T^{-2}$ (0.2)

IV) $E = K_B\theta \quad \Rightarrow \quad [K_B] = [\theta]^{-1}[E] = ML^2T^{-2}K^{-1}$ (0.2)

1.2) Using the Stefan-Boltzmann's law,

$\dfrac{Power}{Area} = \sigma\theta^4$, or any equivalent relation, one obtains:

(0.3)

$[\sigma]K^4 = [E]L^{-2}T^{-1} \Rightarrow [\sigma] = MT^{-3}K^{-4}$. (0.2)

1.3) The Stefan-Boltzmann's constant, up to a numerical coefficient, equals $\sigma = h^\alpha c^\beta G^\gamma k_B{}^\delta$, where $\alpha, \beta, \gamma, \delta$ can be determined by dimensional analysis. Indeed, $[\sigma] = [h]^\alpha [c]^\beta [G]^\gamma [k_B]^\delta$, where e.g. $[\sigma] = MT^{-3}K^{-4}$.

$$MT^{-3}K^{-4} = \left(ML^2T^{-1}\right)^\alpha \left(LT^{-1}\right)^\beta \left(M^{-1}L^3T^{-2}\right)^\gamma \left(ML^2T^{-2}K^{-1}\right)^\delta = M^{\alpha-\gamma+\delta}L^{2\alpha+\beta+3\gamma+2\delta}T^{-\alpha-\beta-2\gamma-2\delta}K^{-\delta},$$

(0.2)

The above equality is satisfied if,

$$\Rightarrow \begin{cases} \alpha - \gamma + \delta = 1, \\ 2\alpha + \beta + 3\gamma + 2\delta = 0, \\ -\alpha - \beta - 2\gamma - 2\delta = -3, \\ -\delta = -4, \end{cases} \quad \text{(Each one (0.1))} \qquad \Rightarrow \begin{cases} \alpha = -3, \\ \beta = -2, \\ \gamma = 0, \\ \delta = 4. \end{cases} \quad \text{(Each one (0.1))}$$

$$\Rightarrow \qquad \sigma = \frac{k_B{}^4}{c^2h^3}.$$

2.1) Since A, the area of the event horizon, is to be calculated in terms of m from a classical theory of relativistic gravity, e.g. the General Relativity, it is a combination of c, characteristic of special relativity, and G characteristic of gravity. Especially, it is independent of the Planck constant h which is characteristic of quantum mechanical phenomena.

$$A = G^{\alpha} c^{\beta} m^{\gamma}$$

Exploiting dimensional analysis,

$$\Rightarrow \quad [A] = [G]^{\alpha}[c]^{\beta}[m]^{\gamma} \quad \Rightarrow \quad L^2 = \left(M^{-1}L^3T^{-2}\right)^{\alpha}\left(LT^{-1}\right)^{\beta} M^{\gamma} = M^{-\alpha+\gamma}L^{3\alpha+\beta}T^{-2\alpha-\beta}$$

(0.2)

The above equality is satisfied if,

$$\Rightarrow \begin{cases} -\alpha+\gamma=0, \\ 3\alpha+\beta=2, \\ -2\alpha-\beta=0, \end{cases} \quad \text{(Each one (0.1))} \quad \Rightarrow \quad \begin{cases} \alpha=2, \\ \beta=-4, \\ \gamma=2, \end{cases} \quad \text{(Each one (0.1))} \Rightarrow$$

$$A = \frac{m^2 G^2}{c^4}.$$

2.2)

From the definition of entropy $dS = \dfrac{dQ}{\theta}$, one obtains $[S] = [E][\theta]^{-1} = ML^2T^{-2}K^{-1}$ (0.2)

2.3) Noting $\eta = S/A$, one verifies that,

$$\begin{cases} [\eta] = [S][A]^{-1} = MT^{-2}K^{-1}, \\ [\eta] = [G]^{\alpha}[h]^{\beta}[c]^{\gamma}[k_B]^{\delta} = M^{-\alpha+\beta+\delta}L^{3\alpha+2\beta+\gamma+2\delta}T^{-2\alpha-\beta-\gamma-2\delta}K^{-\delta}, \end{cases}$$

(0.2)

Using the same scheme as above,

$$\Rightarrow \begin{cases} -\alpha+\beta+\delta=1, \\ 3\alpha+2\beta+\gamma+2\delta=0, \\ -2\alpha-\beta-\gamma-2\delta=-2, \\ \delta=1, \end{cases} \quad \text{(Each one (0.1))} \quad \Rightarrow \begin{cases} \alpha=-1, \\ \beta=-1, \\ \gamma=3, \\ \delta=1, \end{cases} \quad \text{(Each one (0.1))}$$

thus, $\eta = \dfrac{c^3 k_B}{G h}$. (0.1)

3.1)

The first law of thermodynamics is $dE = dQ + dW$. By assumption, $dW = 0$. Using the definition of entropy, $dS = \frac{dQ}{\theta}$, one obtains,

$$dE = \theta_H \, dS + 0,$$ (0.2) + (0.1), for setting $dW = 0$.

Using, $\begin{cases} S = \frac{Gk_B}{ch} m^2, \\ E = mc^2, \end{cases}$ [(0.1) for S]

one obtains, $\theta_H = \frac{dE}{dS} = \left(\frac{dS}{dE}\right)^{-1} = c^2 \left(\frac{dS}{dm}\right)^{-1}$ (0.2)

Therefore, $\theta_H = \left(\frac{1}{2}\right) \frac{c^3 h}{Gk_B} \frac{1}{m}$. (0.1)+(0.1) (for the coefficient)

3.2) The Stefan-Boltzmann's law gives the rate of energy radiation per unit area. Noting that $E = mc^2$ we have:

$$\begin{cases} dE/dt = -\sigma \theta_H{}^4 A, & (0.2) \\ \sigma = \frac{k_B{}^4}{c^2 h^3}, \\ A = \frac{m^2 G^2}{c^4} \\ E = mc^2, \end{cases} \Rightarrow c^2 \frac{dm}{dt} = -\frac{k_B{}^4}{c^2 h^3} \left(\frac{c^3 h}{2G k_B} \frac{1}{m}\right)^4 \frac{m^2 G^2}{c^4}, \quad (0.2)$$

$$\Rightarrow \quad \frac{dm}{dt} = -\frac{1}{16} \frac{c^4 h}{G^2} \frac{1}{m^2}.$$ (0.1) (for simplification) + (0.2) (for the minus sign)

3.3)
By integration:

$$\frac{dm}{dt} = -\frac{1}{16} \frac{c^4 h}{G^2} \frac{1}{m^2}. \quad \Rightarrow \int m^2 dm = -\int \frac{c^4 h}{16G^2} dt$$ (0.3)

$$\Rightarrow m^3(t) - m^3(0) = -\frac{3c^4 h}{16G^2} t,$$ (0.2) + (0.2) (Integration and correct boundary values)

At $t = t^*$ the black hole evaporates completely:

$m(t^*) = 0$ (0.1) $\Rightarrow t^* = \frac{16G^2}{3c^4 h} m^3$ (0.2)+(0.1) (for the coefficient)

3.4) C_V measures the change in E with respect to variation of θ.

$$\begin{cases} C_V = \dfrac{dE}{d\theta}, & \text{(0.2)} \\ E = mc^2, & \text{(0.2)} \\ \theta = \dfrac{c^3 h}{2Gk_B}\dfrac{1}{m} & \end{cases} \Rightarrow \quad C_V = -\frac{2Gk_B}{ch}m^2. \quad \text{0.1)+(0.1) (for the coefficient)}$$

4.1) Again the Stefan-Boltzmann's law gives the rate of energy loss per unit area of the black hole. A similar relation can be used to obtain the energy gained by the black hole due to the background radiation. To justify it, note that in the thermal equilibrium, the total change in the energy is vanishing. The blackbody radiation is given by the Stefan-Boltzmann's law. Therefore the rate of energy gain is given by the same formula.

(0.1) + (0.4) (For the first and the second terms respectively)

$$\begin{cases} \dfrac{dE}{dt} = -\sigma\theta^4 A + \sigma\theta_B{}^4 A \\ E = mc^2, \end{cases} \Rightarrow \quad \frac{dm}{dt} = -\frac{hc^4}{16G^2}\frac{1}{m^2} + \frac{G^2}{c^8 h^3}(k_B\,\theta_B)^4 m^2 \qquad \text{(0.3)}$$

4.2)

Setting $\dfrac{dm}{dt} = 0$, we have:

$$-\frac{hc^4}{16G^2}\frac{1}{m^{*2}} + \frac{G^2}{c^8 h^3}(k_B\,\theta_B)^4 m^{*2} = 0 \qquad \text{(0.2)}$$

and consequently,

$$m^* = \frac{c^3 h}{2G\,k_B}\frac{1}{\theta_B} \qquad \text{(0.2)}$$

4.3)

$$\theta_B = \frac{c^3 h}{2G\,k_B}\frac{1}{m^*} \quad \Rightarrow \quad \frac{dm}{dt} = -\frac{hc^4}{16G^2}\frac{1}{m^2}\left(1 - \frac{m^4}{m^{*4}}\right) \qquad \text{(0.2)}$$

4.4) Use the solution to 4.2,

$m^* = \dfrac{c^3 h}{2G\,k_B}\dfrac{1}{\theta_B}$ (0.2) and 3.1 to obtain, $\theta^* = \dfrac{c^3 h}{2Gk_B}\dfrac{1}{m^*} = \theta_B$ (0.2)

One may also argue that m^* corresponds to thermal equilibrium. Thus for $m = m^*$ the black hole temperature equals θ_B.

Or one may set $\dfrac{dE}{dt} = -\sigma\left(\theta^{*4} - \theta_B{}^4\right)A = 0$ to get $\theta^* = \theta_B$.

4.5) Considering the solution to 4.3, one verifies that it will go away from the equilibrium. (0.6)

$$\frac{dm}{dt} = -\frac{hc^4}{G^2}\frac{1}{m^2}\left(1-\frac{m^4}{m^{*4}}\right) \Rightarrow \begin{cases} m > m^* & \Rightarrow \quad \dfrac{dm}{dt} > 0 \\ m < m^* & \Rightarrow \quad \dfrac{dm}{dt} < 0 \end{cases}$$

In physics, whenever we have an equality relation, both sides of the equation should be of the same type i.e. they must have the same dimensions. For example you cannot have a situation where the quantity on the right-hand side of the equation represents a length and the quantity on the left-hand side represents a time interval. Using this fact, sometimes one can nearly deduce the form of a physical relation without solving the problem analytically. For example if we were asked to find the time it takes for an object to fall from a height of h under the influence of a constant gravitational acceleration g, we could argue that one only needs to build a quantity representing a time interval, using the quantities g and h and the only possible way of doing this is $T = a(h/g)^{1/2}$. Notice that this solution includes an as yet undetermined coefficient a which is *dimensionless* and thus cannot be determined, using this method. This coefficient can be a number such as 1, $1/2$, $\sqrt{3}$, π, or any other real number. This method of deducing physical relations is called *dimensional analysis*. In dimensional analysis the dimensionless coefficients are not important and we do not need to write them. Fortunately in most physical problems these coefficients are of the order of 1 and eliminating them does not change the order of magnitude of the physical quantities. Therefore, by applying the dimensional analysis to the above problem, one obtains $T = (h/g)^{1/2}$.

Generally, the dimensions of a physical quantity are written in terms of the dimensions of four fundamental quantities: M (mass), L (length), T (time), and K (temperature). The dimensions of an arbitrary quantity, x is denoted by $[x]$. As an example, to express the dimensions of velocity v, kinetic energy E_k, and heat capacity C_V we write: $[v] = LT^{-1}$, $[E_k] = ML^2T^{-2}$, $[C_V] = ML^2T^{-2}K^{-1}$.

1 Fundamental Constants and Dimensional Analysis

1.1	Find the dimensions of *the fundamental constants,* i.e. the Planck's constant, h, the speed of light, c, the universal constant of gravitation, G, and the Boltzmann constant, k_B, in terms of the dimensions of length, mass, time, and temperature.	0.8

The Stefan-Boltzmann law states that the black body emissive power which is the total energy radiated per unit surface area of a black body in unit time is equal to $\sigma\theta^4$ where σ is the Stefan-Boltzmann's constant and θ is the absolute temperature of the black body.

1.2	Determine the dimensions of the Stefan-Boltzmann's constant in terms of the dimensions of length, mass, time, and temperature.	0.5

The Stefan-Boltzmann's constant is not a fundamental constant and one can write it in terms of fundamental constants i.e. one can write $\sigma = a\, h^{\alpha} c^{\beta} G^{\gamma} k_B^{\ \delta}$. In this relation a is a dimensionless parameter of the order of 1. As mentioned before, the exact value of a is not significant from our viewpoint, so we will set it equal to 1.

1.3	Find α, β, γ, and δ using dimensional analysis.	1.0

2 Physics of Black Holes

In this part of the problem, we would like to find out some properties of black holes using dimensional analysis. According to a certain theorem in physics known as the *no hair theorem*, all the characteristics of the black hole which we are considering in this problem depend only on the mass of the black hole. One characteristic of a black hole is the area of its *event horizon*. Roughly speaking, the event horizon is the boundary of the black hole. Inside this boundary, the gravity is so strong that even light cannot emerge from the region enclosed by the boundary.

We would like to find a relation between the mass of a black hole, m, and the area of its event horizon, A. This area depends on the mass of the black hole, the speed of light, and the universal constant of gravitation. As in 1.3 we shall write $A = G^{\alpha} c^{\beta} m^{\gamma}$.

2.1	Use dimensional analysis to find α, β, and γ.	0.8

From the result of 2.1 it becomes clear that the area of the event horizon of a black hole increases with its mass. From a classical point of view, nothing comes out of a black hole and therefore in all physical processes the area of the event horizon can only increase. In analogy with the second law of thermodynamics, Bekenstein proposed to assign entropy, S, to a black hole, proportional to the area of its event horizon i.e. $S = \eta A$. This conjecture has been made more plausible using other arguments.

2.2	Use the thermodynamic definition of entropy $dS = dQ/\theta$ to find the dimensions of entropy. dQ is the exchanged heat and θ is the absolute temperature of the system.	0.2

2.3	As in 1.3, express the dimensioned constant η as a function of the fundamental constants h, c, G, and k_B.	1.1

*Do **not** use dimensional analysis for the rest of problem, but you may use the results you have obtained in previous sections.*

3 Hawking Radiation

With a semi-quantum mechanical approach, Hawking argued that contrary to the classical point of view, black holes emit radiation similar to the radiation of a black body at a temperature which is called the *Hawking temperature*.

3.1	Use $E = mc^2$, which gives the energy of the black hole in terms of its mass, and the laws of thermodynamics to express the Hawking temperature θ_H of a black hole in terms of its mass and the fundamental constants. Assume that the black hole does no work on its surroundings.	0.8

3.2	The mass of an isolated black hole will thus change because of the Hawking radiation. Use Stefan-Boltzmann's law to find the dependence of this rate of change on the Hawking temperature of the black hole, θ_H and express it in terms of mass of the black hole and the fundamental constants.	0.7

3.3	Find the time t^*, that it takes an isolated black hole of mass m to evaporate completely i.e. to lose all its mass.	1.1

From the viewpoint of thermodynamics, black holes exhibit certain exotic behaviors. For example the heat capacity of a black hole is negative.

3.4	Find the heat capacity of a black hole of mass m.	0.6

4 Black Holes and the Cosmic Background Radiation

Consider a black hole exposed to the cosmic background radiation. The cosmic background radiation is a black body radiation with a temperature θ_B which fills the entire universe. An object with a total area A will thus receive an energy equal to $\sigma\theta_B^4 \times A$ per unit time. A black hole, therefore, loses energy through Hawking radiation and gains energy from the cosmic background radiation.

4.1	Find the rate of change of a black hole's mass, in terms of the mass of the black hole, the temperature of the cosmic background radiation, and the fundamental constants.	0.8
4.2	At a certain mass, m^*, this rate of change will vanish. Find m^* and express it in terms of θ_B and the fundamental constants.	0.4
4.3	Use your answer to 4.2 to substitute for θ_B in your answer to part 4.1 and express the rate of change of the mass of a black hole in terms of m, m^*, and the fundamental constants.	0.2
4.4	Find the Hawking temperature of a black hole at thermal equilibrium with cosmic background radiation.	0.4
4.5	Is the equilibrium stable or unstable? Why? (Express your answer mathematically)	0.6

Solution (The Experimental Question):

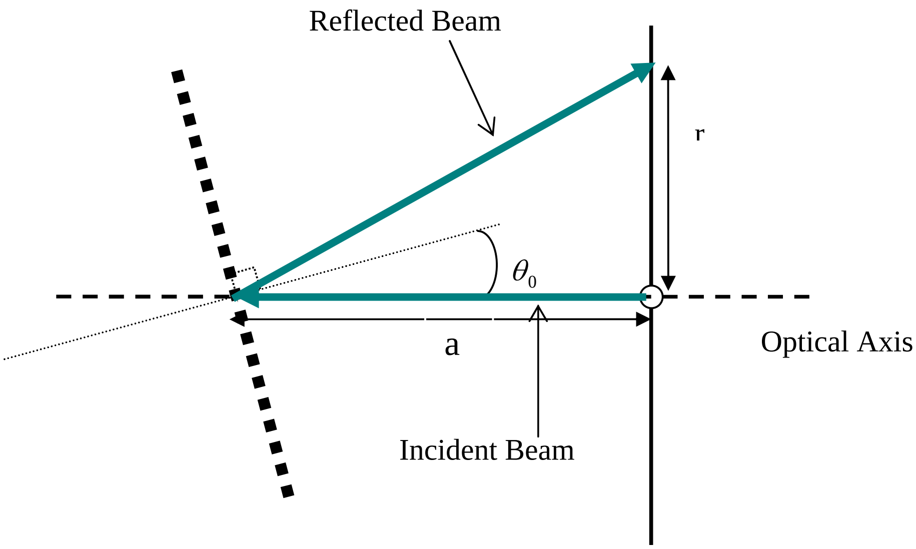
Task 1

1a.

$\Delta\theta_{nominal}$=5´=0.08˚

$\Delta\theta_{nominal}$ (degree)	0.08

1b.



If "a" is the distance between card and the grating and "r" is the distance between the hole and the light spot so we have

$$\Delta f(x_1, x_2, ...) = \sqrt{\left(\frac{\partial f}{\partial x_1}\Delta x_1\right)^2 + \left(\frac{\partial f}{\partial x_2}\Delta x_2\right)^2 + ...}$$

$$\tan(2\theta_0) = \frac{r}{a},\ If\ \theta_0 << 1 \Rightarrow \theta_0 = \frac{r}{2a} \Rightarrow \Delta\theta_0 = \sqrt{\left(\frac{\Delta r}{2a}\right)^2 + \left(\frac{r\,\Delta a}{2a^2}\right)^2}$$

We want θ_0 to be zero i.e. $r = 0 \Rightarrow \Delta\theta_0 = \dfrac{\Delta r}{2a}$

$$\Delta r = 1\,mm,\ \ a = (70 \pm 1)\,mm \Rightarrow \theta_0 = \frac{\Delta r}{2a}\ rad = 0.007\ rad = 0.4^\circ$$

$\Delta\theta_0$	0.4˚
θ range of visible light (degree)	$13^\circ \leq \theta \leq 26^\circ$

1c.

$R_{\min}^{(0)}$	(21.6±0.1) kΩ
$\Delta\varphi_0$	5´ = 0.08˚
$R_{\min}^{(1)}$	R=(192±1) kΩ

$\Delta\varphi_0$=5´ because

θ = 5´ => R= (21.9±0.1) kΩ

θ =-5´ => R= (21.9±0.1) kΩ

1d.

Table 1d. The measured parameters

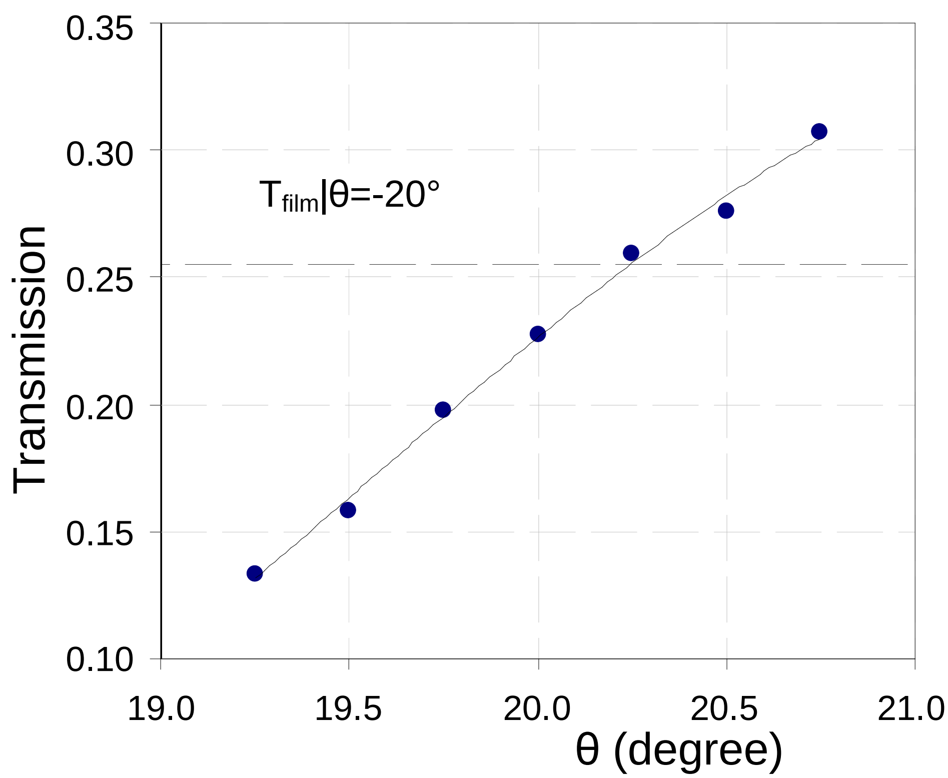
θ (degree)	R_{glass}(MΩ)	ΔR_{glass}(MΩ)	R_{film}(MΩ)	ΔR_{film}(MΩ)
15.00	3.77	0.03	183	3
15.50	2.58	0.02	132	2
16.00	1.88	0.01	87	1
16.50	1.19	0.01	51.5	0.5
17.00	0.89	0.01	33.4	0.3
17.50	0.68	0.01	19.4	0.1
18.00	0.486	0.005	10.4	0.1
18.50	0.365	0.005	5.40	0.03
19.00	0.274	0.003	2.66	0.02
19.50	0.225	0.002	1.42	0.01
20.00	0.200	0.002	0.880	0.005
20.50	0.227	0.002	0.822	0.005
21.00	0.368	0.003	1.123	0.007
21.50	0.600	0.005	1.61	0.01
22.00	0.775	0.005	1.85	0.01
22.50	0.83	0.01	1.87	0.01
23.00	0.88	0.01	1.93	0.02
23.50	1.01	0.01	2.14	0.02
24.00	1.21	0.01	2.58	0.02
24.50	1.54	0.01	3.27	0.02
25.00	1.91	0.01	4.13	0.02
16.25	1.38	0.01	66.5	0.5
16.75	1.00	0.01	40.0	0.3
17.25	0.72	0.01	23.4	0.2
17.75	0.535	0.005	12.8	0.1
18.25	0.391	0.003	6.83	0.05
18.75	0.293	0.003	3.46	0.02
19.25	0.235	0.003	1.76	0.01
19.75	0.195	0.002	0.988	0.005
20.25	0.201	0.002	0.776	0.005
20.75	0.273	0.003	0.89	0.01

1e.

In θ=-20° => R_{glass}= (132± 2) kΩ , R_{film}= (518±5) kΩ

θ	T_{film}	θ	T_{film}
θ = -20°	0.255	19.25	0.134
		19.50	0.158
		19.75	0.197
		20.00	0.227
		20.25	0.259
		20.50	0.276
		20.75	0.307

Graphics



We see that: T(θ = 20.25°) = T(θ = -20°)

δ (degree)	0.25±0.08

Task 2.

2a.

$$\lambda = d\sin\left(\theta - \frac{\delta}{2}\right) \Rightarrow \Delta\lambda = \lambda\sqrt{\left(\frac{\Delta d}{d}\right)^2 + \cot^2\left(\theta - \frac{\delta}{2}\right)\left(\Delta\theta^2 + \frac{\Delta\delta^2}{4}\right)} \approx d\ \cos(\theta)\left(\frac{0.1\pi}{180}\right)$$

where $\Delta\theta = \Delta\delta = 5' = 0.08$ degree

and $d = \frac{1}{600}$ mm

$$\boxed{\Delta\lambda = 2.9\cos(\theta)\ (\text{nm})}$$

$$T_{film} = \frac{R_{glass}}{R_{film}} \Rightarrow \Delta T = T_{film}\sqrt{\left(\frac{\Delta R_{film}}{R_{film}}\right)^2 + \left(\frac{\Delta R_{glass}}{R_{glass}}\right)^2}$$

$$\boxed{\Delta T = \frac{R_{glass}}{R_{film}}\sqrt{\left(\frac{\Delta R_{film}}{R_{film}}\right)^2 + \left(\frac{\Delta R_{glass}}{R_{glass}}\right)^2}}$$

2b.

$$13 \le \theta \le 26$$

$$\boxed{2.6 \le \Delta\lambda \le 2.8\ \text{nm}}$$

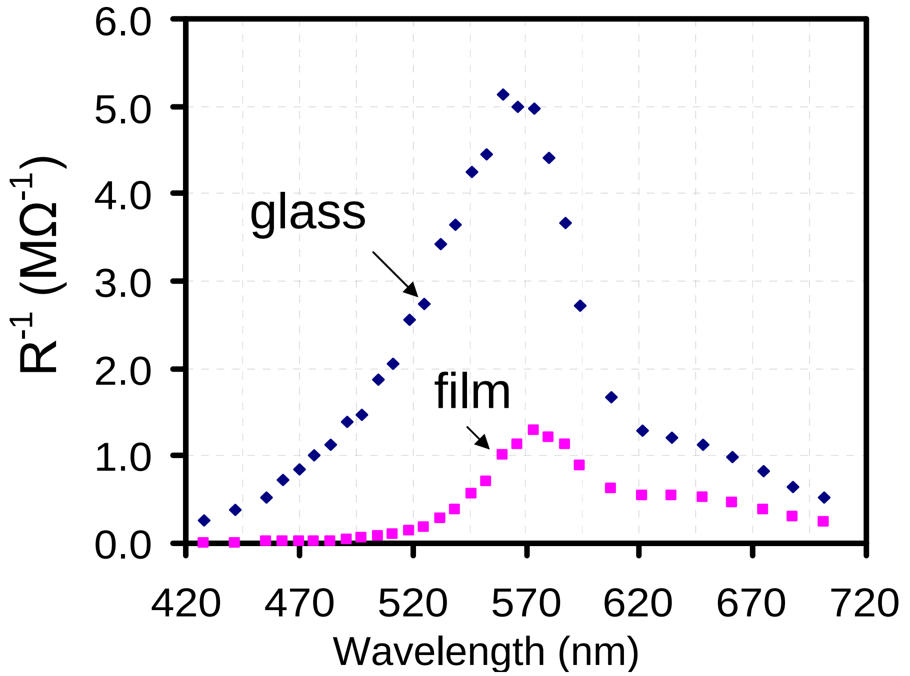
2c.

Table 2c. The calculated parameters using the measured parameters

θ (degree)	λ (nm)	$I_g/C(\lambda)$ ($M\Omega^{-1}$)	$I_s/C(\lambda)$ ($M\Omega^{-1}$)	T_{film}	αt
15.0	428	0.265	0.00546	0.0206	3.88
15.5	442	0.388	0.00758	0.0195	3.94
16.0	456	0.532	0.0115	0.0216	3.83
16.25	463	0.725	0.0150	0.0208	3.88
16.5	470	0.840	0.0194	0.0231	3.77
16.75	477	1.00	0.0250	0.0250	3.69
17.0	484	1.12	0.0299	0.0266	3.63
17.25	491	1.39	0.0427	0.0308	3.48
17.5	498	1.47	0.0515	0.0351	3.35
17.75	505	1.87	0.0781	0.0418	3.17
18.0	512	2.06	0.096	0.0467	3.06
18.25	518	2.56	0.146	0.0572	2.86
18.5	525	2.74	0.185	0.0676	2.69
18.75	532	3.41	0.289	0.0847	2.47
19.0	539	3.65	0.376	0.103	2.27
19.25	546	4.26	0.568	0.134	2.01
19.5	553	4.44	0.704	0.158	1.84
19.75	560	5.13	1.01	0.197	1.62
20.0	567	5.00	1.14	0.227	1.48
20.25	573	4.98	1.29	0.259	1.35
20.5	580	4.41	1.22	0.276	1.29
20.75	587	3.66	1.12	0.307	1.18
21.0	594	2.72	0.890	0.328	1.12
21.5	607	1.67	0.621	0.373	0.99
22.0	621	1.29	0.541	0.419	0.87
22.5	634	1.20	0.535	0.444	0.81
23.0	648	1.14	0.518	0.456	0.79
23.5	661	0.99	0.467	0.472	0.75
24.0	675	0.826	0.388	0.469	0.76
24.5	688	0.649	0.306	0.471	0.75
25.0	701	0.524	0.242	0.462	0.77

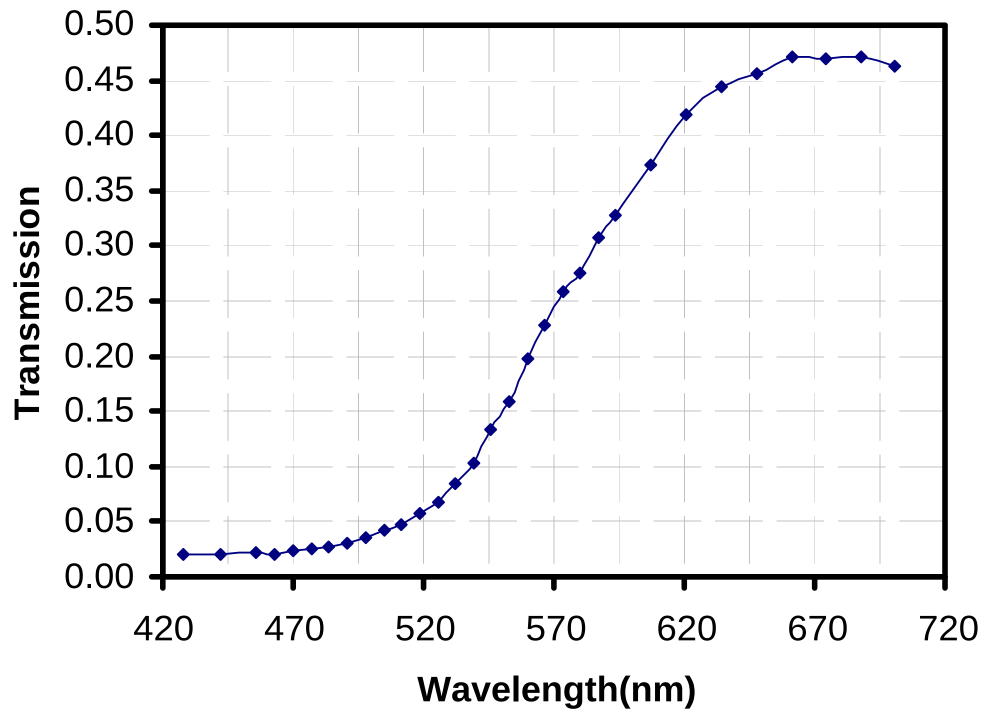
2d.

Graphics



$\lambda_{max}(I_{glass})$	564±5 (nm)
$\lambda_{max}(I_{film})$	573±5 (nm)

2e. Graphics



Task 3.

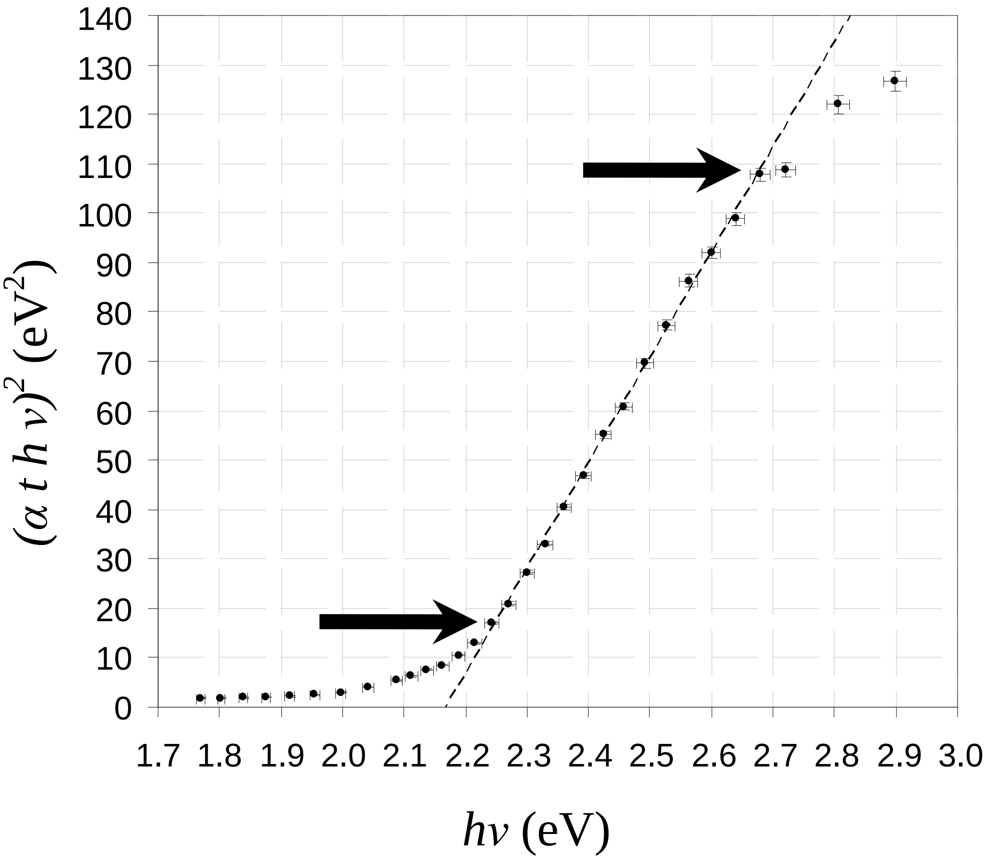
3a.

Table 3a. The calculated parameters for each measured data point

θ (degree)	x (eV)	y (eV^2)
15.00	2.898	126.6
15.50	2.806	121.9
16.00	2.720	108.8
16.25	2.679	107.8
16.50	2.639	98.9
16.75	2.600	92.0
17.00	2.563	86.3
17.25	2.527	77.4
17.50	2.491	69.7
17.75	2.457	60.9
18.00	2.424	55.1
18.25	2.392	46.8
18.50	2.360	40.4
18.75	2.330	33.1
19.00	2.300	27.3
19.25	2.271	20.91
19.50	2.243	17.07
19.75	2.215	12.92
20.00	2.188	10.51
20.25	2.162	8.53
20.50	2.137	7.56
20.75	2.112	6.23
21.00	2.088	5.43
21.50	2.041	4.06
22.00	1.997	3.02
22.50	1.954	2.52
23.00	1.914	2.26
23.50	1.875	1.98
24.00	1.838	1.94
24.50	1.803	1.84
25.00	1.769	1.86

3b.

Graphics



$x_{min} = 2.24(\text{eV})$	$x_{max} = 2.68(\text{eV})$

3c.

$$\alpha h\nu = A(h\nu - E_g)^{\frac{1}{2}} \Rightarrow (\alpha t\, h\nu)^2 = (At)^2(h\nu - E_g)$$

$$\Rightarrow y = (At)^2(x - E_g) \Rightarrow m = (At)^2 \Rightarrow t = \frac{\sqrt{m}}{A}$$

$$\Rightarrow \frac{\Delta t}{t} = \frac{\Delta m}{2m}$$

$t = \dfrac{\sqrt{m}}{A}$
$\Delta t = \dfrac{\Delta m}{2A\sqrt{m}}$

In linear range we have, m=213 (eV), r^2= 0.9986, E_g=2.17 (eV)
and we have A = 0.071 $\left(\mathrm{eV}^{1/2}/\mathrm{nm}\right)$ so we find t= 206 (nm)

$$\Delta m = \sqrt{\frac{(\delta y)^2 + \frac{m^2}{R^2}(\delta x)^2}{\sum_i x_i^2 - N\bar{x}^2}} \approx \sqrt{\frac{(\delta y)^2 + (m\,\delta x)^2}{\sum_i x_i^2 - N\bar{x}^2}} = \sqrt{\frac{(\delta xy)^2}{\sum_i x_i^2 - N\bar{x}^2}},\ (\delta xy)^2 = (\delta y)^2 + (m\,\delta x)^2$$

where δx & δy are the mean of error range of x & y

$$\delta x \approx \sqrt{\frac{\sum_i \delta x_i^{\,2}}{N}} \ \&\ \delta y = \sqrt{\frac{\sum_i \delta y_i^{\,2}}{N}} \quad \text{So} \qquad \delta x \approx 0.014\ (\mathrm{eV}),\ \delta y \approx 0.9(\mathrm{eV})^2$$

→ Δm ≈ 10 (eV) → Δt = t × Δm/(2 m) ≈ 5 (nm)

$$\Delta E_g = \frac{1}{m}\sqrt{\left(\left(\frac{m^2\,\delta x^2 + \delta y^2}{N}\right) + \left(\frac{\bar{y}}{m}\right)^2 \Delta m^2\right)} = \frac{1}{m}\sqrt{\left(\left(\frac{\delta xy^2}{N}\right) + \left(\frac{\bar{y}}{m}\right)^2 \Delta m^2\right)}$$

$$\Delta E_g \approx 0.02(\mathrm{eV})$$

Table 3d. The calculated values of E_g and t using Fig. 3

E_g (eV)	ΔE_g (eV)	t (nm)	Δt (nm)
2.17	0.02	206	5

Description of the Apparatus

In Fig.1 you can see the general view of the apparatus set up on your desk, which will be used in the experiment. The instrument is a spectroscope to be equipped with a detector to act as a simple spectrometer.

To start adjusting the apparatus, you should first pull up the white cover of the box (Fig.1). The cover pivots on one side of the base of the apparatus. In order to establish a dark environment for the detector, the cover should be returned to its initial position and kept tightly closed during the measurement of the spectra. The power cord has a switch that turns the halogen lamp on and off. There are four screws to level the apparatus (a magnified view of which you can see in right inset of Fig.1)

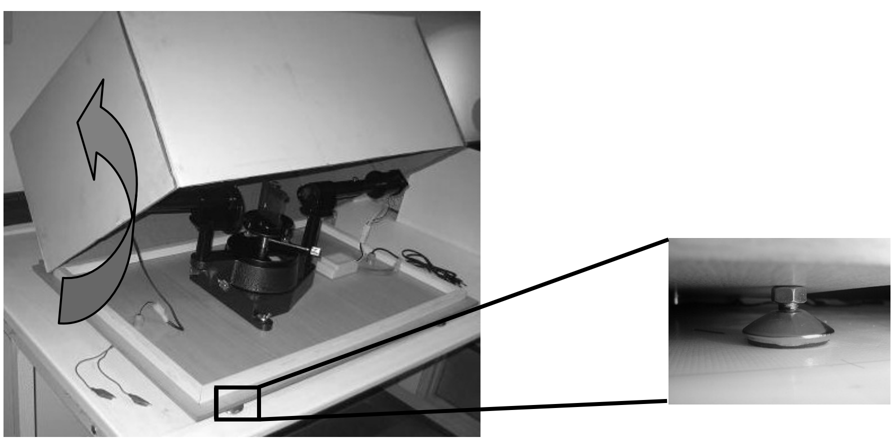


Figure 1. Apparatus of the experiment. One of the level adjusting screws is enlarged in the right inset.

Warning 1: Avoid touching the halogen lamp and its holder which will be ***hot*** after the lamp is turned on!

Warning 2: Do not manipulate the adaptor and its connections. Power is supplied to the apparatus through 220 V outlets!

The top view of the apparatus is shown in Fig.2 . The details are introduced in the figure.

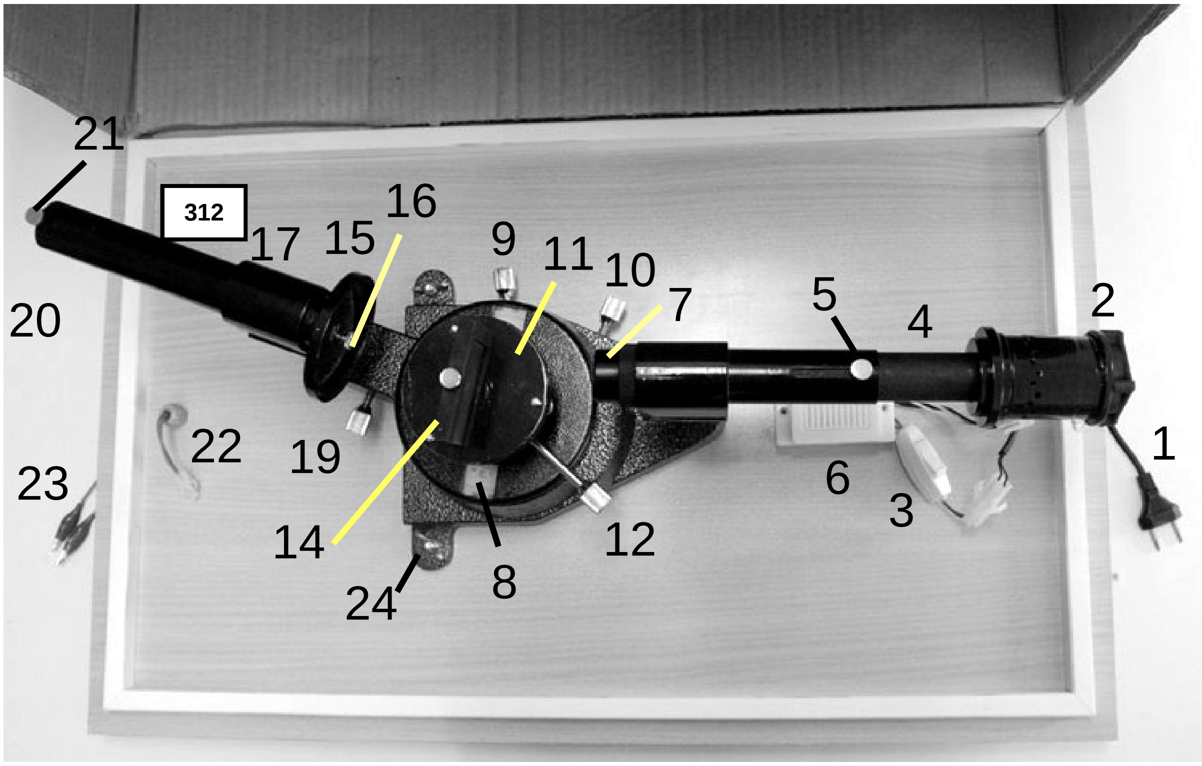


Figure 2.

1. Power cord
2. Halogen lamp and its cooling fan
3. On/Off switch
4. Arm of adjustable length
5. Adjusting screw
6. Adaptor: 220V – less than 12 V
7. Lens
8. Vernier
9. Vernier's lock
10. Fine adjustment screw for the vernier
11. Grating's stage
12. Grating's stage's fixing screw
13. Adjustment screw for leveling the grating's stage (shown in Fig. 4)
14. Grating holder
15. Sample holder
16. Fixing and adjusting screw for the sample and glass holder (Fig. 6)
17. Rotatable arm
18. Rotatable arm's lock (Fig.4)
19. Fine adjustment for the rotatable arm
20. Detector position
21. Fixing screw for the detector
22. Connecting socket for the detector
23. Connection to the multimeter
24. Fixing screw to the base

The number mentioned on the top-left corner, is the **apparatus number**.

The angle, which the rotatable arm makes with the direction of the fixed arm of the apparatus, could be measured by a protractor equipped by a vernier. In this vernier resolution scale is 30' (minutes of arc). This instrument is able to measure an angle with accuracy of 5'.

In addition to the apparatus you should find a box (Figure 3), containing the following elements:

1: a detector in its holder; 2: a 600 line/mm grating; 3: the sample and a glass substrate mounted in a frame.

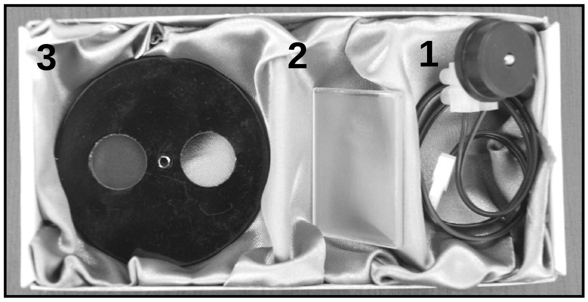


Figure 3. The small box, containing the glass and sample holder, a diffraction grating and a photoresistor.

First, you should take the grating out of its cover and put it into its frame (the grating holder, Fig. 4), carefully.

> CAUTION: Touching the surface of the grating could reduce its diffraction efficiency seriously, or even damage it!

There are three adjustment screws (Fig. 4) for making the grating stand vertically in its position.

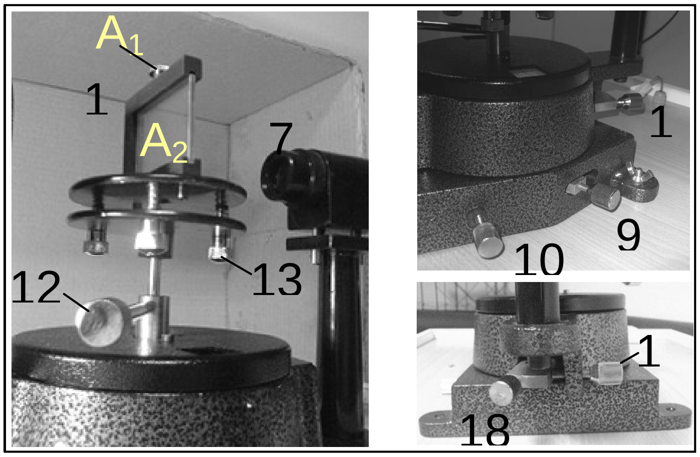


Figure 4. Locking, fixing and adjusting screws of the apparatus. A_1: Fixing screw for the grating; A_2: The grating. 7, 9, 10, 12-14, 18 and 19 are explained in Figure 2.

The detector should be tight to its position, in the end of the rotatable arm, (Figure 5):

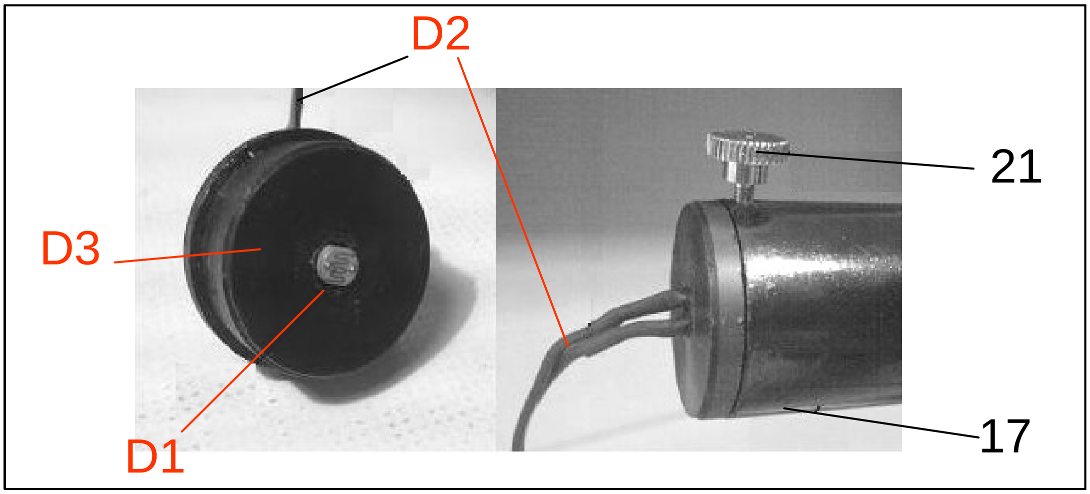


Figure 5 . The detector and its holder. D1: The photoresistor; D2: connecting wire. D3: The detector holder. 17 and 21 are explained in Fig. 2.

The sample and the glass substrate are fixed to a frame (holder) (Fig. 6c), which would be attached to the instrument by a fixing screw (Fig. 6a, item 16). This frame is rotatable and one can put the sample or the glass substrate in front of the entrance hole, by turning the frame around the fixing screw (Fig. 6a).

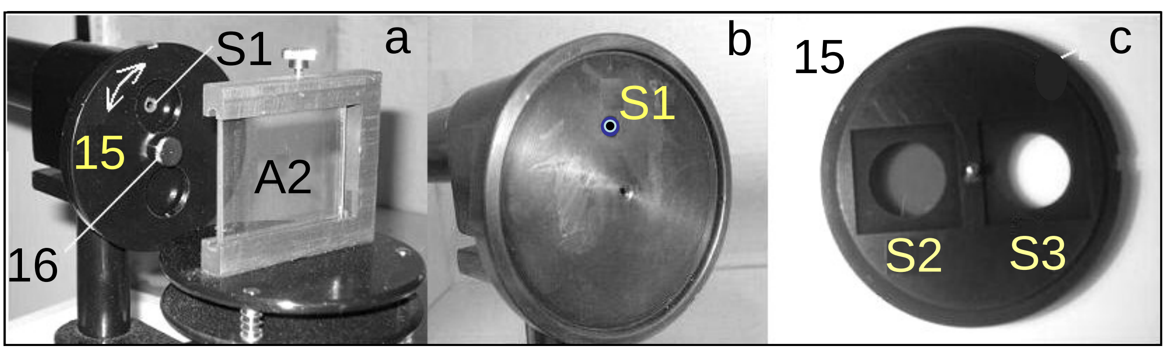


Figure 6 . The Sample and the glass holder. S1: Entrance hole; S2: Sample; S3: Glass substrate. 15 and 16 are explained in Fig. 2.

The Multimeter which you should use for recording the signal detected by the photoresistor is shown in the Fig. 7. This multimeter can measure up to 200 MΩ. The red and black probe wires should be connected to the instrument as is shown in the Fig. 7. The on/off button is placed on the left hand side of the multimeter (Fig. 7, item M1).

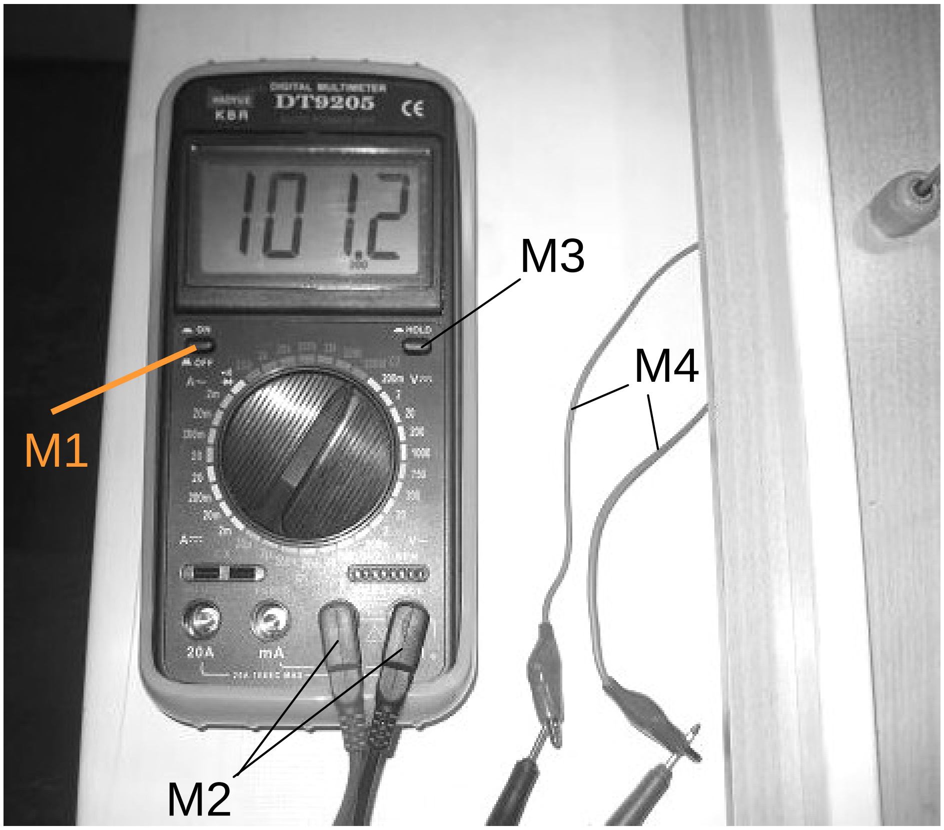


Figure 7. The Multimeter for measuring the resistance of the photoresistor. M1: on/off switch; M2: probe wires; M3: Hold button; M4: connections to the apparatus.

Note: The multimeter has auto-off feature. In the case of auto-off, you should push on/off button (M1) twice, successively.

❖ Hold button should not be active during the experiment.

Experimental Problem

Determination of energy band gap of semiconductor thin films

I. Introduction

Semiconductors can be roughly characterized as materials whose electronic properties fall somewhere between those of conductors and insulators. To understand semiconductor electronic properties, one can start with the *photoelectric effect* as a well-known phenomenon. The photoelectric effect is a quantum electronic phenomenon, in which photoelectrons are emitted from the matter through the absorption of sufficient energy from electromagnetic radiation (i.e. photons). The minimum energy which is required for the emission of an electron from a metal by light irradiation (*photoelectron*) is defined as "*work function"*. Thus, only photons with a frequency ν higher than a characteristic threshold, i.e. with an energy $h\nu$ (h is the Planck's constant) more than the material's work function, are able to knock out the photoelectrons.

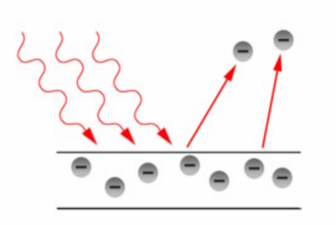


Figure 1. An illustration of photoelectron emission from a metal plate: The incoming photon should have an energy which is more than the work function of the material.

In fact, the concept of work function in the photoelectric process is similar to the concept of the energy band gap of a semiconducting material. In solid state physics, the band gap E_g is the energy difference between the top of the valence band and the bottom of the conduction band of insulators and semiconductors. The valence band is completely filled with electrons, while the conduction band is empty however electrons can go from the valence band to the conduction band if they acquire sufficient energy (at least equal to the band gap energy).The semiconductor's conductivity strongly depends on its energy band gap.

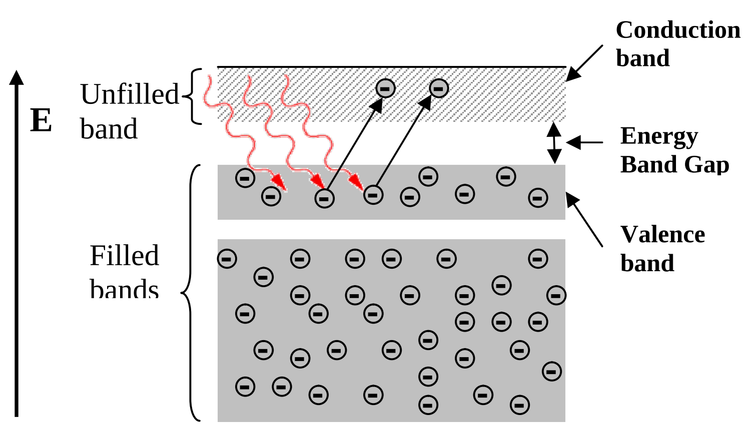


Figure 2. Energy band scheme for a semiconductor.

Band gap engineering is the process of controlling or altering the band gap of a material by controlling the composition of certain semiconductor alloys. Recently, it has been shown that by changing the nanostructure of a semiconductor it is possible to manipulate its band gap.

In this experiment, we are going to obtain the energy band gap of a thin-film semiconductor containing nano-particle chains of iron oxide (Fe_2O_3) by using an optical method. To measure the band gap, we study the optical absorption properties of the transparent film using its optical transmission spectrum. As a rough statement, the absorption spectra shows a sharp increase when the energy of the incident photons equals to the energy band gap.

II. Experimental Setup

You will find the following items on your desk:

1. A large white box containing a spectrometer with a halogen lamp.
2. A small box containing a sample, a glass substrate, a sample-holder, a grating, and a photoresistor.
3. A multimeter.
4. A calculator.
5. A ruler.
6. A card with a hole punched in its center.
7. A set of blank labels.

The spectrometer contains a goniometer with a precision of $5'$. The Halogen lamp acts as the source of radiation and is installed onto the fixed arm of the spectrometer (for detailed information see the enclosed "Description of Apparatus").

The small box contains the following items:

1. A sample-holder with two windows: a glass substrate coated with Fe_2O_3 film mounted on one window and an uncoated glass substrate mounted on the other.
2. A photoresistor mounted on its holder, which acts as a light detector.
3. A transparent diffraction grating (600 line/mm).

Note: *Avoid touching the surface of any component in the small box!*

A schematic diagram of the setup is shown in Figure 3:

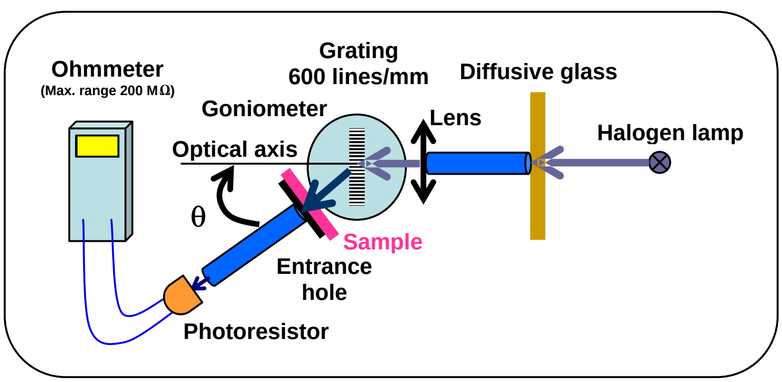


Figure 3. Schematic diagram of the experimental setup.

III. Methods

To obtain the transmission of a film at each wavelength, $T_{film}(\lambda)$, one can use the following formula:

$$T_{film}(\lambda) = I_{film}(\lambda) / I_{glass}(\lambda) \qquad (1)$$

where I_{film} and I_{glass} are respectively the intensity of the light transmitted from the coated glass substrate, and the intensity of the light transmitted from the uncoated glass slide. The value of I can be measured using a light detector such as a photoresistor. In a photoresistor, the electrical resistance decreases when the intensity of the incident light increases. Here, the value of I can be determined from the following relation:

$$I(\lambda) = C(\lambda) R^{-1} \qquad \textbf{(2)}$$

where R is the electrical resistance of the photoresistor, C is a λ-dependent coefficient.

The transparent grating on the spectrometer diffracts different wavelengths of light into different angles. Therefore, to study the variations of T as a function of λ, it is enough to change the angle of the photoresistor (θ') with respect to the optical axis (defined as the direction of the incident light beam on the grating), as shown in Figure 4.

From the principal equation of a diffraction grating:

$$n\lambda = d\,[\sin(\theta' - \theta_0) + \sin\theta_0] \quad (3)$$

one can obtain the angle θ' corresponding to a particular λ: n is an integer number representing the order of diffraction, d is the period of the grating, and θ_o is the angle the normal vector to the surface of grating makes with the optical axis (see Fig. 4). (In this experiment we shall try to place the grating perpendicular to the optical axis making $\theta_o = 0$, but since this cannot be achieved with perfect precision the error associated with this adjustment will be measured in task 1-e.)

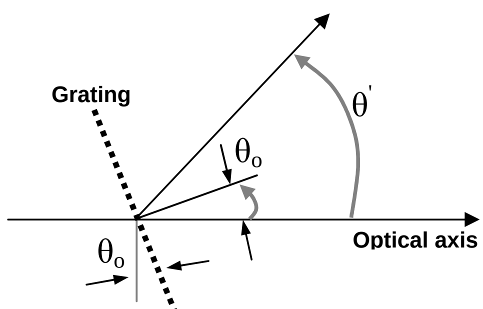


Figure 4. Definition of the angles involved in Equation 3.

Experimentally it has been shown that for photon energies slightly larger than the band gap energy, the following relation holds:

$$\alpha h\nu = A(h\nu - E_g)^{\eta} \quad (4)$$

where α is the absorption coefficient of the film, A is a constant that depends on the film's material, and η is the constant determined by the absorption mechanism of the film's material and structure. Transmission is related to the value of α through the well-known absorption relation:

$$T_{film} = \exp(-\alpha\, t) \qquad (5)$$

where t is thickness of the film.

IV. Tasks:

0. *Your apparatus and sample box (small box containing the sample holder) are marked with numbers. Write down the* ***Apparatus number*** *and* ***Sample number*** *in their appropriate boxes, in the answer sheet.*

1. <u>Adjustments and Measurements:</u>

1-a	Check the vernier scale and report the maximum precision ($\Delta\theta$).	*0.1 pt*

Note: Magnifying glasses are available on request.

Step1:
To start the experiment, turn on the Halogen lamp to warm up. It would be better not to turn off the lamp during the experiment. Since the halogen lamp heats up during the experiment, please be careful not to touch it.

Place the lamp as far from the lens as possible, this will give you a parallel light beam.

We are going to make a rough zero-adjustment of the goniometer without utilizing the photoresistor. Unlock the rotatable arm with screw 18 (underneath the arm), and visually align the rotatable arm with the optical axis. Now, firmly lock the rotatable arm with screw 18. Unlock the vernier with screw 9 and rotate the stage to 0 on the vernier scale. Now firmly lock the vernier with screw 9 and use the vernier fine-adjustment screw (screw 10) to set the zero of the vernier scale. Place the grating inside its holder. Rotate the grating's stage until the diffraction grating is roughly perpendicular to the optical axis. Place the card with a hole in front of the light source and position the hole such that a beam of light is incident on the grating. Carefully rotate the grating so that the spot of reflected light falls onto the hole. Then the reflected light beam coincides with the incident beam. Now lock the grating's stage by tightening screw 12.

1-b	By measuring the distance between the hole and the grating, estimate the precision of this adjustment ($\Delta\theta_o$).	*0.3 pt*
	Now, by rotating the rotatable arm, determine and report the range of angles for which the first-order diffraction of visible light (from blue to red) is observed.	*0.2 pt*

Step 2:
Now, install the photoresistor at the end of the rotatable arm. To align the system optically, by using the photoresistor, loosen the screw 18, and slightly turn the rotatable arm so that the photoresistor shows a minimum resistance. For fine positioning, firmly lock screw 18, and use the fine adjustment screw of the rotatable arm.

Use the vernier fine-adjustment screw to set the zero of the vernier scale.

1-c	Report the measured minimum resistance value ($R_{\min}^{(0)}$).	*0.1 pt*
	Your zero-adjustment is more accurate now, report the precision of this new adjustment ($\Delta\varphi_o$) . Note: $\Delta\varphi_o$ is the error in this alignment i.e. it is a measure of misalignment of the rotatable arm and the optical axis.	*0.1 pt*

➢ **Hint:** After this task you should tighten the fixing screws of the vernier. Moreover, tighten the screw of the photoresistor holder to fix it and do not remove it during the experiment.

Step 3:
Move the rotatable arm to the region of the first-order diffraction. Find the angle at which the resistance of the photoresistor is minimum (maximum light intensity). Using the balancing screws, you can slightly change the *tilt* of the grating's stage, to achieve an even lower resistance value.

1-c	Report the minimum value of the observed resistance ($R_{\min}^{(1)}$) in its appropriate box.	*0.1 pt*

It is now necessary to check the perpendicularity of the grating for zero adjustment, again. For this you must use the reflection-coincidence method of Step 1.

Important: From here onwards carry out the experiment in dark (close the cover).

Measurements: Screw the sample-holder onto the rotatable arm. Before you start the measurements, examine the appearance of your semiconductor film (sample). Place the sample in front of the entrance hole S_1 on the rotatable arm such that a uniformly coated part of the sample covers the hole. To make sure that every time you will be working with the same part of the sample make proper markings on the sample holder and the rotatable arm with blank labels.

Attention: At higher resistance measurements it is necessary to allow the photoresistor to relax, therefore for each measurement in this range wait 3 to 4 minutes before recording your measurement.

1-d	Measure the resistance of the photoresistor for the uncoated glass substrate and the glass substrate coated with semiconductor layer as a function of the angle θ (the value read by the goniometer for the angle between the photoresistor and your specified optical axis). Then fill in Table 1d. Note that you need at least 20 data points in the range you found in Step 1b. Carry out your measurement using the appropriate range of your ohmmeter.	*2.0 pt*
	Consider the error associated with each data point. Base your	*1.0 pt*

	answer only on your direct readings of the ohmmeter.	

Step 4:
The precision obtained so far is still limited since it is impossible to align the rotatable arm with the optical axis and/or position the grating perpendicular to the optical axis with 100% precision. So we still need to find the asymmetry of the measured transmission at both sides of the optical axis (resulting from the deviation of the normal to the grating surface from the optical axis (θ_o)).

To measure this asymmetry, follow these steps:

1-e	First, measure T_{film} at $\theta = -20°$. Then, obtain values for T_{film} at some other angles around $+20°$. Complete Table 1e (you can use the values obtained in Table 1d).	*0.6 pt*
	Draw T_{film} versus θ and visually draw a curve.	*0.6 pt*

On your curve find the angle γ for which the value of T_{film} is equal to the T_{film} that you measured at $\theta = -20^o$ ($\gamma \equiv \theta|_{T_{film}=T_{film}(-20°)}$). Denote the difference of this angle with $+20°$ as δ, in other words:

$$\delta = \gamma - 20° \qquad (6)$$

1-e	Report the value of δ in the specified box.	*0.2 pt*

Then for the first-order diffraction, Eq. (3) can be simplified as follows:

$$\lambda = d\sin(\theta - \delta/2), \qquad (7)$$

where θ is the angle read on the goniometer.

2. <u>Calculations:</u>

2-a	Use Eq. (7) to express $\Delta\lambda$ in terms of the errors of the other parameters (assume d is exact and there is no error is associated with it). Also using Eqs. (1), (2), and (5), express ΔT_{film} in terms of R and ΔR.	*0.6 pt*

2-b	Report the range of values of $\Delta\lambda$ over the region of first-order diffraction.	*0.3 pt*

2-c	Based on the measured parameters in Task 1, complete Table 2c for each θ. Note that the wavelength should be calculated using Eq. (7).	*2.4 pt*

2-d	Plot R_{glass}^{-1} and R_{film}^{-1} as a function of wavelength together on the same diagram. Note that on the basis of Eq. (2) behaviors of R_{glass}^{-1} and R_{film}^{-1} can reasonably give us an indication of the way I_{glass} and I_{film} behave, respectively.	*1.5 pt*
	In Table 2d, report the wavelengths at which R_{glass} and R_{film} attain their minimum values.	*0.4 pt*

2-e	For the semiconductor layer (sample) plot T_{film} as a function of wavelength. This quantity also represents the variation of the film transmission in terms of wavelength.	*1.0 pt*

3. Data analysis:

By substituting $\eta = 1/2$ and $A = 0.071$ ((eV)$^{1/2}$/nm) in Eq. (4) one can find values for E_g and t in units of eV and nm, respectively. This will be accomplished by plotting a suitable diagram in an $x-y$ coordinate system and doing an extrapolation in the region satisfying this equation.

3-a	By assuming $x = h\nu$ and $y = (\alpha t h\nu)^2$ and by using your measurements in Task 1, fill in Table 3a for wavelengths around 530 nm and higher. Express your results (x and y) with the correct number of significant figures (digits), based on the estimation of the error on one single data point. Note that $h\nu$ should be calculated in units of eV and wavelength in units of nm. Write the unit of each variable between the parentheses in the top row of the table.	*2.4 pt*

3-b	Plot y versus x.	*2.6 pt*
	Note that the y parameter corresponds to the absorption of the film. Fit a line to the points in the linear region around 530 nm.	
	Specify the region where Eq. (4) is satisfied, by reporting the values of the smallest and the largest x-coordinates for the data points to which you fit the line.	

3-c	Call the slope of this line m, and find an expression for the film thickness (t) and its error (Δt) in terms of m and A (consider A to have no error).	*0.5 pt*

3-d	Obtain the values of E_g and t and their associated errors in units of eV and nm, respectively. Fill in Table 3d.	*3.0 pt*

❖ Some useful physical constants required for your analysis:

- Speed of the light: $c = 3.00 \times 10^8$ m/s
- Plank's constant: $h = 6.63 \times 10^{-34}$ J.s
- Electron charge: $e = 1.60 \times 10^{-19}$ C

Question "Orange"

1.1)
First of all, we use the Gauss's law for a single plate to obtain the electric field,

$E = \dfrac{\sigma}{\varepsilon_0}$. (0.2)

The density of surface charge for a plate with charge, Q and area, A is

$\sigma = \dfrac{Q}{A}$. (0.2)

Note that the electric field is resulted by two equivalent parallel plates. Hence the contribution of each plate to the electric field is $\frac{1}{2}E$. Force is defined by the electric filed times the charge, then we have

Force = $\dfrac{1}{2}EQ = \dfrac{Q^2}{2\varepsilon_0 A}$ (0.2)+ (0.2) (The ½ coefficient + the final result)

1.2)
The Hook's law for a spring is
$F_m = -k\,x$. (0.2)
In 1.2 we derived the electric force between two plates is

$F_e = \dfrac{Q^2}{2\varepsilon_0 A}$.

The system is stable. The equilibrium condition yields
$F_m = F_e$, (0.2)

$\Rightarrow \qquad x = \dfrac{Q^2}{2\varepsilon_0 A\,k}$ (0.2)

1.3)
The electric field is constant thus the potential difference, V is given by
$V = E(d - x)$ (0.2)
(Other reasonable approaches are acceptable. For example one may use the definition of capacity to obtain V.)
By substituting the electric field obtained from previous section to the above equation, we get, $V = \dfrac{Q\,d}{\varepsilon_0 A}\left(1 - \dfrac{Q^2}{2\varepsilon_0 A\,k\,d}\right)$ (0.2)

1.4)
C is defined by the ratio of charge to potential difference, then

$C = \dfrac{Q}{V}$. (0.1)

Using the answer to 1.3, we get $\frac{C}{C_0} = \left(1 - \frac{Q^2}{2\varepsilon_0 A k d}\right)^{-1}$ (0.2)

1.5)

Note that we have both the mechanical energy due to the spring

$U_m = \frac{1}{2}kx^2$, (0.2)

and the electrical energy stored in the capacitor.

$U_E = \frac{Q^2}{2C}$. (0.2)

Therefore the total energy stored in the system is

$U = \frac{Q^2 d}{2\varepsilon_0 A}\left(1 - \frac{Q^2}{4\varepsilon_0 A k d}\right)$ (0.2)

2.1)

For the given value of x, the amount of charge on each capacitor is

$Q_1 = V\,C_1 = \frac{\varepsilon_0 A V}{d - x}$, (0.2)

$Q_2 = V\,C_2 = \frac{\varepsilon_0 A V}{d + x}$. (0.2)

2.2)

Note that we have two capacitors. By using the answer to 1.1 for each capacitor, we get

$F_1 = \frac{{Q_1}^2}{2\varepsilon_0 A}$,

$F_2 = \frac{{Q_2}^2}{2\varepsilon_0 A}$.

As these two forces are in the opposite directions, the net electric force is

$F_E = F_1 - F_2$, (0.2) $\Rightarrow$ $F_E = \frac{\varepsilon_0 A V^2}{2}\left(\frac{1}{(d-x)^2} - \frac{1}{(d+x)^2}\right)$ (0.2)

2.3)

Ignoring terms of order x^2 in the answer to 2.2., we get

$F_E = \frac{2\varepsilon_0 A V^2}{d^3} x$ (0.2)

2.4)

There are two springs placed in series with the same spring constant, k , then the mechanical force is

$F_m = -2kx$.   (The coefficient (2) has (0.2))

Combining this result with the answer to 2.4 and noticing that these two forces are in the opposite directions, we get

$F = F_m + F_E$, $\Rightarrow$ $F = -2\left(k - \frac{\varepsilon_0 A V^2}{d^3}\right)x$, (Opposite signs of the two forces have (0.3))

$$\Rightarrow \quad k_{eff} = 2\left(k - \frac{\varepsilon_0 A V^2}{d^3}\right) \quad (0.2)$$

2.5)

By using the Newtown's second law,

$F = ma$ (0.2)

and the answer to 2.4, we get

$$a = -\frac{2}{m}\left(k - \frac{\varepsilon_0 A V^2}{d^3}\right)x \quad (0.2)$$

3.1)

Starting with Kirchhoff's laws, for two electrical circuits, we have

$$\begin{cases} \dfrac{Q_S}{C_S} + V - \dfrac{Q_2}{C_2} = 0 \\ -\dfrac{Q_S}{C_S} + V - \dfrac{Q_1}{C_1} = 0 \\ Q_2 - Q_1 + Q_S = 0 \end{cases}$$

(Each has (0.3), Note: the sings may depend on the specific choice made)

Noting that $V_s = \frac{Q_S}{C_S}$ one obtains

$$\Rightarrow V_S = V\frac{\dfrac{2\varepsilon_0 A x}{d^2 - x^2}}{C_S + \dfrac{2\varepsilon_0 A d}{d^2 - x^2}} .$$

((0.4) + (0.2): (0.4) for solving the above equations and (0.2) for final result)

Note: Students may simplify the above relation using the approximation $d^2 >> x^2$. It does not matter in this section.

3.2)
Ignoring terms of order x^2 in the answer to 3.1., we get

$$V_S = V \frac{2\varepsilon_0 A x}{d^2 C_S + 2\varepsilon_0 A d} .$$ (0.2)

4.1)
The ratio of the electrical force to the mechanical (spring) force is

$$\frac{F_E}{F_m} = \frac{\varepsilon_0 A V^2}{k d^3} ,$$

Putting the numerical values:

$$\frac{F_E}{F_m} = 7.6 \times 10^{-9} .$$ ((0.2) + (0.2) + (0.2): (0.2) for order of magnitude, (0.2) for two significant digits and (0.2) for correct answer (7.6 or 7.5)).

As it is clear from this result, we can ignore the electrical forces compared to the electric force.

4.2)
As seen in the previous section, one may assume that the only force acting on the moving plate is due to springs:

$F = 2k x$. (The concept of equilibrium (0.2))

Hence in mechanical equilibrium, the displacement of the moving plate is

$$x = \frac{ma}{2k} .$$

The maximum displacement is twice this amount, like the mass spring system in a gravitational force field, when the mass is let to fall.

$x_{\max} = 2x$ (0.2)

$x_{\max} = \dfrac{ma}{k}$ (0.2)

4.3)
At the acceleration

$a = g$, (0.2)

The maximum displacement is

$$x_{\max} = \frac{mg}{k} .$$

Moreover, from the result obtained in 3.2, we have

$$V_S = V \frac{2\varepsilon_0 A\, x_{\max}}{d^2 C_S + 2\varepsilon_0 A\, d}$$

This should be the same value given in the problem, $0.15\,\mathrm{V}$.

$$\Rightarrow \qquad C_S = \frac{2\varepsilon_0 A}{d}\left(\frac{V\, x_{\max}}{V_S\, d} - 1\right) \qquad (0.2)$$

$$\Rightarrow \qquad C_S = 8.0 \times 10^{-11}\,\mathrm{F} \qquad (0.2)$$

4.4)

Let ℓ be the distance between the driver's head and the steering wheel. It can be estimated to be about

$\ell = 0.4\,m - 1\,m$. (0.2)

Just at the time the acceleration begins, the relative velocity of the driver's head with respect to the automobile is zero.

$\Delta v(t=0) = 0$, (0.2)

then

$$\ell = \frac{1}{2} g\, t_1^{\,2} \qquad \Rightarrow \qquad t_1 = \sqrt{\frac{2\,\ell}{g}} \qquad (0.2)$$

$t_1 = 0.3 - 0.5\ s$ (0.2)

4.5)

The time t_2 is half of period of the harmonic oscillator, hence

$$t_2 = \frac{T}{2}, \qquad (0.3)$$

The period of harmonic oscillator is simply given by

$$T = 2\pi\sqrt{\frac{m}{2k}}\ , \qquad (0.2)$$

therefore,

$t_2 = 0.013\ s$. (0.2)

As $t_1 > t_2$, the airbag activates in time. (0.2)

In this problem we deal with a simplified model of accelerometers designed to activate the safety air bags of automobiles during a collision. We would like to build an electromechanical system in such a way that when the acceleration exceeds a certain limit, one of the electrical parameters of the system such as the voltage at a certain point of the circuit will exceed a threshold and the air bag will be activated as a result.

Note: Ignore gravity in this problem.

1 Consider a capacitor with parallel plates as in Figure 1. The area of each plate in the capacitor is A and the distance between the two plates is d. The distance between the two plates is much smaller than the dimensions of the plates. One of these plates is in contact with a wall through a spring with a spring constant k, and the other plate is fixed. When the distance between the plates is d the spring is neither compressed nor stretched, in other words no force is exerted on the spring in this state. Assume that the permittivity of the air between the plates is that of free vacuum ε_0. The capacitance corresponding to this distance between the plates of the capacitor is $C_0 = \varepsilon_0 A/d$. We put charges $+Q$ and $-Q$ on the plates and let the system achieve mechanical equilibrium.

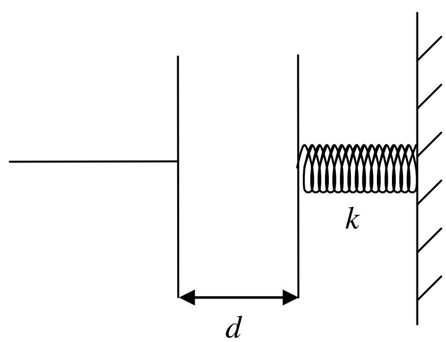


Figure 1

1.1	Calculate the electrical force, F_E, exerted by the plates on each other.	0.8
1.2	Let x be the displacement of the plate connected to the spring. Find x.	0.6
1.3	In this state, what is the electrical potential difference V between the plates of the capacitor in terms of Q, A, d, k?	0.4
1.4	Let C be the capacitance of the capacitor, defined as the ratio of charge to potential difference. Find C/C_0 as a function of Q, A, d and k.	0.3
1.5	What is the total energy, U, stored in the system in terms of Q, A, d and k?	0.6

Figure 2, shows a mass M which is attached to a conducting plate with negligible mass and also to two springs having identical spring constants k. The conducting plate can move back and forth in the space between two fixed conducting plates. All these plates are similar and have the same area A. Thus these three plates constitute two capacitors. As shown in Figure 2, the fixed plates are connected to the given potentials V and $-V$, and the middle plate is connected through a two-state switch to the ground. The wire connected to the movable plate does not disturb the motion of the plate and the three plates will always remain parallel. When the whole complex is not being accelerated, the distance from each fixed plate to the movable plate is d which is much smaller than the dimensions of the plates. The thickness of the movable plate can be ignored.

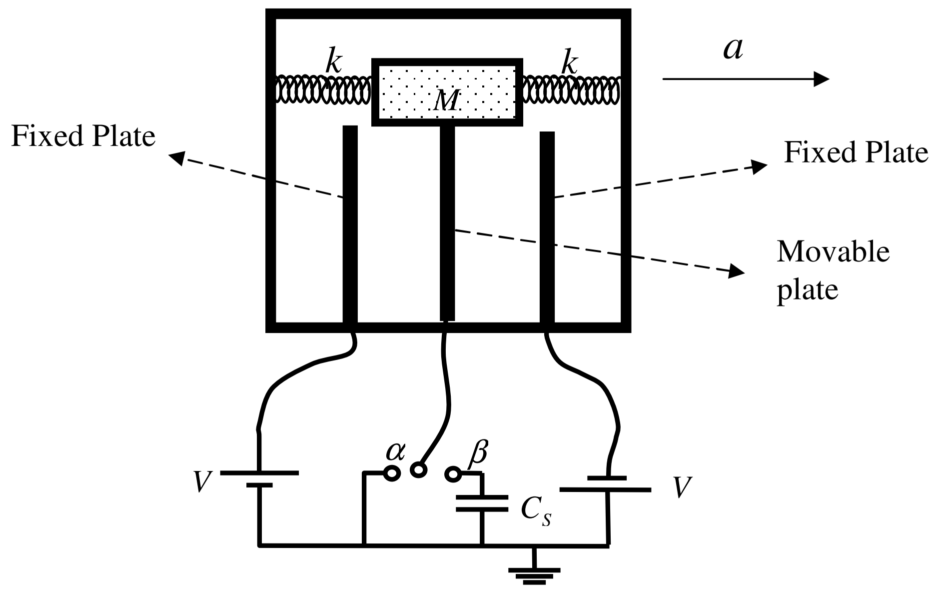


Figure 2

The switch can be in either one of the two states α and β. Assume that the capacitor complex is being accelerated along with the automobile, and the acceleration is constant. Assume that during this constant acceleration the spring does not oscillate and all components of this complex capacitor are in their equilibrium positions, i.e. they do not move with respect to each other, and hence with respect to the automobile.
Due to the acceleration, the movable plate will be displaced a certain amount x from the middle of the two fixed plates.

2 Consider the case where the switch is in state α i.e. the movable plate is connected to the ground through a wire, then

2.1	Find the charge on each capacitor as a function of x.	0.4
2.2	Find the net electrical force on the movable plate, F_E, as a function of x.	0.4
2.3	Assume $d >> x$ and terms of order x^2 can be ignored compared to terms of order d^2. Simplify the answer to the previous part.	0.2
2.4	Write the total force on the movable plate (the sum of the electrical and the spring forces) as $-k_{eff}\,x$ and give the form of k_{eff}.	0.7
2.5	Express the constant acceleration a as a function of x.	0.4

3 Now assume that the switch is in state β i.e. the movable plate is connected to the ground through a capacitor, the capacitance of which is C_S (there is no initial charge on the capacitors). If the movable plate is displaced by an amount x from its central position,

3.1	Find V_S the electrical potential difference across the capacitor C_S as a function of x.	1.5

3.2	Again assume that $d >> x$ and ignore terms of order x^2 compared to terms of order d^2. Simplify your answer to the previous part.	0.2

4 We would like to adjust the parameters in the problem such that the air bag will not be activated in normal braking but opens fast enough during a collision to prevent the driver's head from colliding with the windshield or the steering wheel. As you have seen in Part 2, the force exerted on the movable plate by the springs and the electrical charges can be represented as that of a spring with an effective spring constant k_{eff}. The whole capacitor complex is similar to a *mass and spring* system of mass *M* and spring constant k_{eff} under the influence of a constant acceleration a, which in this problem is the acceleration of the automobile.

Note: In this part of the problem, the assumption that the mass and spring are in equilibrium under a constant acceleration and hence are fixed relative to the automobile, no longer holds.

Ignore friction and consider the following numerical values for the parameters of the problem:

$$d = 1.0\,\text{cm}, \quad A = 2.5\times10^{-2}\,\text{m}^2, \quad k = 4.2\times10^{3}\,\text{N/m}, \quad \varepsilon_0 = 8.85\times10^{-12}\,\text{C}^2/\text{Nm}^2,$$
$$V = 12\,\text{V}, \; M = 0.15\,\text{kg}.$$

4.1	Using this data, find the ratio of the electrical force you calculated in section 2.3 to the force of the springs and show that one can ignore the electrical forces compared to the spring forces.	0.6

Although we did not calculate the electrical forces for the case when the switch is in the state β, it can be shown that in this situation, quite similarly, the electrical forces are as small and can be ignored.

4.2	If the automobile while traveling with a constant velocity, suddenly brakes with a constant acceleration a, what is the maximum displacement of the movable plate? Give your answer in parameter.	0.6

Assume that the switch is in state β and the system has been designed such that when the electrical voltage across the capacitor reaches $V_S = 0.15\text{V}$, the air bag is activated. We would like the air bag not to be activated during normal braking when the automobile's acceleration is less than the acceleration of gravity $g = 9.8\,m/s^2$, but be activated otherwise.

4.3	How much should C_S be for this purpose?	0.6

We would like to find out if the air bag will be activated fast enough to prevent the driver's head from hitting the windshield or the steering wheel. Assume that as a result of collision, the automobile experiences a deceleration equal to g but the driver's head keeps moving at a constant speed.

4.4	By estimating the distance between the driver's head and the steering wheel, find the time t_1 it takes before the driver's head hits the steering wheel.	0.8

4.5	Find the time t_2 before the air bag is activated and compare it to t_1. Is the air bag activated in time? Assume that airbag opens instantaneously.	0.9

Question "Pink"

1.1

Period = 3.0 days = 2.6×10^5 s. (0.4)

Period = $\dfrac{2\pi}{\omega}$ (0.2) $\Rightarrow$ $\omega = 2.4 \times 10^{-5}$ rad s^{-1}. (0.2)

1.2

Calling the minima in the diagram 1, $I_1/I_0 = \alpha = 0.90$ and $I_2/I_0 = \beta = 0.63$, we have:

$$\frac{I_0}{I_1} = 1 + \left(\frac{R_2}{R_1}\right)^2 \left(\frac{T_2}{T_1}\right)^4 = \frac{1}{\alpha} \quad \text{(0.4)}$$

$$\frac{I_2}{I_1} = 1 - \left(\frac{R_2}{R_1}\right)^2 \left(1 - \left(\frac{T_2}{T_1}\right)^4\right) = \frac{\beta}{\alpha} \quad \text{(0.4)} \quad \text{(or equivalent relations)}$$

From above, one finds:

$$\frac{R_1}{R_2} = \sqrt{\frac{\alpha}{1-\beta}} \Rightarrow \frac{R_1}{R_2} = 1.6 \quad \text{(0.2+0.2)} \quad \text{and} \quad \frac{T_1}{T_2} = \sqrt[4]{\frac{1-\beta}{1-\alpha}} \Rightarrow \frac{T_1}{T_2} = 1.4 \quad \text{(0.2+0.2)}$$

2.1)

Doppler-Shift formula:

$\dfrac{\Delta\lambda}{\lambda_0} \cong \dfrac{v}{c}$ (or equivalent relation) (0.4)

Maximum and minimum wavelengths: $\lambda_{1,\max} = 5897.7$ Å, $\lambda_{1,\min} = 5894.1$ Å

$\lambda_{2,\max} = 5899.0$ Å, $\lambda_{2,\min} = 5892.8$ Å

Difference between maximum and minimum wavelengths:

$\Delta\lambda_1 = 3.6$ Å , $\Delta\lambda_2 = 6.2$ Å (All 0.6)

Using the Doppler relation and noting that the shift is due to twice the orbital speed: (Factor of two 0.4)

$$v_1 = c\frac{\Delta\lambda_1}{2\lambda_0} = 9.2 \times 10^4 \text{ m/s} \quad \text{(0.2)}$$

$$v_2 = c\frac{\Delta\lambda_2}{2\lambda_0} = 1.6 \times 10^5 \text{ m/s} \quad \text{(0.2)}$$

The student can use the wavelength of central line and maximum (or minimum) wavelengths. Marking scheme is given in the Excel file.

2.2) As the center of mass is not moving with respect to us: (0.5)

$$\frac{m_1}{m_2} = \frac{v_2}{v_1} = 1.7 \quad (0.2)$$

2.3)

Writing $r_i = \frac{v_i}{\omega}$ for $i = 1,2$, we have (0.4)

$r_1 = 3.8\times10^9$ m, (0.2) $\qquad$ $r_2 = 6.5\times10^9$ m (0.2)

2.4)

$r = r_1 + r_2 = 1.0\times10^{10}$ m (0.2)

3.1)

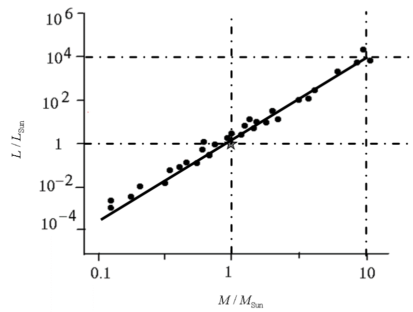
The gravitational force is equal to mass times the centrifugal acceleration

$$G\frac{m_1 m_2}{r^2} = m_1\frac{v_1^{\ 2}}{r_1} = m_2\frac{v_2^{\ 2}}{r_2} \quad (0.7)$$

Therefore,

$$\begin{cases} m_1 = \dfrac{r^2 v_2^{\ 2}}{G r_2} \\ m_2 = \dfrac{r^2 v_1^{\ 2}}{G r_1} \end{cases} \quad (0.1) \quad \Rightarrow \quad \begin{cases} m_1 = 6\times10^{30}\,\text{kg} \\ m_2 = 3\times10^{30}\,\text{kg} \end{cases} \quad (0.2 + 0.2)$$

4.1) As it is clear from the diagram, with one significant digit, $\alpha = 4$. (0.6)



4.2)

As we have found in the previous section: $L_i = L_{Sun}\left(\dfrac{M_i}{M_{Sun}}\right)^4$ (0.2)

So,

$$L_1 = 3\times10^{28}\,\text{Watt} \quad (0.2)$$
$$L_2 = 4\times10^{27}\,\text{Watt} \quad (0.2)$$

4.3) The total power of the system is distributed on a sphere with radius d to produce I_0, that is:

$$I_0 = \frac{L_1 + L_2}{4\pi d^2} \quad (0.5) \qquad \Rightarrow d = \sqrt{\frac{L_1 + L_2}{4\pi I_0}} = 1\times10^{18}\,\text{m} \quad (0.2)$$

$= 100$ ly. (0.2)

4.4) $$\theta \cong \tan\theta = \frac{r}{d} = 1\times10^{-8}\,\text{rad.} \quad (0.2 + 0.2)$$

4.5)
A typical optical wavelength is λ_0. Using uncertainty relation:

$$D = \frac{d\,\lambda_0}{r} \cong 50\text{ m.} \quad (0.2 + 0.2)$$

Two stars rotating around their center of mass form a binary star system. Almost half of the stars in our galaxy are binary star systems. It is not easy to realize the binary nature of most of these star systems from Earth, since the distance between the two stars is much less than their distance from us and thus the stars cannot be resolved with telescopes. Therefore, we have to use either photometry or spectrometry to observe the variations in the intensity or the spectrum of a particular star to find out whether it is a binary system or not.

Photometry of Binary Stars

If we are exactly on the plane of motion of the two stars, then one star will occult (pass in front of) the other star at certain times and the intensity of the whole system will vary with time from our observation point. These binary systems are called ecliptic binaries.

1 Assume that two stars are moving on circular orbits around their common center of mass with a constant angular speed ω and we are exactly on the plane of motion of the binary system. Also assume that the surface temperatures of the stars are T_1 and T_2 $(T_1 > T_2)$, and the corresponding radii are R_1 and R_2 $(R_1 > R_2)$, respectively. The total intensity of light, measured on Earth, is plotted in Figure 1 as a function of time. Careful measurements indicate that the intensities of the incident light from the stars corresponding to the minima are respectively 90 and 63 percent of the maximum intensity, I_0, received from both stars $(I_0 = 4.8\times10^{-9}\ \mathrm{W/m^2})$. The vertical axis in Figure 1 shows the ratio I/I_0 and the horizontal axis is marked in days.

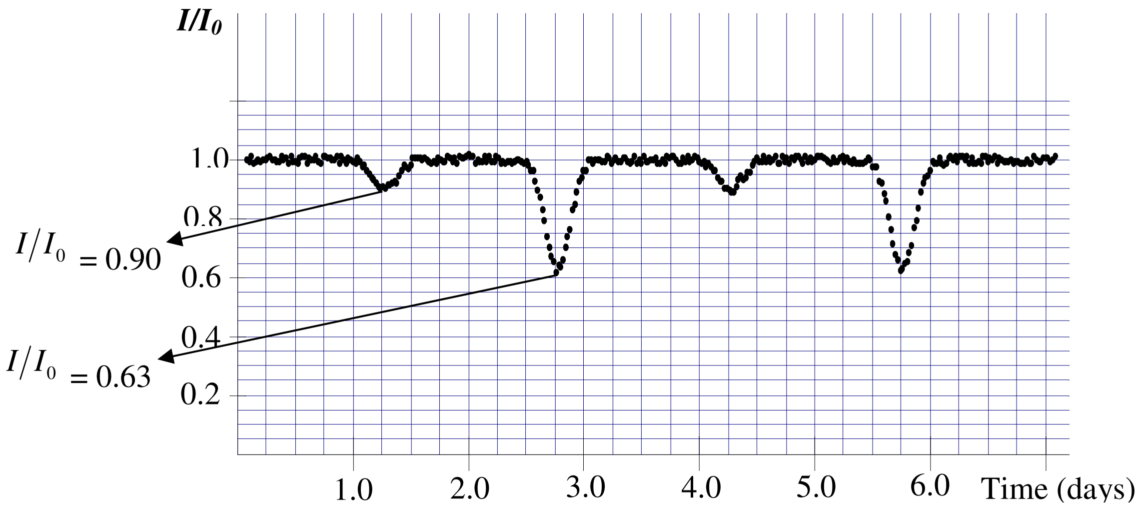


Figure 1. The relative intensity received from the binary star system as a function of time. The vertical axis has been scaled by $I_0 = 4.8\times10^{-9}\ \mathrm{W/m^2}$. Time is given in days.

1.1	Find the period of the orbital motion. Give your answer in **seconds** up to two significant digits. What is the angular frequency of the system in **rad/sec**?	0.8

To a good approximation, the receiving radiation from a star is a uniform black body radiation from a flat disc with a radius equal to the radius of the star. Therefore, the power received from the star is proportional to AT^4 where A is area of the disc and T is the surface temperature of the star.

1.2	Use the diagram in Figure 1 to find the ratios T_1/T_2 and R_1/R_2.	1.6

Spectrometry of Binary Systems

In this section, we are going to calculate the astronomical properties of a binary star by using experimental spectrometric data of the binary system.

Atoms absorb or emit radiation at their certain characteristic wavelengths. Consequently, the observed spectrum of a star contains *absorption lines* due to the atoms in the star's atmosphere. Sodium has a characteristic yellow line spectrum (D_1 line) with a wavelength 5895.9Å (10 Å = 1 nm). We examine the absorption spectrum of atomic Sodium at this wavelength for the binary system of the previous section. The spectrum of the light that we receive from the binary star is Doppler-shifted, because the stars are moving with respect to us. Each star has a different speed. Accordingly the absorption wavelength for each star will be shifted by a different amount. Highly accurate wavelength measurements are required to observe the Doppler shift since the speed of the stars is much less than the speed of light. The speed of the center of mass of the binary system we consider in this problem is much smaller than the orbital velocities of the stars. Hence all the Doppler shifts can be attributed to the orbital velocity of the stars. Table 1 shows the measured spectrum of the stars in the binary system we have observed.

Table 1: Absorption spectrum of the binary star system for the Sodium D_1 line

t/days	0.3	0.6	0.9	1.2	1.5	1.8	2.1	2.4
λ_1 (Å)	5897.5	5897.7	5897.2	5896.2	5895.1	5894.3	5894.1	5894.6
λ_2 (Å)	5893.1	5892.8	5893.7	5896.2	5897.3	5898.7	5899.0	5898.1

t/days	2.7	3.0	3.3	3.6	3.9	4.2	4.5	4.8
λ_1 (Å)	5895.6	5896.7	5897.3	5897.7	5897.2	5896.2	5895.0	5894.3
λ_2 (Å)	5896.4	5894.5	5893.1	5892.8	5893.7	5896.2	5897.4	5898.7

(*Note*: There is no need to make a graph of the data in this table)

2 Using Table 1,

2.1	Let v_1 and v_2 be the orbital velocity of each star. Find v_1 and v_2. The speed of light $c = 3.0\times10^8$ m/s. Ignore all relativistic effects.	1.8

2.2	Find the mass ratio of the stars (m_1/m_2).	0.7

2.3	Let r_1 and r_2 be the distances of each star from their center of mass. Find r_1 and r_2.	0.8

2.4	Let r be the distance between the stars. Find r.	0.2

3 The gravitational force is the only force acting between the stars.

3.1	Find the mass of each star up to one significant digit. The universal gravitational constant $G = 6.7\times10^{-11}\ \mathrm{m^3kg^{-1}s^{-2}}$.	1.2

General Characteristics of Stars

4 Most of the stars generate energy through the same mechanism. Because of this, there is an empirical relation between their mass, M, and their luminosity, L, which is the total radiant power of the star. This relation could be written in the form $L/L_{Sun} = (M/M_{Sun})^{\alpha}$. Here, $M_{Sun} = 2.0\times10^{30}\ \mathrm{kg}$ is the solar mass and, $L_{Sun} = 3.9\times10^{26}\ \mathrm{W}$ is the solar luminosity. This relation is shown in a log-log diagram in Figure 2.

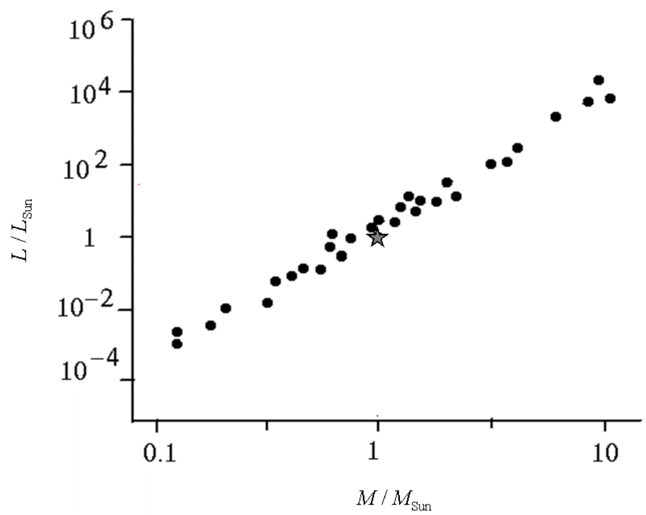


Figure 2. The luminosity of a star versus its mass varies as a power law. The diagram is log-log. The star-symbol represents Sun with a mass of $2.0\times10^{30}\ kg$ and luminosity of $3.9\times10^{26}\ \mathrm{W}$.

4.1	Find α up to one significant digit.	0.6

4.2	Let L_1 and L_2 be the luminosity of the stars in the binary system studied in the previous sections. Find L_1 and L_2.	0.6

4.3	What is the distance, d, of the star system from us in light years? To find the distance you can use the diagram of Figure 1. One light year is the distance light travels in one year.	0.9

4.4	What is the maximum angular distance, θ , between the stars from our observation point?	0.4

4.5	What is the smallest aperture size for an optical telescope, D , that can resolve these two stars?	0.4

WATER-POWERED RICE-POUNDING MORTAR

A. Introduction

Rice is the main staple food of most people in Vietnam. To make white rice from paddy rice, one needs separate of the husk (a process called "hulling") and separate the bran layer ("milling"). The hilly parts of northern Vietnam are abundant with water streams, and people living there use *water-powered rice-pounding mortar* for bran layer separation. Figure 1 shows one of such mortars., Figure 2 shows how it works.

B. Design and operation

1. Design.

The rice-pounding mortar shown in Figure 1 has the following parts:

The mortar, basically a wooden container for rice.

The lever, which is a tree trunk with one larger end and one smaller end. It can rotate around a horizontal axis. A *pestle* is attached perpendicularly to the lever at the smaller end. The length of the pestle is such that it touches the rice in the mortar when the lever lies horizontally. The larger end of the lever is carved hollow to form a bucket. The shape of the bucket is crucial for the mortar's operation.

2. Modes of operation

The mortar has two modes.

Working mode. In this mode, the mortar goes through an operation cycle illustrated in Figure 2.

The rice-pounding function comes from the work that is transferred from the pestle to the rice during stage f) of Figure 2. If, for some reason, the pestle never touches the rice, we say that the mortar is not working.

*Rest mode with the lever lifted up***.** During stage c) of the operation cycle (Figure 2), as the tilt angle α increases, the amount of water in the bucket decreases. At one particular moment in time, the amount of water is just enough to counterbalance the weight of the lever. Denote the tilting angle at this instant by β. If the lever is kept at angle β and the initial angular velocity is zero, then the lever will remain at this position forever. This is the rest mode with the lever lifted up. The stability of this position depends on the flow rate of water into the bucket, Φ. If Φ exceeds some value Φ_2, then this rest mode is stable, and the mortar cannot be in the working mode.

In other words, Φ_2 is the minimal flow rate for the mortar not to work.

Figure 1
A water-powered rice-pounding mortar

OPERATION CYCLE OF A WATER-POWERED RICE-POUNDING MORTAR

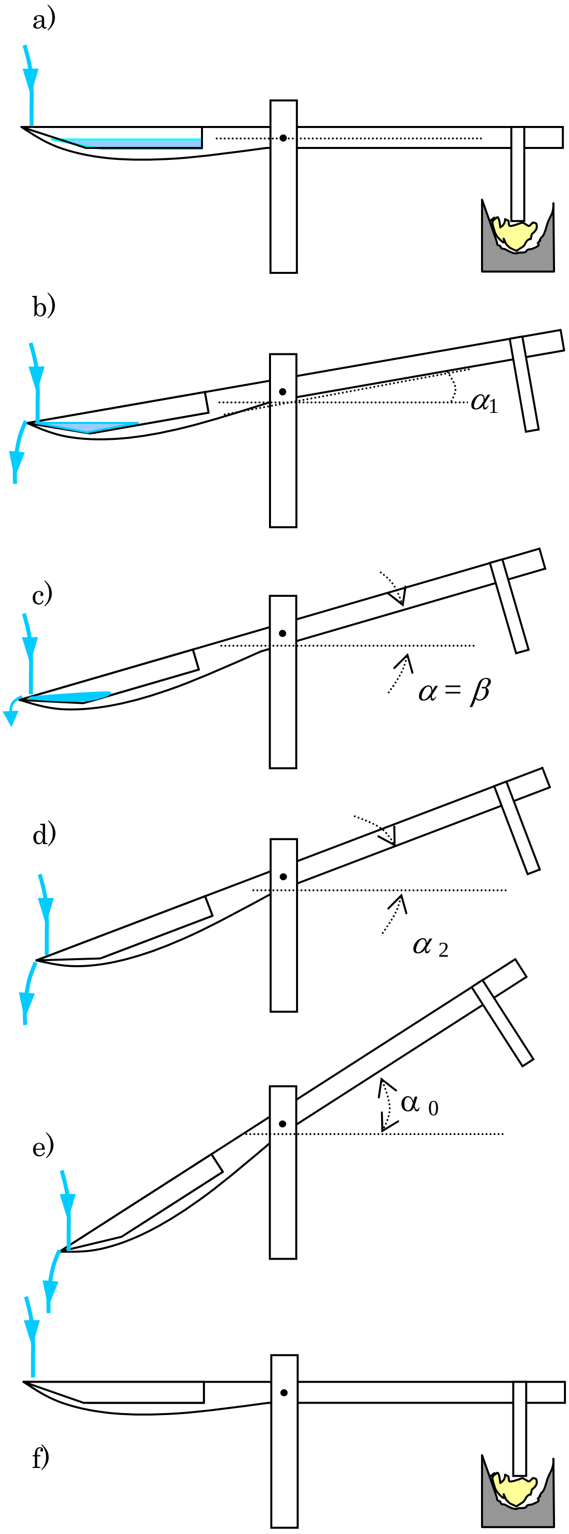


Figure 2

a) At the beginning there is no water in the bucket, the pestle rests on the mortar. Water flows into the bucket with a small rate, but for some time the lever remains in the horizontal position.

b) At some moment the amount of water is enough to lift the lever up. Due to the tilt, water rushes to the farther side of the bucket, tilting the lever more quickly.

Water starts to flow out at $\alpha = \alpha_1$.

c) As the angle α increases, water starts to flow out. At some particular tilt angle, $\alpha = \beta$, the total torque is zero.

d) α continues increasing, water continues to flow out until no water remains in the bucket.

e) α keeps increasing because of inertia. Due to the shape of the bucket, water falls into the bucket but immediately flows out. The inertial motion of the lever continues until α reaches the maximal value α_0.

f) With no water in the bucket, the weight of the lever pulls it back to the initial horizontal position. The pestle gives the mortar (with rice inside) a pound and a new cycle begins.

C. The problem

Consider a water-powered rice-pounding mortar with the following parameters (Figure 3)

The mass of the lever (including the pestle but without water) is $M = 30$ kg,

The center of mass of the lever is G.. The lever rotates around the axis T (projected onto the point T on the figure).

The moment of inertia of the lever around T is $I = 12\ \mathrm{kg \cdot m^2}$.

When there is water in the bucket, the mass of water is denoted as m, the center of mass of the water body is denoted as N.

The tilt angle of the lever with respect to the horizontal axis is α.

The main length measurements of the mortar and the bucket are as in Figure 3.

Neglect friction at the rotation axis and the force due to water falling onto the bucket. In this problem, we make an approximation that the water surface is always horizontal.

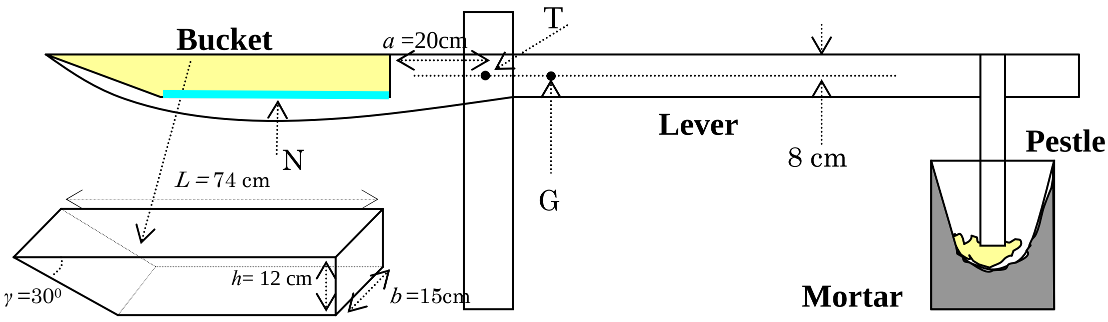


Figure 3 Design and dimensions of the rice-pounding mortar

1. The structure of the mortar

At the beginning, the bucket is empty, and the lever lies horizontally. Then water flows into the bucket until the lever starts rotating. The amount of water in the bucket at this moment is $m = 1.0$ kg.

1.1. Determine the distance from the center of mass G of the lever to the rotation axis T. It is known that GT is horizontal when the bucket is empty.

1.2. Water starts flowing out of the bucket when the angle between the lever and the horizontal axis reaches α_1. The bucket is completely empty when this angle is α_2.

Determine α_1 and α_2.

1.3. Let $\mu(\alpha)$ be the total torque (relative to the axis T) which comes from the weight of the lever and the water in the bucket. $\mu(\alpha)$ is zero when $\alpha = \beta$. Determine β and the mass m_1 of water in the bucket at this instant.

2. Parameters of the working mode

Let water flow into the bucket with a flow rate Φ which is constant and small. The amount of water flowing into the bucket when the lever is in motion is negligible. In this part, neglect the change of the moment of inertia during the working cycle.

2.1. Sketch a graph of the torque μ as a function of the angle α, $\mu(\alpha)$, during one operation cycle. Write down explicitly the values of $\mu(\alpha)$ at angle α_1, α_2, and $\alpha = 0$.

2.2. From the graph found in section 2.1., discuss and give the geometric interpretation of the value of the total energy W_{total} produced by $\mu(\alpha)$ and the work W_{pounding} that is transferred from the pestle to the rice.

2.3. From the graph representing μ versus α, estimate α_0 and W_{pounding} (assume the kinetic energy of water flowing into the bucket and out of the bucket is negligible.) You may replace curve lines by zigzag lines, if it simplifies the calculation.

3. The rest mode

Let water flow into the bucket with a constant rate Φ, but one cannot neglect the amount of water flowing into the bucket during the motion of the lever.

3.1. Assuming the bucket is always overflown with water,

3.1.1. Sketch a graph of the torque μ as a function of the angle α in the vicinity of $\alpha = \beta$. To which kind of equilibrium does the position $\alpha = \beta$ of the lever belong?

3.1.2. Find the analytic form of the torque $\mu(\alpha)$ as a function of $\Delta\alpha$ when $\alpha = \beta + \Delta\alpha$, and $\Delta\alpha$ is small.

3.1.3. Write down the equation of motion of the lever, which moves with zero initial velocity from the position $\alpha = \beta + \Delta\alpha$ ($\Delta\alpha$ is small). Show that the motion is, with good accuracy, harmonic oscillation. Compute the period τ.

3.2. At a given Φ, the bucket is overflown with water at all times only if the lever moves sufficiently slowly. There is an upper limit on the amplitude of harmonic oscillation, which depends on Φ. Determine the minimal value Φ_1 of Φ (in kg/s) so that the lever can make a harmonic oscillator motion with amplitude 1^o.

3.3. Assume that Φ is sufficiently large so that during the free motion of the lever when the tilting angle decreases from α_2 to α_1 the bucket is always overflown with water. However, if Φ is too large the mortar cannot operate. Assuming that the motion of the lever is that of a harmonic oscillator, estimate the minimal flow rate Φ_2 for the rice-pounding mortar to not work.

Solution

1. The structure of the mortar

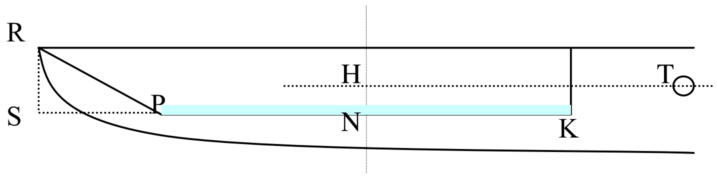
1.1. Calculating the distance TG

The volume of water in the bucket is $V = 1000\,\mathrm{cm}^3 = 10^{-3}\,\mathrm{m}^3$. The length of the bottom of the bucket is $d = L - h\,\mathbf{tan}\,60^0 = (0.74 - 0.12\,\mathbf{tan}\,60^0)\,\mathrm{m} = 0.5322\,\mathrm{m}$.

(as the initial data are given with two significant digits, we shall keep only two significant digits in the final answer, but we keep more digits in the intermediate steps).
The height c of the water layer in the bucket is calculated from the formula:

$$V = bcd + b\frac{c}{2}c\tan 60^0 \Rightarrow c = \frac{(d^2 + 2\sqrt{3}V/b)^{1/2} - d}{\sqrt{3}}$$

Inserting numerical values for V, b and d, we find $c = 0.01228\,\mathrm{m}$.
When the lever lies horizontally, the distance, on the horizontal axis, between the rotation axis and the center of mass of water N, is $\mathrm{TH} \approx a + \frac{d}{2} + \frac{c}{4}\,\mathbf{tan}\,60^\mathrm{o} = 0.4714\,\mathrm{m}$, and $\mathrm{TG} = (m/M)\,\mathrm{TH} = 0.01571\,\mathrm{m}$ (see the figure below).



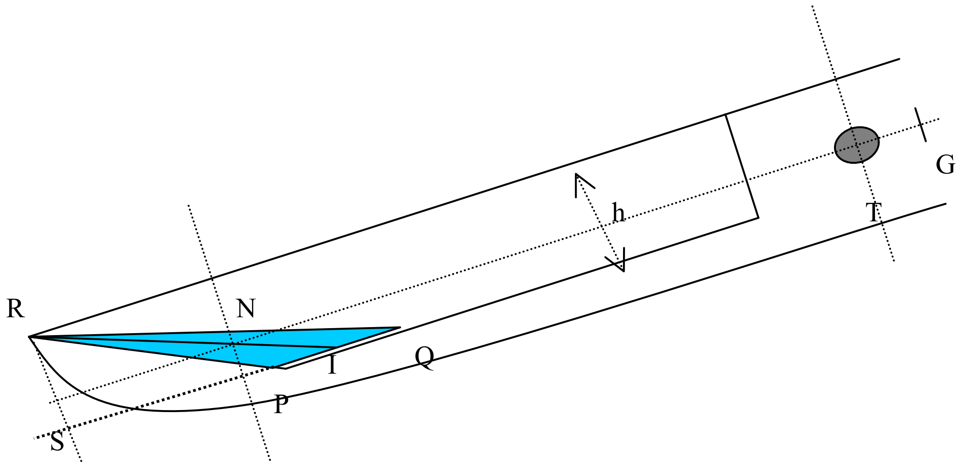
Answer: $\mathrm{TG} = 0.016\,\mathrm{m}$.

1.2. Calculating the values of α_1 and α_2.

When the lever tilts with angle α_1, water level is at the edge of the bucket. At that point the water volume is $10^{-3}\,\mathrm{m}^3$. Assume $\mathrm{PQ} < d$. From geometry $V = hb \times \mathrm{PQ}/2$, from which $\mathrm{PQ} = 0.1111\mathrm{m}$. The assumption $\mathrm{PQ} < d$ is obviously satisfied ($d = 0.5322\,\mathrm{m}$).

To compute the angle α_1, we note that $\mathbf{tan}\,\alpha_1 = h/\mathrm{QS} = h/(\mathrm{PQ} + \sqrt{3}h)$. From this we find $\alpha_1 = 20.6^\mathrm{o}$.

When the tilt angle is 30°, the bucket is empty**:** $\alpha_2 = 30^{\circ}$**.**



1.3. Determining the tilt angle β of the lever and the amount of water in the bucket m when the total torque μ on the lever is equal to zero

Denote $\mathrm{PQ} = x\,(\mathrm{m})$. The amount of water in the bucket is $m = \rho_{\text{water}} \dfrac{xhb}{2} = 9x$ (kg).

$\mu = 0$ when the torque coming from the water in the bucket cancels out the torque coming from the weight of the lever. The cross section of the water in the bucket is the triangle PQR in the figure. The center of mass N of water is located at 2/3 of the meridian RI, therefore NTG lies on a straight line. Then: $mg \times \mathrm{TN} = Mg \times \mathrm{TG}$ or

$$m \times \mathrm{TN} = M \times \mathrm{TG} = 30 \times 0.1571 = 0.4714 \qquad (1)$$

Calculating TN from x then substitute (1) :

$$\mathrm{TN} = L + a - \frac{2}{3}\left(h\sqrt{3} + \frac{x}{2}\right) = 0.94 - 0.08\sqrt{3} - \frac{x}{3} = 0.8014 - \frac{x}{3}$$

which implies $m \times \mathrm{TN} = 9x(0.8014 - x/3) = -3x^2 + 7.213x$ (2)

So we find an equation for x:

$$-3x^2 + 7.213x = 0.4714 \qquad (3)$$

The solutions to (3) are $x = 2.337$ and $x = 0.06723$. Since x has to be smaller than 0.5322, we have to take $x = x_0 = 0.06723$ and $m = 9x_0 = 0.6051\,\mathrm{kg}$.

$$\mathbf{tan}\,\beta = \frac{h}{x + h\sqrt{3}} = 0.4362, \text{ or } \quad \beta = 23.57^{\circ}.$$

Answer: $m = 0.61\,\mathrm{kg}$ and $\beta = 23.6^{\circ}$.

2. Parameters of the working mode

2.1.Graphs of $\mu(\alpha)$, $\alpha(t)$, and $\mu(t)$ during one operation cycle**.**

Initially when there is no water in the bucket, $\alpha = 0$, μ has the largest magnitude equal to $gM \times \mathrm{TG} = 30 \times 9.81 \times 0.01571 = 4.624\,\mathrm{N \cdot m}$. Our convention will be that the sign of this torque is negative as it tends to decrease α.

As water flows into the bucket, the torque coming from the water (which carries positive sign) makes μ increase until μ is slightly positive, when the lever starts to lift up. From that moment, by assumption, the amount of water in the bucket is constant.

The lever tilts so the center of mass of water moves away from the rotation axis, leading to an increase of μ, which reaches maximum when water is just about to overflow the edge of the bucket. At this moment $\alpha = \alpha_1 = 20.6^{\mathrm{o}}$.

A simple calculation shows that

$$\mathrm{SI} = \mathrm{SP} + \mathrm{PQ}/2 = 0.12 \times 1.732 + 0.1111/2 = 0.2634\,\mathrm{m}.$$

$$\mathrm{TN} = 0.20 + 0.74 - \frac{2}{3}\mathrm{SI} = 0.7644\,\mathrm{m}.$$

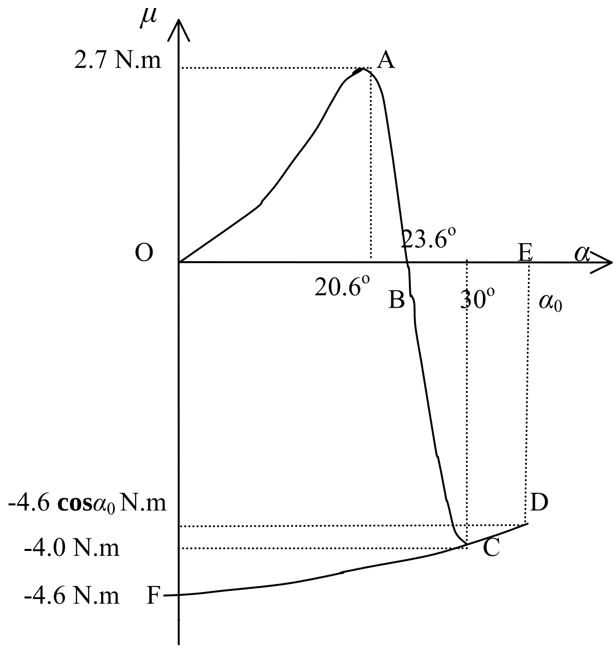
$$\mu_{\max} = \mathbf{(}1\mathbf{.}0 \times \mathrm{TN} - 30 \times \mathrm{TG}\mathbf{)}g\,\mathbf{cos}\,20.6^{\mathrm{o}}$$

$$= \mathbf{(}1\mathbf{.}0 \times 0\mathbf{.}7644 - 30 \times 0\mathbf{.}01571\mathbf{)} \times 9\mathbf{.}81 \times \mathbf{cos}\,20.6^{\mathrm{o}} = 2\mathbf{.}690\ \mathrm{N \cdot m}\mathbf{.}$$

Therefore $\mu_{\max} = 2.7\ \mathrm{N \cdot m}$**.**

As the bucket tilts further, the amount of water in the bucket decreases, and when $\alpha = \beta$, $\mu = 0$. Due to inertia, α keeps increasing and μ keeps decreasing. The bucket is empty when $\alpha = 30^{\mathrm{o}}$, when μ equals $-30 \times g \times \mathrm{TG} \times \mathbf{cos}\,30^{\mathrm{o}} = -4\mathbf{.}0\,\mathrm{N \cdot m}$. After that α keeps increasing due to inertia to α_0 ($\mu = -gM\,\mathrm{TG}\,\mathbf{cos}\,\alpha_0 = -4.62\,\mathbf{cos}\,\alpha_0\,\mathrm{N \cdot m}$), then quickly decreases to 0 ($\mu = -4.62\,\mathrm{N \cdot m}$).

On this basis we can sketch the graphs of $\alpha(t)$, $\mu(t)$, and $\mu(\alpha)$ as in the figure below



2.2. The infinitesimal work produced by the torque $\mu(\alpha)$ is $dW = \mu(\alpha)d\alpha$. The energy obtained by the lever during one cycle due to the action of $\mu(\alpha)$ is $W = \oint \mu(\alpha)d\alpha$, which is the area limited by the line $\mu(\alpha)$. Therefore W_{total} is equal to the area enclosed by the curve (OABCDFO) on the graph $\mu(\alpha)$.

The work that the lever transfers to the mortar is the energy the lever receives as it moves from the position $\alpha = \alpha_o$ to the horizontal position $\alpha = 0$. We have W_{pounding} equals to the area of (OEDFO) on the graph $\mu(\alpha)$. It is equal to $gM \times \text{TG} \times \mathbf{sin}\,\alpha_0 = \mathbf{4.6\,sin}\,\alpha_0$ (J).

2.3. The magnitudes of α_0 can be estimated from the fact that at point D the energy of the lever is zero. We have

area (OABO) = area (BEDCB)

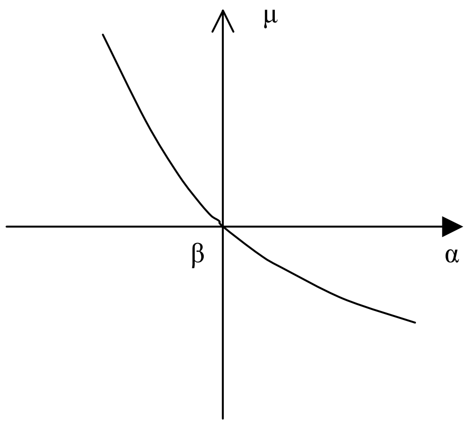
Approximating OABO by a triangle, and BEDCB by a trapezoid, we obtain:

$$23.6 \times 2.7 \times (1/2) = 4.0 \times [(\alpha_0 - 23.6) + (\alpha_0 - 30)] \times (1/2),$$

which implies $\alpha_0 = 34.7^{\text{o}}$. From this we find

$$W_{\text{pounding}} = \text{area (OEDFO)} = \int_{34.76}^{0} -Mg \times \text{TG} \times \mathbf{cos}\,\alpha\, d\alpha = \mathbf{4.62} \times \mathbf{sin}\,34.7^{\text{o}} = \mathbf{2.63}$$

Thus we find $W_{\text{pounding}} \approx 2.6$ J.



3. The rest mode

3.1.

3.1.1. The bucket is always overflown with water. The two branches of $\mu(\alpha)$ in the vicinity of $\alpha = \beta$ corresponding to increasing and decreasing α coincide with each other.

The graph implies that $\alpha = \beta$ is a stable equilibrium of the mortar.

3.1.2. Find the expression for the torque μ when the tilt angle is $\alpha = \beta + \Delta\alpha$ ($\Delta\alpha$ is small **).**

The mass of water in bucket when the lever tilts with angle α is $m = (1/2)\rho bh\,\mathrm{PQ}$, where $\mathrm{PQ} = h\left(\frac{1}{\tan\alpha} - \frac{1}{\tan 30^0}\right)$**.** A simple calculation shows that when α increases from β to $\beta + \Delta\alpha$, the mass of water increases by $\Delta m = -\frac{bh^2\rho}{2\sin^2\alpha}\Delta\alpha \approx -\frac{bh^2\rho}{2\sin^2\beta}\Delta\alpha$. The torque μ acting on the lever when the tilt is $\beta + \Delta\alpha$ equals the torque due to Δm.

We have $\mu = \Delta m \times g \times \mathrm{TN} \times \cos(\beta + \Delta\alpha)$. TN is found from the equilibrium condition of the lever at tilting angle β:

$$\mathrm{TN} = M \times \mathrm{TG}/m = 30 \times 0.01571/0.605 = 0.779\,\mathrm{m}.$$

We find at the end $\mu = -47.2 \times \Delta\alpha\ \ \mathrm{N\cdot m} \approx -47 \times \Delta\alpha\ \ \mathrm{N\cdot m}$**.**

3.1.3. Equation of motion of the lever

$\mu = I\frac{d^2\alpha}{dt^2}$ where $\mu = -47 \times \Delta\alpha$, $\alpha = \beta + \Delta\alpha$, and I is the sum of moments of inertia of the lever and of the water in bucket relative to the axis T. Here I is not constant the amount of water in the bucket depends on α. When $\Delta\alpha$ is small, one can consider the amount and the shape of water in the bucket to be constant, so I is approximatey a constant. Consider water in bucket as a material point with mass 0.6 kg, a simple calculation gives $I = 12 + 0.6 \times 0.78^2 = 12.36 \approx 12.4\ \mathrm{kg\,m^2}$. We have $-47 \times \Delta\alpha = 12.4 \times \frac{d^2\Delta\alpha}{dt^2}$. That is the equation for a harmonic oscillator with period

$\tau = 2\pi\sqrt{\dfrac{12.4}{47}} = 3.227$. The answer is therefore $\tau = 3.2\,\mathrm{s}$.

3.2. Harmonic oscillation of lever (around $\alpha = \beta$) when bucket is always overflown. Assume the lever oscillate harmonically with amplitude $\Delta\alpha_0$ around $\alpha = \beta$. At time $t = 0$, $\Delta\alpha = 0$, the bucket is overflown. At time dt the tilt changes by $d\alpha$. We are interested in the case $d\alpha < 0$, i.e., the motion of lever is in the direction of decreasing α, and one needs to add more water to overflow the bucket. The equation of motion is:

$\Delta\alpha = -\Delta\alpha_0 \sin(2\pi t/\tau)$, therefore $d(\Delta\alpha) = d\alpha = -\Delta\alpha_0 (2\pi/\tau)\cos(2\pi t/\tau)dt$.

For the bucket to be overflown, during this time the amount of water falling to the bucket should be at least $dm = -\dfrac{bh^2\rho}{2\sin^2\beta}d\alpha = \dfrac{2\Delta\alpha_0 \pi bh^2\rho dt}{2\tau\sin^2\beta}\cos\left(\dfrac{2\pi t}{\tau}\right)$; dm is maximum at $t = 0$, $dm_0 = \dfrac{\pi bh^2\rho\Delta\alpha_0}{\tau\sin^2\beta}dt$.

The amount of water falling to the bucket is related to flow rate Φ; $dm_0 = \Phi dt$, therefore $\Phi = \dfrac{\pi bh^2\rho\Delta\alpha_0}{\tau\sin^2\beta}$.

An overflown bucket is the necessary condition for harmonic oscillations of the lever, therefore the condition for the lever to have harmonic oscillations with ampltude 1^o or $2\pi/360\,\mathrm{rad}$ is $\Phi \geq \Phi_1$ with

$$\Phi_1 = \frac{\pi bh^2\rho 2\pi}{360\tau\sin^2\beta} = 0.2309\,\mathrm{kg/s}$$

So $\Phi_1 = 0.23\,\mathrm{kg/s}$.

3.3 Determination of Φ_2

If the bucket remains overflown when the tilt decreases to 20.6^o, then the amount of water in bucket should reach 1 kg at this time, and the lever oscillate harmonically with amplitude equal $23.6^\mathrm{o} - 20.6^\mathrm{o} = 3^\mathrm{o}$. The flow should exceed $3\Phi_1$, therefore

$\Phi_2 = 3 \times 0.23 \approx 0.7\,\mathrm{kg/s}$.

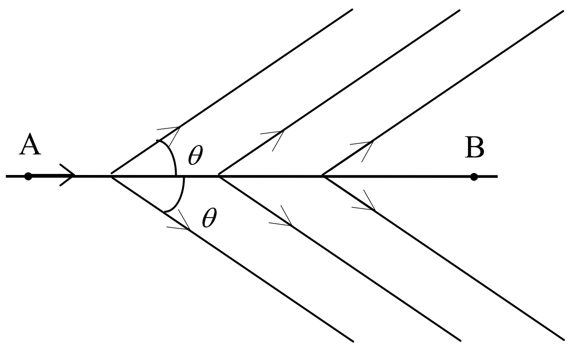
This is the minimal flow rate for the rice-pounding mortar not to work.

CHERENKOV LIGHT AND RING IMAGING COUNTER

Light propagates in vacuum with the speed c. There is no particle which moves with a speed higher than c. However, it is possible that in a transparent medium a particle moves with a speed v higher than the speed of the light in the same medium $\frac{c}{n}$, where n is the refraction index of the medium. Experiment (Cherenkov, 1934) and theory (Tamm and Frank, 1937) showed that a charged particle, moving with a speed v in a transparent medium with refractive index n such that $v > \frac{c}{n}$, radiates light, called *Cherenkov light*, in directions forming with the trajectory an angle

$$\theta = \mathbf{arccos}\frac{1}{\beta n} \qquad (1)$$

where $\beta = \frac{v}{c}$.



1. To establish this fact, consider a particle moving at constant velocity $v > \frac{c}{n}$ on a straight line. It passes A at time 0 and B at time t_1. As the problem is symmetric with respect to rotations around AB, it is sufficient to consider light rays in a plane containing AB.

At any point C between A and B, the particle emits a spherical light wave, which propagates with velocity $\frac{c}{n}$. We define the wave front at a given time t as the envelope of all these spheres at this time.

1.1. Determine the wave front at time t_1 and draw its intersection with a plane containing the trajectory of the particle.

1.2. Express the angle φ between this intersection and the trajectory of the particle in terms of n and β.

2. Let us consider a beam of particles moving with velocity $v > \frac{c}{n}$, such that the angle θ is small, along a straight line IS. The beam crosses a concave spherical mirror of focal length f and center C, at point S. SC makes with SI a small angle α (see the figure in the Answer Sheet). The particle beam creates a ring image in the focal plane of the mirror.

Explain why with the help of a sketch illustrating this fact. Give the position of the center O and the radius r of the ring image.

This set up is used in *ring imaging Cherenkov counters* (RICH) and the medium which the particle traverses is called the *radiator*.

Note: in all questions of the present problem, terms of second order and higher in α and θ will be neglected.

3. A beam of particles of known momentum $p = 10.0\ \mathrm{GeV}/c$ consists of three types of particles: protons, kaons and pions, with rest mass $M_\mathrm{p} = 0.94\ \mathrm{GeV}/c^2$, $M_\kappa = 0.50\ \mathrm{GeV}/c^2$ and $M_\pi = 0.14\ \mathrm{GeV}/c^2$, respectively. Remember that pc and Mc^2 have the dimension of an energy, and 1 eV is the energy acquired by an electron after being accelerated by a voltage 1 V, and 1 GeV = 10^9 eV, 1 MeV = 10^6 eV.

The particle beam traverses an air medium (the radiator) under the pressure P. The refraction index of air depends on the air pressure P according to the relation $n = 1 + aP$ where $a = 2.7\times10^{-4}\ \mathrm{atm}^{-1}$

3.1. Calculate for each of the three particle types the minimal value $P_{\min}$ of the air pressure such that they emit Cherenkov light.

3.2. Calculate the pressure $P_{\frac{1}{2}}$ such that the ring image of kaons has a radius equal to one half of that corresponding to pions. Calculate the values of θ_κ and θ_π in this case.

Is it possible to observe the ring image of protons under this pressure?

4. Assume now that the beam is not perfectly monochromatic: the particles momenta are distributed over an interval centered at 10 GeV/c having a half width at half height Δp. This makes the ring image broaden, correspondingly θ distribution has a half width at half height $\Delta\theta$. The pressure of the radiator is $P_{\frac{1}{2}}$ determined in 3.2.

4.1. Calculate $\dfrac{\Delta\theta_\kappa}{\Delta p}$ and $\dfrac{\Delta\theta_\pi}{\Delta p}$, the values taken by $\dfrac{\Delta\theta}{\Delta p}$ in the pions and kaons cases.

4.2. When the separation between the two ring images, $\theta_\pi - \theta_\kappa$, is greater than 10 times the half-width sum $\Delta\theta = \Delta\theta_{\kappa} + \Delta\theta_{\pi}$, that is $\theta_{\pi} - \theta_{\kappa} > 10\ \Delta\theta$, it is possible to distinguish well the two ring images. Calculate the maximal value of Δp such that the two ring images can still be well distinguished.

5. Cherenkov first discovered the effect bearing his name when he was observing a bottle of water located near a radioactive source. He saw that the water in the bottle emitted light.

5.1. Find out the minimal kinetic energy $T_{\min}$ of a particle with a rest mass M moving in water, such that it emits Cherenkov light. The index of refraction of water is $n = 1.33$.

5.2. The radioactive source used by Cherenkov emits either α particles (i.e. helium nuclei) having a rest mass $M_{\alpha} = 3.8\ \mathrm{GeV}/c^2$ or β particles (i.e. electrons) having a rest mass $M_{\mathrm{e}} = 0.51\ \mathrm{MeV}/c^2$. Calculate the numerical values of $T_{\min}$ for α particles and β particles.

Knowing that the kinetic energy of particles emitted by radioactive sources never exceeds a few MeV, find out which particles give rise to the radiation observed by Cherenkov.

6. In the previous sections of the problem, the dependence of the Cherenkov effect on wavelength λ has been ignored. We now take into account the fact that the Cherenkov radiation of a particle has a broad continuous spectrum including the visible range (wavelengths from 0.4 μm to 0.8 μm). We know also that the index of refraction n of the radiator decreases linearly by 2% of $n-1$ when λ increases over this range.

6.1. Consider a beam of pions with definite momentum of $10.0\ \mathrm{GeV}/c$ moving in air at pressure 6 atm. Find out the angular difference $\delta\theta$ associated with the two ends of the visible range.

6.2. On this basis, study qualitatively the effect of the dispersion on the ring image of pions with momentum distributed over an interval centered at $p = 10\ \mathrm{GeV}/c$ and having a half width at half height $\Delta p = 0.3\ \mathrm{GeV}/c$.

6.2.1. Calculate the broadening due to dispersion (varying refraction index) and that due to achromaticity of the beam (varying momentum).

6.2.2. Describe how the color of the ring changes when going from its inner to outer edges by checking the appropriate boxes in the Answer Sheet.

Solution

1.

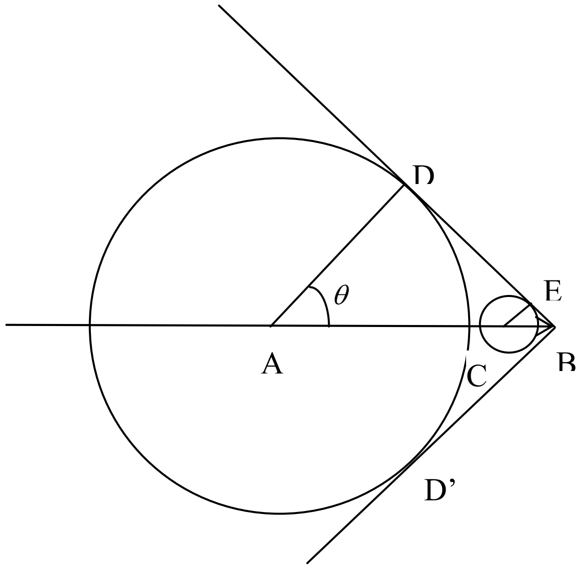


Figure 1

Let us consider a plane containing the particle trajectory. At $t=0$, the particle position is at point A. It reaches point B at $t=t_1$. According to the Huygens principle, at moment $0<t<t_1$, the radiation emitted at A reaches the circle with a radius equal to AD and the one emitted at C reaches the circle of radius CE. The radii of the spheres are proportional to the distance of their centre to B:

$$\frac{\mathrm{CE}}{\mathrm{CB}}=\frac{c\left(t_1-t\right)/n}{v\left(t_1-t\right)}=\frac{1}{\beta n}=\mathrm{const}$$

The spheres are therefore transformed into each other by homothety of vertex B and their envelope is the cone of summit B and half aperture $\varphi=\mathbf{Arcsin}\frac{1}{\beta n}=\frac{\pi}{2}-\theta$, where θ is the angle made by the light ray CE with the particle trajectory.

1.1. The intersection of the wave front with the plane is two straight lines, BD and BD'.

1.2. They make an angle $\varphi=\mathbf{Arcsin}\frac{1}{\beta n}$ with the particle trajectory.

2. The construction for finding the ring image of the particles beam is taken in the plane containing the trajectory of the particle and the optical axis of the mirror.

We adopt the notations:

S – the point where the beam crosses the spherical mirror

F – the focus of the spherical mirror

C – the center of the spherical mirror

IS – the straight-line trajectory of the charged particle making a small angle α with the optical axis of the mirror.

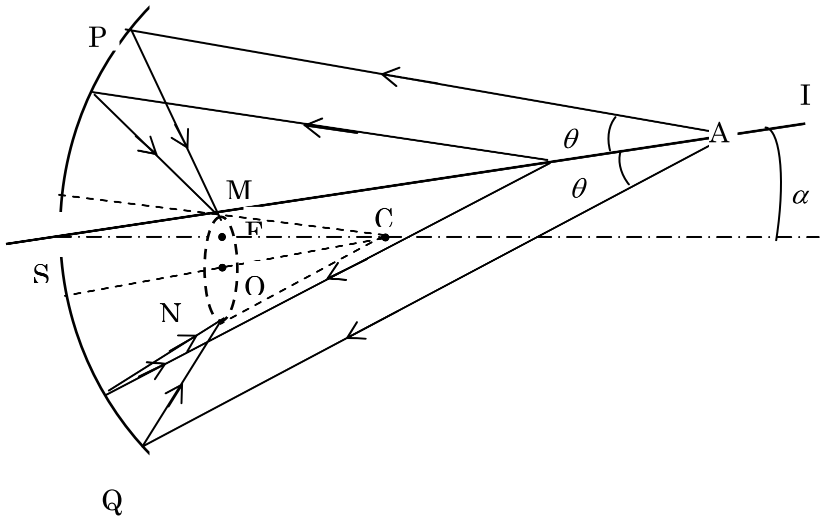


Figure 2

CF = FS = f

CO//IS

CM//AP

CN//AQ

$$\widehat{\mathrm{FCO}} = \alpha \Rightarrow \mathrm{FO} = f \times \alpha$$

$$\widehat{\mathrm{MCO}} = \widehat{\mathrm{OCN}} = \theta \Rightarrow \mathrm{MO} = f \times \theta$$

We draw a straight line parallel to IS passing through the center C. The line intersects the focal plane at O. We have FO $\approx f \times \alpha$.

Starting from C, we draw two lines in both sides of the line CO making with it an angle θ. These two lines intersect the focal plane at M and N, respectively. All the rays of Cherenkov radiation in the plane of the sketch, striking the mirror and being reflected, intersect at M or N.

In three-dimension case, the Cherenkov radiation gives a ring in the focal plane with the center at O (FO $\approx f\times\alpha$) and with the radius MO $\approx f\times\theta$.

In the construction, all the lines are in the plane of the sketch. Exceptionally, the ring is illustrated spatially by a dash line.

3.

3.1. For the Cherenkov effect to occur it is necessary that $n>\frac{c}{v}$, that is $n_{\min}=\frac{c}{v}$.

Putting $\zeta = n-1 = 2.7\times10^{-4}P$, we get

$$\zeta_{\min} = 2.7\times10^{-4}P_{\min} = \frac{c}{v}-1 = \frac{1}{\beta}-1 \qquad (1)$$

Because

$$\frac{Mc^2}{pc} = \frac{Mc}{p} = \frac{Mc}{\frac{Mv}{\sqrt{1-\beta^2}}} = \frac{\sqrt{1-\beta^2}}{\beta} = K \qquad (2)$$

then K = 0.094 ; 0.05 ; 0.014 for proton, kaon and pion, respectively.

From (2) we can express β through K as

$$\beta = \frac{1}{\sqrt{1+K^2}} \qquad (3)$$

Since $K^2 << 1$ for all three kinds of particles we can neglect the terms of order higher than 2 in K. We get

$$1-\beta = 1-\frac{1}{\sqrt{1+K^2}} \approx \frac{1}{2}K^2 = \frac{1}{2}\left(\frac{Mc}{p}\right)^2 \qquad (3a)$$

$$\frac{1}{\beta}-1 = \sqrt{1+K^2}-1 \approx \frac{1}{2}K^2 = \frac{1}{2}\left(\frac{Mc}{p}\right)^2 \qquad (3b)$$

Putting (3b) into (1), we obtain

$$P_{\min} = \frac{1}{2.7\times10^{-4}}\times\frac{1}{2}K^2 \qquad (4)$$

We get the following numerical values of the minimal pressure:

$P_{\min} = 16$ atm for protons,

$P_{\min} = 4.6$ atm for kaons,

$P_{\min} = 0.36$ atm for pions.

3.2. For $\theta_\pi = 2\theta_\kappa$ we have

$$\cos\theta_\pi = \cos 2\theta_\kappa = 2\cos^2\theta_\kappa - 1 \qquad (5)$$

We denote

$$\varepsilon = 1-\beta = 1-\frac{1}{\sqrt{1+K^2}} \approx \frac{1}{2}K^2 \qquad (6)$$

From (5) we obtain

$$\frac{1}{\beta_\pi n} = \frac{2}{\beta_\kappa^2 n^2} - 1 \qquad (7)$$

Substituting $\beta = 1-\varepsilon$ and $n = 1+\zeta$ into (7), we get approximately:

$$\zeta_{\frac{1}{2}} = \frac{4\varepsilon_\kappa - \varepsilon_\pi}{3} = \frac{1}{6}\left(4K_\kappa^2 - K_\pi^2\right) = \frac{1}{6}\left[4.(0.05)^2 - (0.014)^2\right],$$

$$P_{\frac{1}{2}} = \frac{1}{2.7\times10^{-4}}\zeta_{\frac{1}{2}} = 6 \text{ atm}.$$

The corresponding value of refraction index is $n = 1.00162$. We get:

$\theta_\kappa = 1.6^\circ$; $\qquad \theta_\pi = 2\theta_\kappa = 3.2^\circ$.

We do not observe the ring image of protons since

$P_{\frac{1}{2}} = 6\,\text{atm} < 16\,\text{atm} = P_{\min}$ for protons.

4.

4.1. Taking logarithmic differentiation of both sides of the equation $\cos\theta = \frac{1}{\beta n}$, we obtain

$$\frac{\sin\theta \times \Delta\theta}{\cos\theta} = \frac{\Delta\beta}{\beta} \tag{8}$$

Logarithmically differentiating equation (3a) gives

$$\frac{\Delta\beta}{1-\beta} = 2\frac{\Delta p}{p} \tag{9}$$

Combining (8) and (9), taking into account (3b) and putting approximately $\tan\theta = \theta$, we derive

$$\frac{\Delta\theta}{\Delta p} = \frac{2}{\theta} \times \frac{1-\beta}{p\beta} = \frac{K^2}{\theta p} \tag{10}$$

We obtain

-for kaons $K_\kappa = 0.05$, $\theta_\kappa = 1.6^o = 1.6\frac{\pi}{180}$ rad, and so, $\frac{\Delta\theta_\kappa}{\Delta p} = 0.51\frac{1^o}{\text{GeV}/c}$,

-for pions $K_\pi = 0.014$, $\theta_\pi = 3.2^o$, and $\frac{\Delta\theta_\pi}{\Delta p} = 0.02\frac{1^o}{\text{GeV}/c}$.

4.2. $\frac{\Delta\theta_\kappa + \Delta\theta_\pi}{\Delta p} \equiv \frac{\Delta\theta}{\Delta p} = (0.51 + 0.02)\frac{1^o}{\text{GeV}/c} = 0.53\frac{1^o}{\text{GeV}/c}$.

The condition for two ring images to be distinguishable is $\Delta\theta < 0.1(\theta_\pi - \theta_\kappa) = 0.16^o$.

It follows $\Delta p < \frac{1}{10} \times \frac{1.6}{0.53} = 0.3\ \text{GeV}/c$.

5.

5.1. The lower limit of β giving rise to Cherenkov effect is

$$\beta = \frac{1}{n} = \frac{1}{1.33}. \qquad (11)$$

The kinetic energy of a particle having rest mass M and energy E is given by the expression

$$T = E - Mc^2 = \frac{Mc^2}{\sqrt{1-\beta^2}} - Mc^2 = Mc^2\left[\frac{1}{\sqrt{1-\beta^2}} - 1\right]. \qquad (12)$$

Substituting the limiting value (11) of β into (12), we get the minimal kinetic energy of the particle for Cherenkov effect to occur:

$$T_{\min} = Mc^2\left[\frac{1}{\sqrt{1-\left(\frac{1}{1.33}\right)^2}} - 1\right] = 0.517\ Mc^2 \qquad (13)$$

5.2.

For α particles, $T_{\min} = 0.517 \times 3.8\ \text{GeV} = 1.96\ \text{GeV}$.

For electrons, $T_{\min} = 0.517 \times 0.51\ \text{MeV} = 0.264\ \text{MeV}$.

Since the kinetic energy of the particles emitted by radioactive source does not exceed a few MeV, these are electrons which give rise to Cherenkov radiation in the considered experiment.

6. For a beam of particles having a definite momentum the dependence of the angle θ on the refraction index n of the medium is given by the expression

$$\cos\theta = \frac{1}{n\beta} \qquad (14)$$

6.1. Let $\delta\theta$ be the difference of θ between two rings corresponding to two wavelengths limiting the visible range, i.e. to wavelengths of 0.4 μm (violet) and 0.8 μm (red), respectively. The difference in the refraction indexes at these wavelengths is $n_v - n_r = \delta n = 0.02(n-1)$.

Logarithmically differentiating both sides of equation (14) gives

$$\frac{\sin\theta \times \delta\theta}{\cos\theta} = \frac{\delta n}{n} \qquad (15)$$

Corresponding to the pressure of the radiator $P = 6$ atm we have from 4.2. the values $\theta_\pi = 3.2^{\circ}$, $n = 1.00162$.

Putting approximately $\tan\theta = \theta$ and $n = 1$, we get $\delta\theta = \frac{\delta n}{\theta} = 0.033^{\circ}$.

6.2.

6.2.1. The broadening due to dispersion in terms of half width at half height is, according to (6.1), $\frac{1}{2}\delta\theta = 0.017^{\circ}$.

6.2.2. The broadening due to achromaticity is, from 4.1., $0.02\frac{1^{\circ}}{\text{GeV/c}} \times 0.3 \text{ GeV/c} = 0.006^{\circ}$, that is three times smaller than above.

6.2.3. The color of the ring changes from red to white then blue from the inner edge to the outer one.

CHANGE OF AIR TEMPERATURE WITH ALTITUDE, ATMOSPHERIC STABILITY AND AIR POLLUTION

Vertical motion of air governs many atmospheric processes, such as the formation of clouds and precipitation and the dispersal of air pollutants. If the atmosphere is *stable*, vertical motion is restricted and air pollutants tend to be accumulated around the emission site rather than dispersed and diluted. Meanwhile, in an *unstable* atmosphere, vertical motion of air encourages the vertical dispersal of air pollutants. Therefore, the pollutants' concentrations depend not only on the strength of emission sources but also on the *stability* of the atmosphere.

We shall determine the atmospheric stability by using the concept of *air parcel* in meteorology and compare the temperature of the air parcel rising or sinking adiabatically in the atmosphere to that of the surrounding air. We will see that in many cases an air parcel containing air pollutants and rising from the ground will come to rest at a certain altitude, called a *mixing height*. The greater the mixing height, the lower the air pollutant concentration. We will evaluate the mixing height and the concentration of carbon monoxide emitted by motorbikes in the Hanoi metropolitan area for a morning rush hour scenario, in which the vertical mixing is restricted due to a temperature inversion (air temperature increases with altitude) at elevations above 119 m.

Let us consider the air as an ideal diatomic gas, with molar mass $\mu = 29$ g/mol.

Quasi equilibrium adiabatic transformation obey the equation $pV^{\gamma} = \text{const}$, where

$\gamma = \dfrac{c_p}{c_V}$ is the ratio between isobaric and isochoric heat capacities of the gas.

The student may use the following data if necessary:

The universal gas constant is $R = 8.31$ J/(mol.K).

The atmospheric pressure on ground is $p_0 = 101.3$ kPa

The acceleration due to gravity is constant, $g = 9.81$ m/s^2

The molar isobaric heat capacity is $c_p = \dfrac{7}{2}R$ for air.

The molar isochoric heat capacity is $c_V = \dfrac{5}{2}R$ for air.

Mathematical hints

a. $\int \frac{dx}{A+Bx} = \frac{1}{B}\int \frac{d(A+Bx)}{A+Bx} = \frac{1}{B}\mathbf{ln}(A+Bx)$

b. The solution of the differential equation $\frac{dx}{dt} + Ax = B$ (with A and B constant) is $x(t) = x_1(t) + \frac{B}{A}$ where $x_1(t)$ is the solution of the differential equation $\frac{dx}{dt} + Ax = 0$.

c. $\mathbf{lim}_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$

1. Change of pressure with altitude.

1.1. Assume that the temperature of the atmosphere is uniform and equal to T_0. Write down the expression giving the atmospheric pressure p as a function of the altitude z.

1.2. Assume that the temperature of the atmosphere varies with the altitude according to the relation

$$T(z) = T(0) - \Lambda z$$

where Λ is a constant, called the *temperature lapse rate* of the atmosphere (the vertical gradient of temperature is - Λ).

1.2.1. Write down the expression giving the atmospheric pressure p as a function of the altitude z.

1.2.2. A process called free convection occurs when the air density increases with altitude. At which values of Λ does the free convection occur?

2. Change of the temperature of an air parcel in vertical motion

Consider an air parcel moving upward and downward in the atmosphere. An air parcel is a body of air of sufficient dimension, several meters across, to be treated as an independent thermodynamical entity, yet small enough for its temperature to be considered uniform. The vertical motion of an air parcel can be treated as a quasi adiabatic process, i.e. the exchange of heat with the surrounding air is negligible. If the air parcel rises in the atmosphere, it expands and cools. Conversely, if it moves downward, the increasing outside pressure will compress the air inside the parcel and its temperature will increase.

As the size of the parcel is not large, the atmospheric pressure at different points on the parcel boundary can be considered to have the same value $p(z)$, with z - the altitude of the parcel center. The temperature in the parcel is uniform and equals to $T_{\text{parcel}}(z)$, which is generally different from the temperature of the surrounding air $T(z)$. In parts 2.1 and 2.2, we do not make any assumption about the form of $T(z)$.

2.1. The change of the parcel temperature T_{parcel} with altitude is defined by $\frac{dT_{\text{parcel}}}{dz} = -G$. Derive the expression of $G(T, T_{\text{parcel}})$.

2.2. Consider a special atmospheric condition in which at any altitude z the temperature T of the atmosphere equals to that of the parcel T_{parcel}, $T(z) = T_{\text{parcel}}(z)$. We use Γ to denote the value of G when $T = T_{\text{parcel}}$, that is $\Gamma = -\frac{dT_{\text{parcel}}}{dz}$ (with $T = T_{\text{parcel}}$). Γ is called *dry adiabatic lapse rate*.

2.2.1. Derive the expression of Γ

2.2.2. Calculate the numerical value of Γ.

2.2.3. Derive the expression of the atmospheric temperature $T(z)$ as a function of the altitude.

2.3. Assume that the atmospheric temperature depends on altitude according to the relation $T(z) = T(0) - \Lambda z$, where Λ is a constant. Find the dependence of the parcel temperature $T_{\text{parcel}}(z)$ on altitude z.

2.4. Write down the approximate expression of $T_{\text{parcel}}(z)$ when $|\Lambda z| << T(0)$ and $T(0) \approx T_{\text{parcel}}(0)$.

3. The atmospheric stability.

In this part, we assume that T changes linearly with altitude.

3.1. Consider an air parcel initially in equilibrium with its surrounding air at altitude

z_0, i.e. it has the same temperature $T(z_0)$ as that of the surrounding air. If the parcel is moved slightly up and down (e.g. by atmospheric turbulence), one of the three following cases may occur:

- The air parcel finds its way back to the original altitude z_0, the equilibrium of the parcel is stable. The atmosphere is said to be stable.
- The parcel keeps moving in the original direction, the equilibrium of the parcel is unstable. The atmosphere is unstable.
- The air parcel remains at its new position, the equilibrium of the parcel is indifferent. The atmosphere is said to be neutral.

What is the condition on Λ for the atmosphere to be stable, unstable or neutral?

3.2. A parcel has its temperature on ground $T_{\text{parcel}}(0)$ higher than the temperature $T(0)$ of the surrounding air. The buoyancy force will make the parcel rise. Derive the expression for the maximal altitude the parcel can reach in the case of a stable atmosphere in terms of Λ and Γ.

4. The mixing height

4.1. Table 1 shows air temperatures recorded by a radio sounding balloon at 7: 00 am on a November day in Hanoi. The change of temperature with altitude can be approximately described by the formula $T(z) = T(0) - \Lambda z$ with different lapse rates Λ in the three layers $0 < z < 96$ m, 96 m $< z <$ 119 m and 119 m$< z <$ 215 m.

Consider an air parcel with temperature $T_{\text{parcel}}(0) =$ 22$^{\text{o}}$C ascending from ground. On the basis of the data given in Table 1 and using the above linear approximation, calculate the temperature of the parcel at the altitudes of 96 m and 119 m.

4.2. Determine the maximal elevation H the parcel can reach, and the temperature $T_{\text{parcel}}(H)$ of the parcel.

H is called the mixing height. Air pollutants emitted from ground can mix with the air in the atmosphere (e.g. by wind, turbulence and dispersion) and become diluted within this layer.

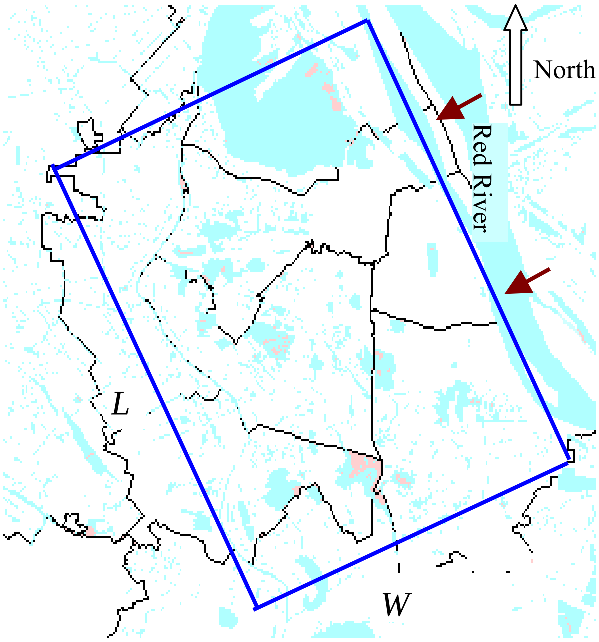
Table 1

Data recorded by a radio sounding balloon at 7:00 am on a November day in Hanoi.

Altitude, m	Temperature, °C
5	21.5
60	20.6
64	20.5
69	20.5
75	20.4
81	20.3
90	20.2
96	20.1
102	20.1
109	20.1
113	20.1
119	20.1
128	20.2
136	20.3
145	20.4
153	20.5
159	20.6
168	20.8
178	21.0
189	21.5
202	21.8
215	22.0
225	22.1
234	22.2
246	22.3
257	22.3

5. Estimation of carbon monoxide (CO) pollution during a morning motorbike rush hour in Hanoi.

Hanoi metropolitan area can be approximated by a rectangle with base dimensions L and W as shown in the figure, with one side taken along the south-west bank of the Red River.



It is estimated that during the morning rush hour, from 7:00 am to 8:00 am, there are 8×10^5 motorbikes on the road, each running on average 5 km and emitting 12 g of CO per kilometer. The amount of CO pollutant is approximately considered as emitted uniformly in time, at a constant rate M during the rush hour. At the same time, the clean north-east wind blows perpendicularly to the Red River (i.e. perpendicularly to the sides L of the rectangle) with velocity u, passes the city with the same velocity, and carries a part of the CO-polluted air out of the city atmosphere.

Also, we use the following rough approximate model:

- The CO spreads quickly throughout the entire volume of the mixing layer above the Hanoi metropolitan area, so that the concentration $C(t)$ of CO at time t can be assumed to be constant throughout that rectangular box of dimensions L, W and H.
- The upwind air entering the box is clean and no pollution is assumed to be lost from the box through the sides parallel to the wind.
- Before 7:00 am, the CO concentration in the atmosphere is negligible.

5.1. Derive the differential equation determining the CO pollutant concentration $C(t)$ as a function of time.

5.2. Write down the solution of that equation for $C(t)$.

5.3. Calculate the numerical value of the concentration $C(t)$ at 8:00 a.m.

Given $L = 15$ km, $W = 8$ km, $u = 1$ m/s.

Solution

1. For an altitude change dz , the atmospheric pressure change is :

$$dp = -\rho g dz \tag{1}$$

where g is the acceleration of gravity, considered constant, ρ is the specific mass of air, which is considered as an ideal gas:

$$\rho = \frac{m}{V} = \frac{p\mu}{RT}$$

Put this expression in (1) :

$$\frac{dp}{p} = -\frac{\mu g}{RT}dz$$

1.1. If the air temperature is uniform and equals T_0, then

$$\frac{dp}{p} = -\frac{\mu g}{RT_0}dz$$

After integration, we have :

$$p(z) = p(0)\mathbf{e}^{-\frac{\mu g}{RT_0}z} \tag{2}$$

1.2. If

$$T(z) = T(0) - \Lambda z \tag{3}$$

then

$$\frac{dp}{p} = -\frac{\mu g}{R\left[T(0) - \Lambda z\right]}dz \tag{4}$$

1.2.1. Knowing that :

$$\int \frac{dz}{T(0) - \Lambda z} = -\frac{1}{\Lambda}\int \frac{d\left[T(0) - \Lambda z\right]}{T(0) - \Lambda z} = -\frac{1}{\Lambda}\mathbf{ln}\left(T(0) - \Lambda z\right)$$

by integrating both members of (4), we obtain :

$$\mathbf{ln}\frac{p(z)}{p(0)} = \frac{\mu g}{R\Lambda}\mathbf{ln}\frac{T(0) - \Lambda z}{T(0)} = \frac{\mu g}{R\Lambda}\mathbf{ln}\left(1 - \frac{\Lambda z}{T(0)}\right)$$

$$p(z) = p(0)\left(1 - \frac{\Lambda z}{T(0)}\right)^{\frac{\mu g}{R\Lambda}} \tag{5}$$

1.2.2. The free convection occurs if:

$$\frac{\rho(z)}{\rho(0)} > 1$$

The ratio of specific masses can be expressed as follows:

$$\frac{\rho(z)}{\rho(0)} = \frac{p(z)}{p(0)}\frac{T(0)}{T(z)} = \left(1 - \frac{\Lambda z}{T(0)}\right)^{\frac{\mu g}{R\Lambda}-1}$$

The last term is larger than unity if its exponent is negative:

$$\frac{\mu g}{R\Lambda} - 1 < 0$$

Then :

$$\Lambda > \frac{\mu g}{R} = \frac{0.029 \times 9.81}{8.31} = 0.034\ \frac{\mathrm{K}}{\mathrm{m}}$$

2. In vertical motion, the pressure of the parcel always equals that of the surrounding air, the latter depends on the altitude. The parcel temperature T_{parcel} depends on the pressure.

2.1. We can write:

$$\frac{dT_{\text{parcel}}}{dz} = \frac{dT_{\text{parcel}}}{dp}\frac{dp}{dz}$$

p is simultaneously the pressure of air in the parcel and that of the surrounding air.

Expression for $\frac{dT_{\text{parcel}}}{dp}$

By using the equation for adiabatic processes $pV^{\gamma} = \text{const}$ and equation of state, we can deduce the equation giving the change of pressure and temperature in a quasi-equilibrium adiabatic process of an air parcel:

$$T_{\text{parcel}}\, p^{\frac{1-\gamma}{\gamma}} = \text{const} \qquad (6)$$

where $\gamma = \frac{c_p}{c_V}$ is the ratio of isobaric and isochoric thermal capacities of air. By logarithmic differentiation of the two members of (6), we have:

$$\frac{dT_{\text{parcel}}}{T_{\text{parcel}}} + \frac{1-\gamma}{\gamma}\frac{dp}{p} = 0$$

Or

$$\frac{dT_{\text{parcel}}}{dp} = \frac{T_{\text{parcel}}}{p}\frac{\gamma-1}{\gamma} \qquad (7)$$

Note: we can use the first law of thermodynamic to calculate the heat received by the parcel in an elementary process: $dQ = \frac{m}{\mu}c_V dT_{\text{parcel}} + pdV$, this heat equals zero in an adiabatic process. Furthermore, using the equation of state for air in the parcel $pV = \frac{m}{\mu}RT_{\text{parcel}}$ we can derive (6)

Expression for $\frac{dp}{dz}$

From (1) we can deduce:

$$\frac{dp}{dz} = -\rho g = -\frac{pg\mu}{RT}$$

where T is the temperature of the surrounding air.

On the basis of these two expressions, we derive the expression for dT_{parcel}/dz :

$$\frac{dT_{\text{parcel}}}{dz} = -\frac{\gamma-1}{\gamma}\frac{\mu g}{R}\frac{T_{\text{parcel}}}{T} = -G \qquad (8)$$

In general, G is not a constant.

2.2.

2.2.1. If at any altitude, $T = T_{\text{parcel}}$, then instead of G in (8), we have :

$$\Gamma = \frac{\gamma-1}{\gamma}\frac{\mu g}{R} = \text{const} \qquad (9)$$

or

$$\Gamma = \frac{\mu g}{c_p} \qquad (9')$$

2.2.2. Numerical value:

$$\Gamma = \frac{1.4-1}{1.4}\frac{0.029\times 9.81}{8.31} = 0.00978\,\frac{\text{K}}{\text{m}} \approx 10^{-2}\,\frac{\text{K}}{\text{m}}$$

2.2.3. Thus, the expression for the temperature at the altitude z in this special atmosphere (called adiabatic atmosphere) is :

$$T(z) = T(0) - \Gamma z \qquad (10)$$

2.3. Search for the expression of $T_{\text{parcel}}(z)$

Substitute T in (7) by its expression given in (3), we have:

$$\frac{dT_{\text{parcel}}}{T_{\text{parcel}}} = -\frac{\gamma-1}{\gamma}\frac{\mu g}{R}\frac{dz}{T(0)-\Lambda z}$$

Integration gives:

$$\mathbf{ln}\frac{T_{\text{parcel}}(z)}{T_{\text{parcel}}(0)} = -\frac{\gamma-1}{\gamma}\frac{\mu g}{R}\left(-\frac{1}{\Lambda}\right)\mathbf{ln}\frac{T(0)-\Lambda z}{T(0)}$$

Finally, we obtain:

$$T_{\text{parcel}}(z) = T_{\text{parcel}}(0)\left(\frac{T(0)-\Lambda z}{T(0)}\right)^{\frac{\Gamma}{\Lambda}} \qquad (11)$$

2.4.

From (11) we obtain

$$T_{\text{parcel}}(z) = T_{\text{parcel}}(0)\left(1-\frac{\Lambda z}{T(0)}\right)^{\frac{\Gamma}{\Lambda}}$$

If $\Lambda z << T(0)$, then by putting $x = \frac{-T(0)}{\Lambda z}$, we obtain

$$T_{\text{parcel}}(z) = T_{\text{parcel}}(0)\left(\left(1+\frac{1}{x}\right)^{x}\right)^{-\frac{\Gamma z}{T(0)}}$$

$$\approx T_{\text{parcel}}(0)\mathbf{e}^{-\frac{\Gamma z}{T(0)}} \approx T_{\text{parcel}}(0)\left(1-\frac{\Gamma z}{T(0)}\right) \approx T_{\text{parcel}}(0) - \Gamma z$$

hence,

$$T_{\text{parcel}}(z) \approx T_{\text{parcel}}(0) - \Gamma z \qquad (12)$$

3. Atmospheric stability

In order to know the stability of atmosphere, we can study the stability of the equilibrium of an air parcel in this atmosphere.

At the altitude z_0, where $T_{\text{parcel}}(z_0) = T(z_0)$, the air parcel is in equilibrium. Indeed, in this case the specific mass ρ of air in the parcel equals ρ'- that of the surrounding air in the atmosphere. Therefore, the buoyant force of the surrounding air on the parcel equals the weight of the parcel. The resultant of these two forces is zero.

Remember that the temperature of the air parcel $T_{\text{parcel}}(z)$ is given by (7), in which we can assume approximately $G = \Gamma$ at any altitude z near $z = z_0$.

Now, consider the stability of the air parcel equilibrium:

Suppose that the air parcel is lifted into a higher position, at the altitude $z_0 + d$ (with d>0), $T_{\text{parcel}}(z_0 + d) = T_{\text{parcel}}(z_0) - \Gamma d$ and $T(z_0 + d) = T(z_0) - \Lambda d$.

- In the case the atmosphere has temperature lapse rate $\Lambda > \Gamma$, we have $T_{\text{parcel}}(z_0 + d) > T(z_0 + d)$, then $\rho < \rho'$. The buoyant force is then larger than the air parcel weight, their resultant is oriented upward and tends to push the parcel away from the equilibrium position.

Conversely, if the air parcel is lowered to the altitude $z_0 - d$ (d>0), $T_{\text{parcel}}(z_0 - d) < T(z_0 - d)$ and then $\rho > \rho'$.

The buoyant force is then smaller than the air parcel weight; their resultant is oriented downward and tends to push the parcel away from the equilibrium position (see Figure 1)

So the equilibrium of the parcel is unstable, and we found that: *An atmosphere with a temperature lapse rate* $\Lambda > \Gamma$ *is unstable.*

- In an atmosphere with temperature lapse rate $\Lambda < \Gamma$, if the air parcel is lifted to a higher position, at altitude $z_0 + d$ (with d>0), $T_{\text{parcel}}(z_0 + d) < T(z_0 + d)$, then

$\rho > \rho'$. The buoyant force is then smaller than the air parcel weight, their resultant is oriented downward and tends to push the parcel back to the equilibrium position.

Conversely, if the air parcel is lowered to altitude $z_0 - d$ ($d > 0$), $T_{\text{parcel}}(z_0 - d) > T(z_0 - d)$ and then $\rho < \rho'$. The buoyant force is then larger than the air parcel weight, their resultant is oriented upward and tends to push the parcel also back to the equilibrium position (see Figure 2).

So the equilibrium of the parcel is stable, and we found that: *An atmosphere with a temperature lapse rate* $\Lambda < \Gamma$ *is stable.*

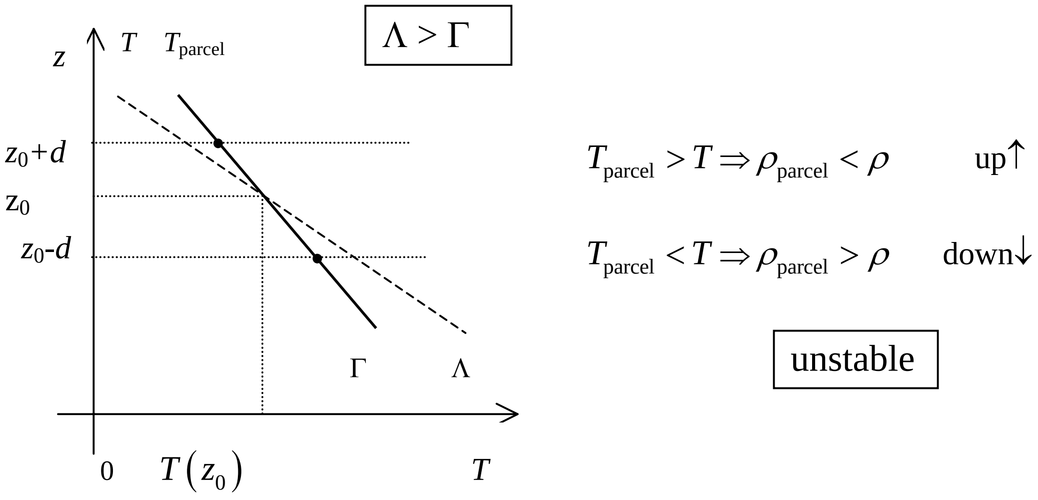


Figure 1

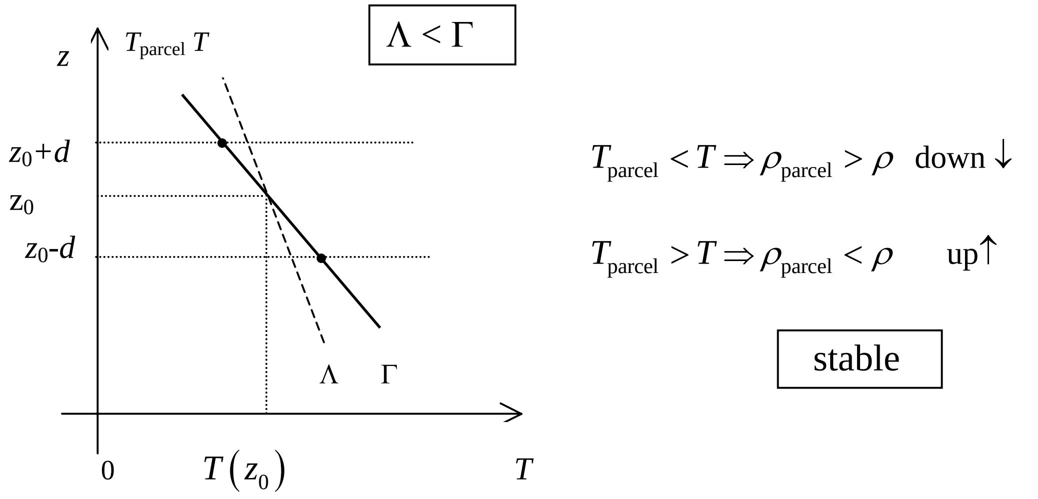


Figure 2

• In an atmosphere with lapse rate $\Lambda = \Gamma$, if the parcel is brought from equilibrium position and put in any other position, it will stay there, the equilibrium is indifferent. *An atmosphere with a temperature lapse rate* $\Lambda = \Gamma$ *is neutral*

3.2. In a stable atmosphere, with $\Lambda < \Gamma$, a parcel, which on ground has temperature $T_{\text{parcel}}(0) > T(0)$ and pressure $p(0)$ equal to that of the atmosphere, can rise and reach a maximal altitude h, where $T_{\text{parcel}}(h) = T(h)$.

In vertical motion from the ground to the altitude h, the air parcel realizes an adiabatic quasi-static process, in which its temperature changes from $T_{\text{parcel}}(0)$ to $T_{\text{parcel}}(h) = T(h)$. Using (11), we can write:

$$\left(1 - \frac{\Lambda h}{T(0)}\right)^{-\frac{\Gamma}{\Lambda}} = \frac{T_{\text{parcel}}(0)}{T(h)} = \frac{T_{\text{parcel}}(0)}{T(0)\left(1 - \frac{\Lambda h}{T(0)}\right)}$$

$$\left(1 - \frac{\Lambda h}{T(0)}\right)^{1-\frac{\Gamma}{\Lambda}} = T_{\text{parcel}}(0) \times T^{-1}(0)$$

$$1 - \frac{\Lambda h}{T(0)} = T_{\text{parcel}}^{\frac{\Lambda}{\Lambda\text{-}\Gamma}}(0) \times T^{-\frac{\Lambda}{\Lambda\text{-}\Gamma}}(0)$$

$$h = \frac{1}{\Lambda} T(0)\left[1 - T_{\text{parcel}}^{\frac{\Lambda}{\Lambda\text{-}\Gamma}}(0) \times T^{-\frac{\Lambda}{\Lambda\text{-}\Gamma}}(0)\right]$$

$$= \frac{1}{\Lambda}\left[T(0) - T_{\text{parcel}}^{-\frac{\Lambda}{\Lambda-\Gamma}}(0)\, T^{\frac{\Gamma}{\Gamma-\Lambda}}(0)\right]$$

So that the maximal altitude h has the following expression:

$$h = \frac{1}{\Lambda}\left[T(0) - \left(\frac{\left(T(0)\right)^{\Gamma}}{\left(T_{\text{parcel}}(0)\right)^{\Lambda}}\right)^{\frac{1}{\Gamma-\Lambda}}\right] \tag{13}$$

4.

Using data from the Table, we obtain the plot of z versus T shown in Figure 3.

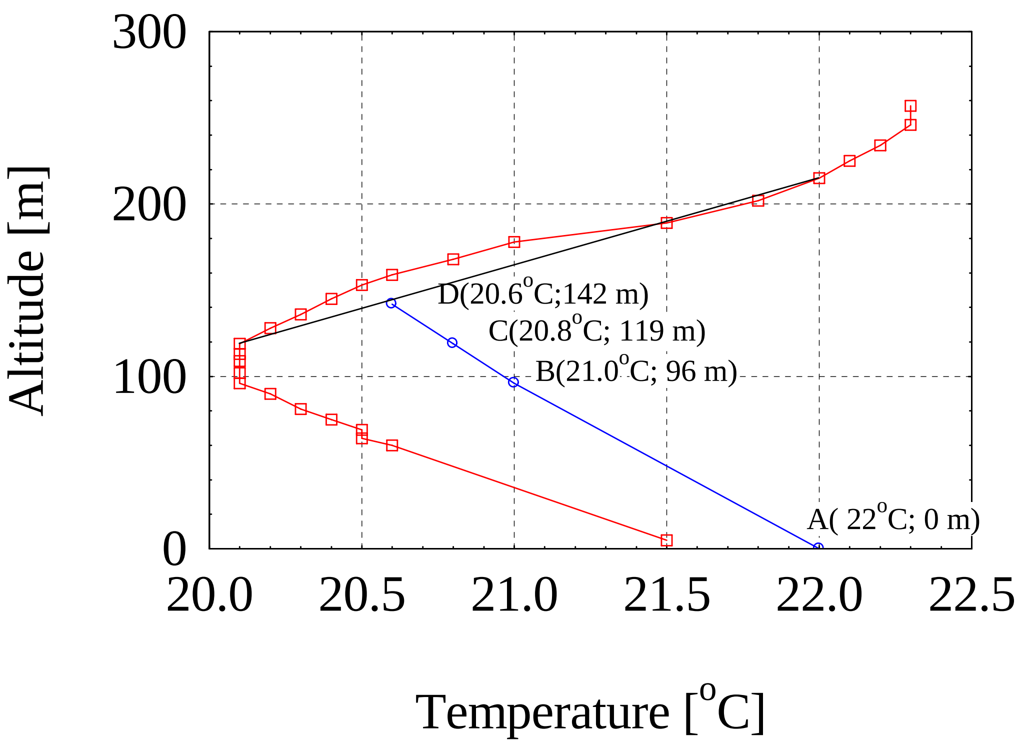


Figure 3

4.1. We can divide the atmosphere under 200m into three layers, corresponding to the following altitudes:

1) $0 < z < 96$ m, $\quad \Lambda_1 = \dfrac{21.5-20.1}{91} = 15.4\times10^{-3}\,\dfrac{\text{K}}{\text{m}}$.

2) $96\text{ m} < z < 119$ m, $\quad \Lambda_2 = 0$, isothermal layer.

3) $119\text{ m} < z < 215$ m, $\quad \Lambda_3 = -\dfrac{22-20.1}{215-119} = -0.02\,\dfrac{\text{K}}{\text{m}}$.

In the layer 1), the parcel temperature can be calculated by using (11)

$$T_{\text{parcel}}(96\,\text{m}) = 294.04\text{ K} \approx 294.0\text{ K} \quad \text{that is } 21.0^{\circ}\text{C}$$

In the layer 2), the parcel temperature can be calculated by using its expression in isothermal atmosphere $T_{\text{parcel}}(z) = T_{\text{parcel}}(0)\,\mathbf{exp}\left[-\dfrac{\Gamma z}{T(0)}\right]$.

The altitude 96 m is used as origin, corresponding to 0 m. The altitude 119 m corresponds to 23 m. We obtain the following value for parcel temperature:

$$T_{\text{parcel}}(119\text{ m}) = 293.81\text{ K} \quad \text{that is } 20.8^{\text{o}}\text{C}$$

4.2. In the layer 3), starting from 119 m, by using (13) we find the maximal elevation h = 23 m, and the corresponding temperature 293.6 K (or 20.6 $^{\text{o}}$C).

Finally, the mixing height is

$$H = 119 + 23 = 142\text{ m}.$$

And

$$T_{\text{parcel}}(142\text{ m}) = 293.6\text{ K} \quad \text{that is} \quad 20.6^{\text{o}}\text{C}$$

From this relation, we can find $T_{\text{parcel}}(119\text{ m}) \approx 293.82\text{ K}$ and $h = 23\text{ m}$.

Note: By using approximate expression (12) we can easily find $T_{\text{parcel}}(z) = 294$ K and 293.8 K at elevations 96 m and 119 m, respectively. At 119 m elevation, the difference between parcel and surrounding air temperatures is 0.7 K (= 293.8 – 293.1), so that the maximal distance the parcel will travel in the third layer is $0.7/(\Gamma - \Lambda_3) = 0.7/0.03 = 23$ m.

5.

Consider a volume of atmosphere of Hanoi metropolitan area being a parallelepiped with height H, base sides L and W. The emission rate of CO gas by motorbikes from 7:00 am to 8:00 am

$$M = 800\,000 \times 5 \times 12 / 3600 = 13\,300 \text{ g/s}$$

The CO concentration in air is uniform at all points in the parallelepiped and denoted by $C(t)$.

5.1. After an elementary interval of time dt, due to the emission of the motorbikes, the mass of CO gas in the box increases by Mdt. The wind blows parallel to the short sides W, bringing away an amount of CO gas with mass $LHC(t)udt$. The remaining part raises the CO concentration by a quantity dC in all over the box. Therefore:

$$Mdt - LHC(t)udt = LWHdC$$

or

$$\frac{dC}{dt}+\frac{u}{W}C(t)=\frac{M}{LWH} \tag{14}$$

5.2. The general solution of (14) is :

$$C(t)=K\,\mathbf{exp}\left(-\frac{ut}{W}\right)+\frac{M}{LHu} \tag{15}$$

From the initial condition $C(0)=0$, we can deduce :

$$C(t)=\frac{M}{LHu}\left[1-\mathbf{exp}\left(-\frac{ut}{W}\right)\right] \tag{16}$$

5.3. Taking as origin of time the moment 7:00 am, then 8:00 am corresponds to t =3600 s. Putting the given data in (15), we obtain :

$$C(3600\text{ s})=6.35\times(1-0.64)=2.3\text{ mg/m}^3$$

THEORETICAL PROBLEM No. 1

EVOLUTION OF THE EARTH-MOON SYSTEM

Scientists can determine the distance Earth-Moon with great precision. They achieve this by bouncing a laser beam on special mirrors deposited on the Moon´s surface by astronauts in 1969, and measuring the round travel time of the light (see Figure 1).

Figure 1. A laser beam sent from an observatory is used to measure accurately the distance between the Earth and the Moon.

With these observations, they have directly measured that the Moon is slowly receding from the Earth. That is, the Earth-Moon distance is increasing with time. This is happening because due to tidal torques the Earth is transferring angular momentum to the Moon, see Figure 2. In this problem you will derive the basic parameters of the phenomenon.

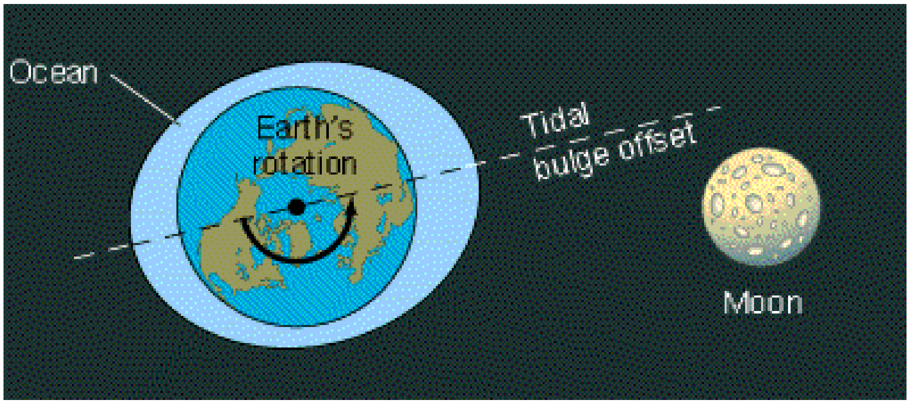


Figure 2. The Moon's gravity produces tidal deformations or "bulges" in the Earth. Because of the Earth's rotation, the line that goes through the bulges is not aligned with the line between the Earth and the Moon. This misalignment produces a torque that transfers angular momentum from the Earth's rotation to the Moon's translation. The drawing is not to scale.

1. Conservation of Angular Momentum.

Let L_1 be the present total angular momentum of the Earth-Moon system. Now, make the following assumptions: i) L_1 is the sum of the rotation of the Earth around its axis and the translation of the Moon in its orbit around the Earth only. ii) The Moon's orbit is circular and the Moon can be taken as a point. iii) The Earth's axis of rotation and the Moon's axis of revolution are parallel. iv) To simplify the calculations, we take the motion to be around the center of the Earth and not the center of mass. Throughout the problem, all moments of inertia, torques and angular momenta are defined around the axis of the Earth. v) Ignore the influence of the Sun.

1a	Write down the equation for the present total angular momentum of the Earth-Moon system. Set this equation in terms of I_E, the moment of inertia of the Earth; ω_{E1}, the present angular frequency of the Earth's rotation; I_{M1}, the present moment of inertia of the Moon with respect to the Earth´s axis; and ω_{M1}, the present angular frequency of the Moon's orbit.	0.2

This process of transfer of angular momentum will end when the period of rotation of the Earth and the period of revolution of the Moon around the Earth have the same duration. At this point the tidal bulges produced by the Moon on the Earth will be aligned with the line between the Moon and the Earth and the torque will disappear.

1b	Write down the equation for the final total angular momentum L_2 of the Earth-Moon system. Make the same assumptions as in Question 1a. Set this equation in terms of I_E, the moment of inertia of the Earth; ω_2, the final angular frequency of the Earth's rotation and Moon's translation; and I_{M2}, the final moment of inertia of the Moon.	0.2

1c	Neglecting the contribution of the Earth´s rotation to the final total angular momentum, write down the equation that expresses the angular momentum conservation for this problem.	0.3

2. Final Separation and Final Angular Frequency of the Earth-Moon System.

Assume that the gravitational equation for a circular orbit (of the Moon around the Earth) is always valid. Neglect the contribution of the Earth´s rotation to the final total angular momentum.

2a	Write down the gravitational equation for the circular orbit of the Moon around the Earth, at the final state, in terms of M_E, ω_2, G and the final separation D_2 between the Earth and the Moon. M_E is the mass of the Earth and G is the gravitational constant.	0.2

2b	Write down the equation for the final separation D_2 between the Earth and the Moon in terms of the known parameters, L_1, the total angular momentum of the system, M_E and M_M, the masses of the Earth and Moon, respectively, and G.	0.5

2c	Write down the equation for the final angular frequency ω_2 of the Earth-Moon system in terms of the known parameters L_1, M_E, M_M and G.	0.5

Below you will be asked to find the numerical values of D_2 and ω_2. For this you need to know the moment of inertia of the Earth.

2d	Write down the equation for the moment of inertia of the Earth I_E assuming it is a sphere with inner density ρ_i from the center to a radius r_i, and with outer density ρ_o from the radius r_i to the surface at a radius r_o (see Figure 3).	0.5

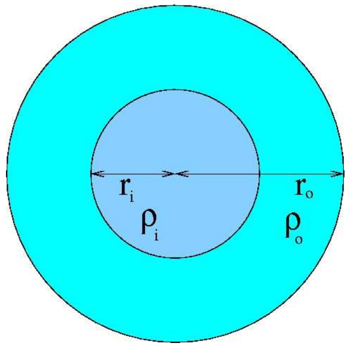


Figure 3. The Earth as a sphere with two densities, ρ_i and ρ_o.

Determine the numerical values requested in this problem always to *two significant digits*.

2e	Evaluate the moment of inertia of the Earth I_E, using $\rho_i = 1.3\times10^4$ kg m^{-3}, $r_i = 3.5\times10^6$ m, $\rho_o = 4.0\times10^3$ kg m^{-3}, and $r_o = 6.4\times10^6$ m.	0.2

The masses of the Earth and Moon are $M_E = 6.0\times10^{24}$ kg and $M_M = 7.3\times10^{22}$ kg, respectively. The present separation between the Earth and the Moon is $D_1 = 3.8\times10^8$ m. The present angular frequency of the Earth's rotation is $\omega_{E1} = 7.3\times10^{-5}$ s^{-1}. The present angular frequency of the Moon's translation around the Earth is $\omega_{M1} = 2.7\times10^{-6}$ s^{-1}, and the gravitational constant is $G = 6.7\times10^{-11}$ m^3 kg^{-1} s^{-2}.

2f	Evaluate the numerical value of the total angular momentum of the system, L_1.	0.2

2g	Find the final separation D_2 in meters and in units of the present separation D_1.	0.3

2h	Find the final angular frequency ω_2 in s^{-1}, as well as the final duration of the day in units of present days.	0.3

Verify that the assumption of neglecting the contribution of the Earth´s rotation to the final total angular momentum is justified by finding the ratio of the final angular momentum of the Earth to that of the Moon. This should be a small quantity.

2i	Find the ratio of the final angular momentum of the Earth to that of the Moon.	0.2

3. How much is the Moon receding per year?

Now, you will find how much the Moon is receding from the Earth each year. For this, you will need to know the equation for the torque acting at present on the Moon. Assume that the tidal bulges can be approximated by two point masses, each of mass m, located on the surface of the Earth, see Fig. 4. Let θ be the angle between the line that goes through the bulges and the line that joins the centers of the Earth and the Moon.

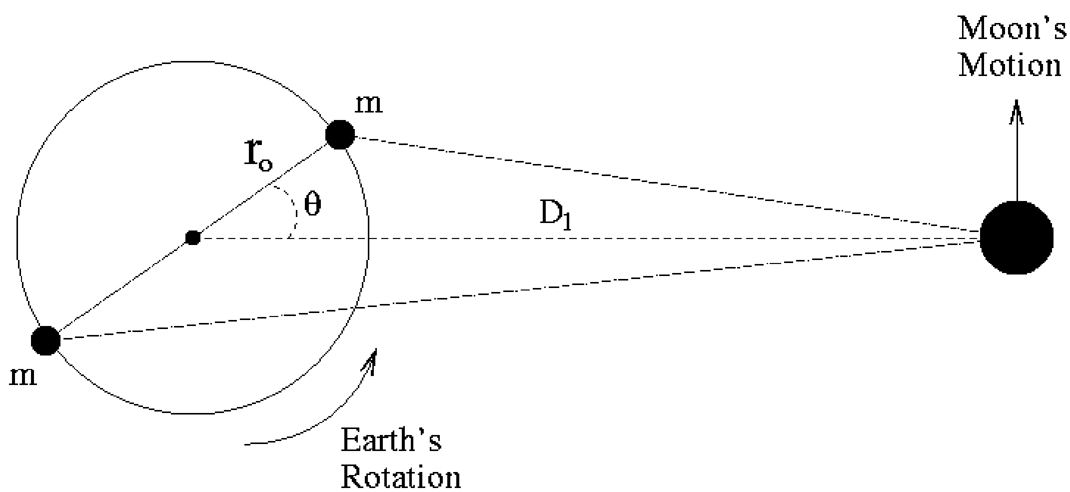


Figure 4. Schematic diagram to estimate the torque produced on the Moon by the bulges on the Earth. The drawing is not to scale.

3a	Find F_c, the magnitude of the force produced on the Moon by the closest point mass.	0.4

3b	Find F_f, the magnitude of the force produced on the Moon by the farthest point mass.	0.4

You may now evaluate the torques produced by the point masses.

3c	Find the magnitude of τ_c, the torque produced by the closest point mass.	0.4

3d	Find the magnitude of τ_f, the torque produced by the farthest point mass.	0.4

3e	Find the magnitude of the total torque τ produced by the two masses. Since $r_o << D_1$ you should approximate your expression to lowest significant order in r_o / D_1. You may use that $(1+x)^a \approx 1 + ax$, if $x<<1$.	1.0

3f	Calculate the numerical value of the total torque τ, taking into account that $\theta = 3^\circ$ and that $m = 3.6\times10^{16}$ kg (note that this mass is of the order of 10^{-8} times the mass of the Earth).	0.5

Since the torque is the rate of change of angular momentum with time, find the increase in the distance Earth-Moon at present, per year. For this step, express the angular momentum of the Moon in terms of M_M, M_E, D_1 and G only.

3g	Find the increase in the distance Earth-Moon at present, per year.	1.0

Finally, estimate how much the length of the day is increasing each year.

3h	Find the decrease of ω_{E1} per year and how much is the length of the day at present increasing each year.	1.0

4. Where is the energy going?

In contrast to the angular momentum, that is conserved, the total (rotational plus gravitational) energy of the system is not. We will look into this in this last section.

4a	Write down an equation for the total (rotational plus gravitational) energy of the Earth-Moon system at present, E. Put this equation in terms of I_E, ω_{E1}, M_M, M_E, D_1 and G only.	0.4

4b	Write down an equation for the change in E, ΔE, as a function of the changes in D_1 and in ω_{E1}. Evaluate the numerical value of ΔE for a year, using the values of changes in D_1 and in ω_{E1} found in questions 3g and 3h.	0.4

Verify that this loss of energy is consistent with an estimate for the energy dissipated as heat in the tides produced by the Moon on the Earth. Assume that the tides rise, on the average by 0.5 m, a layer of water $h =$ 0.5 m deep that covers the surface of the Earth (for simplicity assume that all the surface of the Earth is covered with water). This happens twice a day. Further assume that 10% of this gravitational energy is dissipated as heat due to viscosity when the water descends. Take the density of water to be $\rho_{water} = 10^3$ kg m^{-3}, and the gravitational acceleration on the surface of the Earth to be $g = 9.8$ m s^{-2}.

4c	What is the mass of this surface layer of water?	0.2

4d	Calculate how much energy is dissipated in a year? How does this compare with the energy lost per year by the Earth-Moon system at present?	0.3

THEORETICAL PROBLEM No. 1

EVOLUTION OF THE EARTH-MOON SYSTEM

SOLUTIONS

1. Conservation of Angular Momentum

1a	$L_1 = I_E \omega_{E1} + I_{M1}\omega_{M1}$	0.2

1b	$L_2 = I_E \omega_2 + I_{M2}\omega_2$	0.2

1c	$I_E \omega_{E1} + I_{M1}\omega_{M1} = I_{M2}\omega_2 = L_1$	0.3

2. Final Separation and Angular Frequency of the Earth-Moon System.

2a	$\omega_2^2 D_2^3 = GM_E$	0.2

2b	$D_2 = \dfrac{L_1^2}{GM_E M_M^2}$	0.5

2c	$\omega_2 = \dfrac{G^2 M_E^2 M_M^3}{L_1^3}$	0.5
2d	The moment of inertia of the Earth will be the addition of the moment of inertia of a sphere with radius r_o and density ρ_o and of a sphere with radius r_i and density $\rho_i - \rho_o$: $I_E = \frac{2}{5}\frac{4\pi}{3}[r_o^5 \rho_o + r_i^5(\rho_i - \rho_o)]$.	0.5

2e	$I_E = \frac{2}{5}\frac{4\pi}{3}[r_o^5 \rho_o + r_i^5(\rho_i - \rho_o)] = 8.0\times10^{37}$ kg m^2	0.2

2f	$L_1 = I_E \omega_{E1} + I_{M1}\omega_{M1} = 3.4\times10^{34}$ kg m^2 s^{-1}	0.2

2g	$D_2 = 5.4 \times 10^8$ m, that is $D_2 = 1.4 D_1$	0.3

2h	$\omega_2 = 1.6 \times 10^{-6}$ s^{-1}, that is, a period of 46 days.	0.3

2i	Since $I_E\, \omega_2 = 1.3 \times 10^{32}$ kg m^2 s^{-1} and $I_{M2}\, \omega_2 = 3.4 \times 10^{34}$ kg m^2 s^{-1}, the approximation is justified since the final angular momentum of the Earth is 1/260 of that of the Moon.	0.2

3. How much is the Moon receding per year?

3a	Using the law of cosines, the magnitude of the force produced by the mass m closest to the Moon will be: $F_c = \dfrac{G\, m\, M_M}{D_1^2 + r_o^2 - 2 D_1 r_o \cos(\theta)}$	0.4

3b	Using the law of cosines, the magnitude of the force produced by the mass m farthest to the Moon will be: $F_f = \dfrac{G\, m\, M_M}{D_1^2 + r_o^2 + 2 D_1 r_o \cos(\theta)}$	0.4

3c	Using the law of sines, the torque will be $\tau_c = F_c \dfrac{\sin(\theta)\, r_0\, D_1}{[D_1^2 + r_o^2 - 2\, D_1\, r_o \cos(\theta)]^{1/2}} = \dfrac{G\, m\, M_M \sin(\theta)\, r_0\, D_1}{[D_1^2 + r_o^2 - 2 D_1 r_o \cos(\theta)]^{3/2}}$	0.4

3d	Using the law of sines, the torque will be $\tau_f = F_f \dfrac{\sin(\theta)\, r_0\, D_1}{[D_1^2 + r_o^2 + 2\, D_1\, r_o \cos(\theta)]^{1/2}} = \dfrac{G\, m\, M_M \sin(\theta)\, r_0\, D_1}{[D_1^2 + r_o^2 + 2 D_1 r_o \cos(\theta)]^{3/2}}$	0.4

3e	$\tau_c - \tau_f = G\, m\, M_M \sin(\theta)\, r_0\, D_1^{-2} \left(1 - \dfrac{3 r_o^2}{2 D_1^2} + \dfrac{3 r_o \cos(\theta)}{D_1} - 1 + \dfrac{3 r_o^2}{2 D_1^2} + \dfrac{3 r_o \cos(\theta)}{D_1}\right)$ $= \dfrac{6 G\, m\, M_M\, r_o^2 \sin(\theta) \cos(\theta)}{D_1^3}$	1.0

3f	$\tau = \dfrac{6 G m M_M r_o^2 \sin(\theta)\cos(\theta)}{D_1^3} = 4.1\times 10^{16}$ N m	0.5

3g	Since $\omega_{M1}^2 D_1^3 = G M_E$, we have that the angular momentum of the Moon is $I_{M1}\,\omega_{M1} = M_M\, D_1^2 \left[\dfrac{G\, M_E}{D_1^3}\right]^{1/2} = M_M\,[D_1\, G\, M_E]^{1/2}$ The torque will be: $\tau = \dfrac{M_M\,[G M_E]^{1/2}\,\Delta(D_1^{1/2})}{\Delta t} = \dfrac{M_M\,[G M_E]^{1/2}\ \Delta D_1}{2[D_1]^{1/2}\Delta t}$ So, we have that $\Delta D_1 = \dfrac{2\,\tau\,\Delta t}{M_M}\left[\dfrac{D_1}{G M_E}\right]^{1/2}$ That for $\Delta t = 3.1\times 10^7$ s = 1 year, gives $\Delta D_1 = 0.034$ m. This is the yearly increase in the Earth-Moon distance.	1.0

3h	We now use that $\tau = -\dfrac{I_E\,\Delta\omega_{E1}}{\Delta t}$ from where we get $\Delta\omega_{E1} = -\dfrac{\tau\,\Delta t}{I_E}$ that for $\Delta t = 3.1\times 10^7$ s = 1 year gives $\Delta\omega_{E1} = -1.6\times 10^{-14}$ s^{-1}. If P_E is the period of time considered, we have that: $\dfrac{\Delta P_E}{P_E} = -\dfrac{\Delta\omega_{E1}}{\omega_E}$ since $P_E = 1\ day = 8.64\times 10^4$ s, we get $\Delta P_E = 1.9\times 10^{-5}$ s. This is the amount of time that the day lengthens in a year.	1.0

4. Where is the energy going?

4a	The present total (rotational plus gravitational) energy of the system is: $E = \dfrac{1}{2} I_E\,\omega_{E1}^2 + \dfrac{1}{2} I_M\,\omega_{M1}^2 - \dfrac{G M_E M_M}{D_1}$. Using that $\omega_{M1}^2\, D_1^3 = G\, M_E$, we get	0.4

	$E=\frac{1}{2}I_E\,\omega_{E1}^2-\frac{1}{2}\frac{GM_E M_M}{D_1}$	

4b	$\Delta E=I_E\,\omega_{E1}\,\Delta\omega_{E1}+\frac{1}{2}\frac{GM_E M_M}{D_1^2}\Delta D_1$, that gives $\Delta E=-9.0\times10^{19}\,\text{J}$	0.4

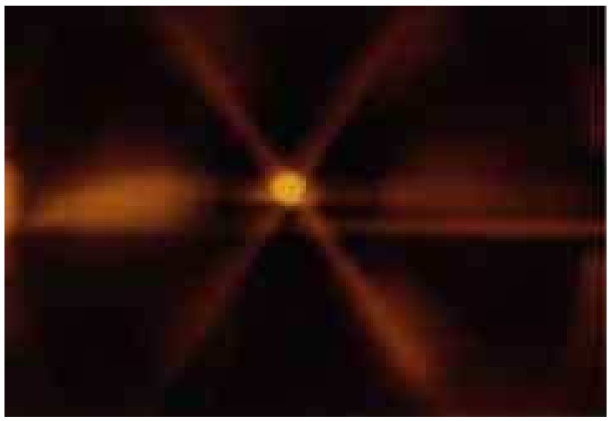
4c	$M_{water}=4\pi r_o^2\times h\times\rho_{water}$ kg = 2.6×10^{17} kg.	0.2

4d	$\Delta E_{water}=-g\,M_{water}\times0.5\,m\times2\,day^{-1}\times365\,days\times0.1=-9.3\times10^{19}$ J. Then, the two energy estimates are comparable.	0.3

THEORETICAL PROBLEM 2

DOPPLER LASER COOLING AND OPTICAL MOLASSES

The purpose of this problem is to develop a simple theory to understand the so-called "laser cooling" and "optical molasses" phenomena. This refers to the cooling of a beam of neutral atoms, typically alkaline, by counterpropagating laser beams with the same frequency. This is part of the Physics Nobel Prize awarded to S. Chu, P. Phillips and C. Cohen-Tannoudji in 1997.



The image above shows sodium atoms (the bright spot in the center) trapped at the intersection of three orthogonal pairs of opposing laser beams. The trapping region is called "optical molasses" because the dissipative optical force resembles the viscous drag on a body moving through molasses.

In this problem you will analyze the basic phenomenon of the interaction between a photon incident on an atom and the basis of the dissipative mechanism in one dimension.

PART I: BASICS OF LASER COOLING

Consider an atom of mass m moving in the $+x$ direction with velocity v. For simplicity, we shall consider the problem to be one-dimensional, namely, we shall ignore the y and z directions (see figure 1). The atom has two internal energy levels. The energy of the lowest state is considered to be zero and the energy of the excited state to be $\hbar\omega_0$, where $\hbar = h/2\pi$. The atom is initially in the lowest state. A laser beam with frequency ω_L in the laboratory is directed in the $-x$ direction and it is incident on the atom. Quantum mechanically the laser is composed of a large number of photons, each with energy $\hbar\omega_L$ and momentum $-\hbar q$. A photon can be absorbed by the atom and later spontaneously emitted; this emission can occur with equal probabilities along the $+x$ and $-x$ directions. Since the atom moves at non-relativistic speeds, $v/c << 1$ (with c the speed of light) keep terms up to first order in this quantity only. Consider also $\hbar q / mv << 1$, namely, that the momentum of the atom is much larger than the momentum of a single photon. In writing your answers, keep only corrections linear in either of the above quantities.

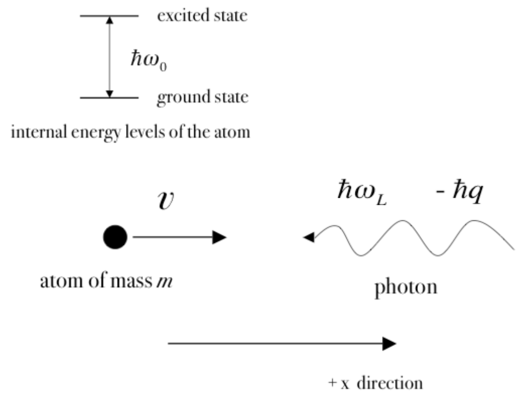


Fig.1 Sketch of an atom of mass m with velocity v in the $+x$ direction, colliding with a photon with energy $\hbar\omega_L$ and momentum $-\hbar q$. The atom has two internal states with energy difference $\hbar\omega_0$.

Assume that the laser frequency ω_L is tuned such that, as seen by the moving atom, it is in resonance with the internal transition of the atom. Answer the following questions:

1. Absorption.

1a	Write down the resonance condition for the absorption of the photon.	0.2

1b	Write down the momentum p_{at} of the atom after absorption, as seen in the laboratory.	0.2

1c	Write down the total energy ε_{at} of the atom after absorption, as seen in the laboratory.	0.2

2. Spontaneous emission of a photon in the $-x$ direction.

At some time after the absorption of the incident photon, the atom may emit a photon in the $-x$ direction.

2a	Write down the energy of the emitted photon, ε_{ph}, after the emission process in the $-x$ direction, as seen in the laboratory.	0.2

2b	Write down the momentum of the emitted photon p_{ph}, after the emission process in the $-x$ direction, as seen in the laboratory.	0.2

2c	Write down the momentum of the atom p_{at}, after the emission process in the $-x$ direction, as seen in the laboratory.	0.2

2d	Write down the total energy of the atom ε_{at}, after the emission process in the $-x$ direction, as seen in the laboratory.	0.2

3. Spontaneous emission of a photon in the $+x$ direction.

At some time after the absorption of the incident photon, the atom may instead emit a photon in the $+x$ direction.

3a	Write down the energy of the emitted photon, ε_{ph}, after the emission process in the $+x$ direction, as seen in the laboratory.	0.2

3b	Write down the momentum of the emitted photon p_{ph}, after the emission process in the $+x$ direction, as seen in the laboratory.	0.2

3c	Write down the momentum of the atom p_{at}, after the emission process in the $+x$ direction, as seen in the laboratory.	0.2

3d	Write down the total energy of the atom ε_{at}, after the emission process in the $+x$ direction, as seen in the laboratory.	0.2

4. Average emission after the absorption.

The spontaneous emission of a photon in the $-x$ or in the $+x$ directions occurs with the same probability. Taking this into account, answer the following questions.

4a	Write down the average energy of an emitted photon, ε_{ph}, after the emission process.	0.2

4b	Write down the average momentum of an emitted photon p_{ph}, after the emission process.	0.2

4c	Write down the average total energy of the atom ε_{at}, after the emission process.	0.2

4d	Write down the average momentum of the atom p_{at}, after the emission process.	0.2

5. Energy and momentum transfer.

Assuming a complete one-photon absorption-emission process only, as described above, there is a net average momentum and energy transfer between the laser radiation and the atom.

5a	Write down the average energy change $\Delta\varepsilon$ of the atom after a complete one-photon absorption-emission process.	0.2

5b	Write down the average momentum change Δp of the atom after a complete one-photon absorption-emission process.	0.2

6. Energy and momentum transfer by a laser beam along the $+x$ direction.

Consider now that a laser beam of frequency ω_L' is incident on the atom along the $+x$ direction, while the atom moves also in the $+x$ direction with velocity v. Assuming a resonance condition between the internal transition of the atom and the laser beam, as seen by the atom, answer the following questions:

6a	Write down the average energy change $\Delta\varepsilon$ of the atom after a complete one-photon absorption-emission process.	0.3

6b	Write down the average momentum change Δp of the atom after a complete one-photon absorption-emission process.	0.3

PART II: DISSIPATION AND THE FUNDAMENTALS OF OPTICAL MOLASSES

Nature, however, imposes an inherent uncertainty in quantum processes. Thus, the fact that the atom can spontaneously emit a photon in a *finite* time after absorption, gives as a result that the resonance condition does not have to be obeyed *exactly* as in the discussion above. That is, the frequency of the laser beams ω_L and ω_L' may have any value and the absorption-emission process can still occur. These will happen with different (quantum) probabilities and, as one should expect, the maximum probability is found at the exact resonance condition. On the average, the time elapsed between a single process of absorption and emission is called the lifetime of the excited energy level of the atom and it is denoted by Γ^{-1}.

Consider a collection of N atoms at *rest* in the laboratory frame of reference, and a laser beam of frequency ω_L incident on them. The atoms absorb and emit continuously such that there is, on average, N_{exc} atoms in the excited state (and therefore, $N - N_{exc}$ atoms in the ground state). A quantum mechanical calculation yields the following result:

$$N_{exc} = N \frac{\Omega_R^2}{(\omega_0 - \omega_L)^2 + \frac{\Gamma^2}{4} + 2\Omega_R^2}$$

where ω_0 is the resonance frequency of the atomic transition and Ω_R is the so-called Rabi frequency; Ω_R^2 is proportional to the *intensity* of the laser beam. As mentioned above, you can see that this number is different from zero even if the resonance frequency ω_0 is different from the frequency of the laser beam ω_L. An alternative way of expressing the previous result is that the number of absorption-emission processes per unit of time is $N_{exc}\Gamma$.

Consider the physical situation depicted in Figure 2, in which two counter propagating laser beams with the *same* but *arbitrary* frequency ω_L are incident on a gas of N atoms that move in the $+x$ direction with velocity v.

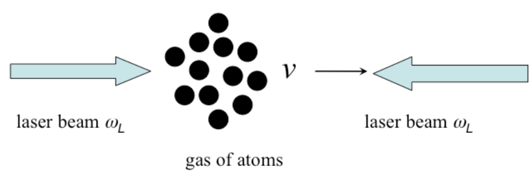


Figure 2. Two counter propagating laser beams with the *same* but *arbitrary* frequency ω_L are incident on a gas of N atoms that move in the $+x$ direction with velocity v.

7. Force on the atomic beam by the lasers.

7a	With the information found so far, find the force that the lasers exert on the atomic beam. You should assume that $mv >> \hbar q$.	1.5

8. Low velocity limit.

Assume now that the velocity of the atoms is small enough, such that you can expand the force up to first order in v.

8a	Find an expression for the force found in Question (7a), in this limit.	1.5

Using this result, you can find the conditions for speeding up, slowing down, or no effect at all on the atoms by the laser radiation.

8b	Write down the condition to obtain a positive force (speeding up the atoms).	0.25

8c	Write down the condition to obtain a zero force.	0.25

8d	Write down the condition to obtain a negative force (slowing down the atoms).	0.25

8e	Consider now that the atoms are moving with a velocity $-v$ (in the $-x$ direction). Write down the condition to obtain a slowing down force on the atoms.	0.25

9. Optical molasses.

In the case of a negative force, one obtains a frictional dissipative force. Assume that initially, at $t=0$, the gas of atoms has velocity v_0.

9a	In the limit of low velocities, find the velocity of the atoms after the laser beams have been on for a time τ.	1.5

9b	Assume now that the gas of atoms is in thermal equilibrium at a temperature T_0. Find the temperature T after the laser beams have been on for a time τ.	0.5

This model does not allow you to go to arbitrarily low temperatures.

THEORETICAL PROBLEM 2

SOLUTION

DOPPLER LASER COOLING AND OPTICAL MOLASSES

The key to this problem is the Doppler effect (to be precise, the longitudinal Doppler effect): The frequency of a monochromatic beam of light detected by an observer depends on its state of motion relative to the emitter, i.e. the observed frequency is

$$\omega' = \omega\sqrt{\frac{1 \pm v/c}{1 \mp v/c}} \approx \omega\left(1 \pm \frac{v}{c}\right)$$

where v is the relative speed of emitter and observer and ω the frequency of the emitter. The upper-lower signs correspond, respectively, when source and observer move towards or away from each other. The second equality holds in the limit of low velocities (non-relativistic limit).

The frequency of the laser in the lab is ω_L; ω_0 is the transition frequency of the atom; the atom moves with speed v towards the incident direction of the laser:

It is important to point out that the results must be given to first significant order in v/c or $\hbar q/mv$.

PART I: BASICS OF LASER COOLING

1. Absorption.

1a	Write down the resonance condition for the absorption of the photon. $\omega_0 \approx \omega_L\left(1 + \frac{v}{c}\right)$	0.2

1b	Write down the momentum p_{at} of the atom after absorption, as seen in the laboratory $p_{at} = p - \hbar q \approx mv - \frac{\hbar\omega_L}{c}$	0.2

1c	Write down the energy ε_{at} of the atom after absorption, as seen in the laboratory $\varepsilon_{at} = \frac{p_{at}^2}{2m} + \hbar\omega_0 \approx \frac{mv^2}{2} + \hbar\omega_L$	0.2

2. Spontaneous emission in the $-x$ direction.

First, one calculates the energy of the emitted photon, as seen in the lab reference frame. One must be careful to keep the correct order; this is because the velocity of the atom changes after the absorption, however, this is second order correction for the emitted frequency:

$$\omega_{ph} \approx \omega_0\left(1-\frac{v'}{c}\right) \quad \text{with} \quad v' \approx v-\frac{\hbar q}{m}$$

thus,

$$\begin{aligned}\omega_{ph} &\approx \omega_0\left(1-\frac{v}{c}+\frac{\hbar q}{mc}\right)\\ &\approx \omega_L\left(1+\frac{v}{c}\right)\left(1-\frac{v}{c}+\frac{\hbar q}{mc}\right)\\ &\approx \omega_L\left(1+\frac{\hbar q}{mc}\right)\\ &\approx \omega_L\left(1+\left(\frac{\hbar q}{mv}\right)\left(\frac{v}{c}\right)\right)\\ &\approx \omega_L\end{aligned}$$

2a	Write down the energy of the emitted photon, ε_{ph}, after the emission process in the $-x$ direction, as seen in the laboratory. $\varepsilon_{ph} \approx \hbar\omega_L$	0.2

2b	Write down the momentum of the emitted photon p_{ph}, after the emission process in the $-x$ direction, as seen in the laboratory. $p_{ph} \approx -\hbar\omega_L / c$	0.2

Use conservation of momentum (see 1b):

$$p_{at} + p_{ph} \approx p - \hbar q$$

2c	Write down the momentum of the atom p_{at}, after the emission process in the $-x$ direction, as seen in the laboratory. $p_{at} \approx p = mv$	0.2

2d	Write down the energy of the atom ε_{at}, after the emission process in the $-x$ direction, as seen in the laboratory. $\varepsilon_{at} \approx \dfrac{p^2}{2m} = \dfrac{mv^2}{2}$	0.2

3. Spontaneous emission in the $+x$ direction.

The same as in the previous questions, keeping the right order

3a	Write down the energy of the emitted photon, ε_{ph}, after the emission process in the $+x$ direction, as seen in the laboratory. $\varepsilon_{ph} \approx \hbar\omega_0\left(1+\frac{v}{c}\right) \approx \hbar\omega_L\left(1+\frac{v}{c}\right)\left(1+\frac{v}{c}\right) \approx \hbar\omega_L\left(1+2\frac{v}{c}\right)$	0.2

3b	Write down the momentum of the emitted photon p_{ph}, after the emission process in the $+x$ direction, as seen in the laboratory. $p_{ph} \approx \frac{\hbar\omega_L}{c}\left(1+2\frac{v}{c}\right)$	0.2

3c	Write down the momentum of the atom p_{at}, after the emission process in the $+x$ direction, as seen in the laboratory. $p_{at} = p - \hbar q - p_{ph} \approx p - \hbar q - \frac{\hbar\omega_L}{c}\left(1+2\frac{v}{c}\right) \approx mv - 2\frac{\hbar\omega_L}{c}$	0.2

3d	Write down the energy of the atom ε_{at}, after the emission process in the $+x$ direction, as seen in the laboratory. $\varepsilon_{at} = \frac{p_{at}^2}{2m} \approx \frac{mv^2}{2}\left(1-2\frac{\hbar q}{mv}\right)$	0.2

4. Average emission after absorption.

The spontaneous emission processes occur with equal probabilities in both directions.

4a	Write down the average energy of an emitted photon, ε_{ph}, after the emission process. $\varepsilon_{ph} = \frac{1}{2}\varepsilon_{ph}^{+} + \frac{1}{2}\varepsilon_{ph}^{-} \approx \hbar\omega_L\left(1+\frac{v}{c}\right)$	0.2

4b	Write down the average momentum of an emitted photon p_{ph}, after the emission process. $p_{ph} = \frac{1}{2}p_{ph}^{+} + \frac{1}{2}p_{ph}^{-} \approx \frac{\hbar\omega_L}{c}\frac{v}{c} = mv\left(\frac{\hbar q}{mv}\frac{v}{c}\right) \approx 0$ second order	0.2

4c	Write down the average energy of the atom ε_{at}, after the emission process. $\varepsilon_{at} = \frac{1}{2}\varepsilon_{at}^{+} + \frac{1}{2}\varepsilon_{at}^{-} \approx \frac{mv^2}{2}\left(1-\frac{\hbar q}{mv}\right)$	0.2

4d	Write down the average momentum of the atom p_{at}, after the emission process. $p_{at} = \frac{1}{2} p_{at}^{+} + \frac{1}{2} p_{at}^{-} \approx p - \frac{\hbar\omega_L}{c}$	0.2

5. Energy and momentum transfer.

Assuming a complete one-photon absorption-emission process only, as described above, there is a net average momentum and energy transfer between the laser and the atom.

5a	Write down the average energy change $\Delta\varepsilon$ of the atom after a complete one-photon absorption-emission process. $\Delta\varepsilon = \varepsilon_{at}^{after} - \varepsilon_{at}^{before} \approx -\frac{1}{2}\hbar q v = -\frac{1}{2}\hbar\omega_L \frac{v}{c}$	0.2

5b	Write down the average momentum change Δp of the atom after a complete one-photon absorption-emission process. $\Delta p = p_{at}^{after} - p_{at}^{before} \approx -\hbar q = -\frac{\hbar\omega_L}{c}$	0.2

6. Energy and momentum transfer by a laser beam along the $+x$ direction.

6a	Write down the average energy change $\Delta\varepsilon$ of the atom after a complete one-photon absorption-emission process. $\Delta\varepsilon = \varepsilon_{at}^{after} - \varepsilon_{at}^{before} \approx +\frac{1}{2}\hbar q v = +\frac{1}{2}\hbar\omega_L' \frac{v}{c}$	0.3

6b	Write down the average momentum change Δp of the atom after a complete one-photon absorption-emission process. $\Delta p = p_{at}^{after} - p_{at}^{before} \approx +\hbar q = +\frac{\hbar\omega_L'}{c}$	0.3

PART II: DISSIPATION AND THE FUNDAMENTALS OF OPTICAL MOLASSES

Two counterpropagating laser beams with the *same* but *arbitrary* frequency ω_L are incident on a beam of N atoms that move in the $+x$ direction with (average) velocity v.

7. Force on the atomic beam by the lasers.

On the average, the fraction of atoms found in the excited state is given by,

$$P_{exc} = \frac{N_{exc}}{N} = \frac{\Omega_R^2}{(\omega_0 - \omega_L)^2 + \frac{\Gamma^2}{4} + 2\Omega_R^2}$$

where ω_0 is the resonance frequency of the atoms and Ω_R is the so-called Rabi frequency; Ω_R^2 is proportional to the *intensity* of the laser beam. The lifetime of the excited energy level of the atom is Γ^{-1}.

The force is calculated as the number of absorption-emission cycles, times the momentum exchange in each event, divided by the time of each event. CAREFUL! One must take into account the Doppler shift of each laser, as seen by the atoms:

7a	With the information found so far, find the force that the lasers exert on the atomic beam. You must assume that $mv >> \hbar q$. $F = N\Delta p^- P_{exc}^- \Gamma + N\Delta p^+ P_{exc}^+ \Gamma$ $= \left(\frac{\Omega_R^2}{\left(\omega_0 - \omega_L + \omega_L \frac{v}{c}\right)^2 + \frac{\Gamma^2}{4} + 2\Omega_R^2} - \frac{\Omega_R^2}{\left(\omega_0 - \omega_L - \omega_L \frac{v}{c}\right)^2 + \frac{\Gamma^2}{4} + 2\Omega_R^2} \right) N\Gamma\hbar q$	1.5

8. Low velocity limit.

Assume now the velocity to be small enough in order to expand the force to first order in v.

8a	Find an expression for the force found in Question (7a), in this limit. $F \approx -\frac{4N\hbar q^2 \Omega_R^2 \Gamma}{\left((\omega_0 - \omega_L)^2 + \frac{\Gamma^2}{4} + 2\Omega_R^2\right)^2} (\omega_0 - \omega_L) v$	1.5

8b	Write down the condition to obtain a positive force (speeding up the atom). $\omega_0 < \omega_L$	0.25

8c	Write down the condition to obtain a zero force. $\omega_0 = \omega_L$	0.25

8d	Write down the condition to obtain a negative force (slowing down the atom). $\omega_0 > \omega_L$ … this is the famous rule "tune below resonance for cooling down"	0.25

8e	Consider now that the atoms are moving with a velocity $-v$ (in the $-x$ direction). Write down the condition to obtain a slowing down force on the atoms. $\omega_0 > \omega_L$ … i.e. independent of the direction motion of the atom.	0.25

9. Optical molasses

In the case of a negative force, one obtains a frictional dissipative force. Assume that initially, $t=0$, the gas of atoms has velocity v_0.

9a	In the limit of low velocities, find the velocity of the atoms after the laser beams have been on for a time τ . $F=-\beta v \Rightarrow m\frac{dv}{dt} \approx -\beta v$ β can be read from (8a) $\Rightarrow v = v_0 e^{-\beta t/m}$	1.5

9b	Assume now that the gas of atoms is in thermal equilibrium at a temperature T_0. Find the temperature T after the laser beams have been on for a time τ. Recalling that $\frac{1}{2}mv^2 = \frac{1}{2}kT$ in 1 dimension, and using v as the average thermal velocity in the equation of (9a), we can write down $T = T_0 e^{-2\beta t/m}$	0.5

THEORETICAL PROBLEM No. 3

WHY ARE STARS SO LARGE?

The stars are spheres of hot gas. Most of them shine because they are fusing hydrogen into helium in their central parts. In this problem we use concepts of both classical and quantum mechanics, as well as of electrostatics and thermodynamics, to understand why stars have to be big enough to achieve this fusion process and also derive what would be the mass and radius of the smallest star that can fuse hydrogen.

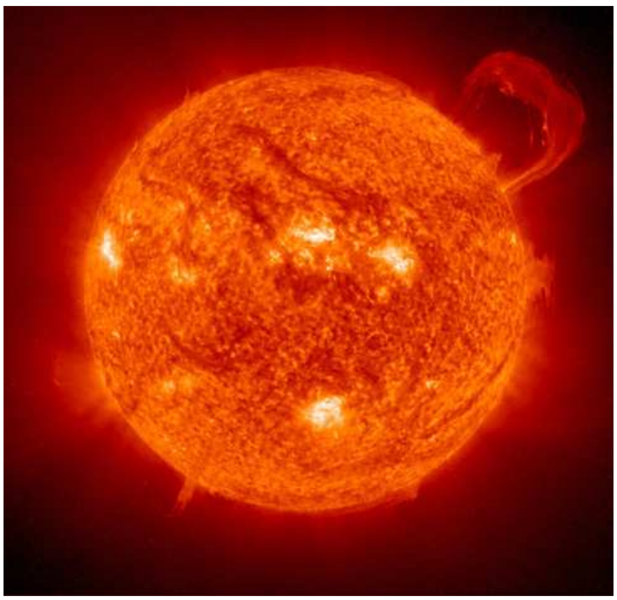


Figure 1. Our Sun, as most stars, shines as a result of thermonuclear fusion of hydrogen into helium in its central parts.

USEFUL CONSTANTS

Gravitational constant = $G = 6.7\times10^{-11}$ m^3 kg^{-1} s^2

Boltzmann's constant = $k = 1.4\times10^{-23}$ J K^{-1}

Planck's constant = $h = 6.6\times10^{-34}$ m^2 kg s^{-1}

Mass of the proton = $m_p = 1.7\times10^{-27}$ kg

Mass of the electron = $m_e = 9.1\times10^{-31}$ kg

Unit of electric charge = $q = 1.6\times10^{-19}$ C

Electric constant (vacuum permittivity) = $\varepsilon_0 = 8.9\times10^{-12}$ C^2 N^{-1} m^{-2}

Radius of the Sun = $R_S = 7.0\times10^{8}$ m

Mass of the Sun = $M_S = 2.0\times10^{30}$ kg

1. A classical estimate of the temperature at the center of the stars.

Assume that the gas that forms the star is pure ionized hydrogen (electrons and protons in equal amounts), and that it behaves like an ideal gas. From the classical point of view, to fuse two protons, they need to get as close as 10^{-15} m for the short range strong nuclear force, which is attractive, to become dominant. However, to bring them together they have to overcome first the repulsive action of Coulomb's force. Assume classically that the two protons (taken to be point sources) are moving in an antiparallel way, each with velocity v_{rms}, the root-mean-square (rms) velocity of the protons, in a one-dimensional frontal collision.

1a	What has to be the temperature of the gas, T_c, so that the distance of closest approach of the protons, d_c, equals 10^{-15} m? Give this and all numerical values in this problem up to two significant figures.	1.5

2. Finding that the previous temperature estimate is wrong.

To check if the previous temperature estimate is reasonable, one needs an independent way of estimating the central temperature of a star. The structure of the stars is very complicated, but we can gain significant understanding making some assumptions. Stars are in equilibrium, that is, they do not expand or contract because the inward force of gravity is balanced by the outward force of pressure (see Figure 2). For a slab of gas the equation of hydrostatic equilibrium at a given distance r from the center of the star, is given by

$$\frac{\Delta P}{\Delta r} = -\frac{G M_r \rho_r}{r^2},$$

where P is the pressure of the gas, G the gravitational constant, M_r the mass of the star within a sphere of radius r, and ρ_r is the density of the gas in the slab.

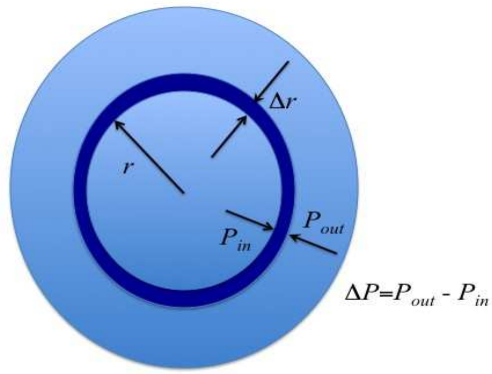


Figure 2. The stars are in hydrostatic equilibrium, with the pressure difference balancing gravity.

An order of magnitude estimate of the central temperature of the star can be obtained with values of the parameters at the center and at the surface of the star, making the following approximations:

$\Delta P \approx P_o - P_c$,

where P_c and P_o are the pressures at the center and surface of the star, respectively. Since $P_c >> P_o$, we can assume that

$\Delta P \approx -P_c$.

Within the same approximation, we can write

$\Delta r \approx R$,

where R is the total radius of the star, and

$M_r \approx M_R = M$,

with M the total mass of the star.

The density may be approximated by its value at the center,

$\rho_r \approx \rho_c$.

You can assume that the pressure is that of an ideal gas.

2a	Find an equation for the temperature at the center of the star, T_c, in terms of the radius and mass of the star and of physical constants only.	0.5

We can use now the following prediction of this model as a criterion for its validity:

2b	Using the equation found in (2a) write down the ratio M/R expected for a star in terms of physical constants and T_c only.	0.5

2c	Use the value of T_c derived in section (1a) and find the numerical value of the ratio M/R expected for a star.	0.5

2d	Now, calculate the ratio $M(Sun)/R(Sun)$, and verify that this value is much smaller than the one found in (2c).	0.5

3. A quantum mechanical estimate of the temperature at the center of the stars

The large discrepancy found in (2d) suggests that the classical estimate for T_c obtained in (1a) is not correct. The solution to this discrepancy is found when we consider quantum mechanical effects, that tell us that the protons behave as waves and that a single proton is smeared on a size of the order of λ_p, the de Broglie wavelength. This implies that if d_c, the distance of closest approach of the protons is of the order of λ_p, the protons in a quantum mechanical sense overlap and can fuse.

3a	Assuming that $d_c = \frac{\lambda_p}{2^{1/2}}$ is the condition that allows fusion, for a proton with velocity v_{rms}, find an equation for T_c in terms of physical constants only.	1.0

3b	Evaluate numerically the value of T_c obtained in (3a).	0.5

3c	Use the value of T_c derived in (3b) to find the numerical value of the ratio M/R expected for a star, using the formula derived in (2b). Verify that this value is quite similar to the ratio $M(Sun)/R(Sun)$ observed.	0.5

Indeed, stars in the so-called *main sequence* (fusing hydrogen) approximately do follow this ratio for a large range of masses.

4. The mass/radius ratio of the stars.

The previous agreement suggests that the quantum mechanical approach for estimating the temperature at the center of the Sun is correct.

4a	Use the previous results to demonstrate that for any star fusing hydrogen, the ratio of mass M to radius R is the same and depends only on physical constants. Find the equation for the ratio M / R for stars fusing hydrogen.	0.5

5. The mass and radius of the smallest star.

The result found in (4a) suggests that there could be stars of any mass as long as such a relationship is fulfilled; however, this is not true.

The gas inside normal stars fusing hydrogen is known to behave approximately as an ideal gas. This means that d_e, the typical separation *between electrons* is on the average larger that λ_e, their typical de Broglie wavelength. If closer, the electrons would be in a so-called degenerate state and the stars would behave differently. Note the distinction in the ways we treat protons and electrons inside the star. For protons, their de Broglie waves should overlap closely as they collide in order to fuse, whereas for electrons their de Broglie waves should not overlap in order to remain as an ideal gas.

The density in the stars increases with decreasing radius. Nevertheless, for this order-of-magnitude estimate assume they are of uniform density. You may further use that $m_p >> m_e$.

5a	Find an equation for n_e, the average electron number density inside the star.	0.5

5b	Find an equation for d_e, the typical separation between electrons inside the star.	0.5

5c	Use the $d_e \geq \frac{\lambda_e}{2^{1/2}}$ condition to write down an equation for the radius of the smallest normal star possible. Take the temperature at the center of the star as typical for all the stellar interior.	1.5

5d	Find the numerical value of the radius of the smallest normal star possible, both in meters and in units of solar radius.	0.5

5e	Find the numerical value of the mass of the smallest normal star possible, both in kg and in units of solar masses.	0.5

6. Fusing helium nuclei in older stars.

As stars get older they will have fused most of the hydrogen in their cores into helium (He), so they are forced to start fusing helium into heavier elements in order to continue shining. A helium nucleus has two protons and two neutrons, so it has twice the charge and approximately four times the mass of a proton. We saw before that $d_c = \frac{\lambda_p}{2^{1/2}}$ is the condition for the protons to fuse.

6a	Set the equivalent condition for helium nuclei and find $v_{rms}(He)$, the rms velocity of the helium nuclei and $T(He)$, the temperature needed for helium fusion.	0.5

Answers

Theoretical Problem No. 3

Why are stars so large?

1) A first, classic estimate of the temperature at the center of the stars.

1a	We equate the initial kinetic energy of the two protons to the electric potential energy at the distance of closest approach: $2(\frac{1}{2}m_p\, v_{rms}^2) = \frac{q^2}{4\pi\varepsilon_0 d_c}$; and since $\frac{3}{2}kT_c = \frac{1}{2}m_p\, v_{rms}^2$, we obtain $T_c = \frac{q^2}{12\pi\varepsilon_0 d_c k} = 5.5\times10^9\ \mathrm{K}$	1.5

2) Finding that the previous temperature estimate is wrong.

2a	Since we have that $\frac{\Delta P}{\Delta r} = -\frac{G M_r \rho_r}{r^2}$, making the assumptions given above, we obtain that: $P_c = \frac{G M \rho_c}{R}$. Now, the pressure of an ideal gas is $P_c = \frac{2\rho_c k T_c}{m_p}$, where k is Boltzmann´s constant, T_c is the central temperature of the star, and m_p is the proton mass. The factor of 2 in the previous equation appears because we have two particles (one proton and one electron) per proton mass and that both contribute equally to the pressure. Equating the two previous equations, we finally obtain that: $T_c = \frac{G M m_p}{2kR}$	0.5

2b	From section (2a) we have that: $\frac{M}{R} = \frac{2kT_c}{G m_p}$	0.5

2c	From section (2b) we have that, for $T_c = 5.5 \times 10^9$ K: $\frac{M}{R} = \frac{2kT_c}{G m_p} = 1.4 \times 10^{24}$ kg m^{-1}.	0.5

2d	For the Sun we have that: $\frac{M(Sun)}{R(Sun)} = 2.9 \times 10^{21}$ kg m^{-1} , that is, three orders of magnitude smaller.	0.5

3) A quantum mechanical estimate of the temperature at the center of the stars

3a	We have that $\lambda_p = \frac{h}{m_p\, v_{rms}}$, and since $\frac{3}{2} k T_c = \frac{1}{2} m_p\, v_{rms}^2$, and $T_c = \frac{q^2}{12\pi\varepsilon_0 d_c k}$, we obtain: $T_c = \frac{q^4 m_p}{24\pi^2 \varepsilon_0^2 k h^2}$.	1.0

3b	$T_c = \frac{q^4 m_p}{24\pi^2 \varepsilon_0^2 k h^2} = 9.7 \times 10^6$ K.	0.5

3c	From section (2b) we have that, for $T_c = 9.7 \times 10^6$ K: $\frac{M}{R} = \frac{2kT_c}{G m_p} = 2.4 \times 10^{21}$ kg m^{-1} ; while for the Sun we have that: $\frac{M(Sun)}{R(Sun)} = 2.9 \times 10^{21}$ kg m^{-1}.	0.5

4) The mass/radius ratio of the stars.

4a	Taking into account that	0.5

	$\frac{M}{R}=\frac{2kT_c}{G\,m_p}$, and that $T_c=\frac{q^4\,m_p}{24\pi^2\,\varepsilon_0^2\,k\,h^2}$, we obtain: $\frac{M}{R}=\frac{q^4}{12\pi^2\,\varepsilon_0^2\,G\,h^2}$.	

5) The mass and radius of the smallest star.

5a	$n_e=\frac{M}{(4/3)\pi R^3\,m_p}$	0.5

5b	$d_e=n_e^{-1/3}=\left(\frac{M}{(4/3)\pi R^3\,m_p}\right)^{-1/3}$	0.5

5c	We assume that $d_e\geq\frac{\lambda_e}{2^{1/2}}$. Since $\lambda_e=\frac{h}{m_e\,v_{rms}(electron)}$, $\frac{3}{2}kT_c=\frac{1}{2}m_e\,v_{rms}^2(electron)$, $T_c=\frac{q^4\,m_p}{24\pi^2\,\varepsilon_0^2\,k\,h^2}$, $\frac{M}{R}=\frac{q^4}{12\pi^2\,\varepsilon_0^2\,G\,h^2}$, and $d_e=\left(\frac{M}{(4/3)\pi R^3\,m_p}\right)^{-1/3}$, we get that $R\geq\frac{\varepsilon_o^{1/2}\,h^2}{4^{1/4}\,q\,m_e^{3/4}\,m_p^{5/4}\,G^{1/2}}$	1.5

5d	$R\geq\frac{\varepsilon_o^{1/2}\,h^2}{4^{1/4}\,q\,m_e^{3/4}\,m_p^{5/4}\,G^{1/2}}=6.9\times10^7$ m = 0.10 $R(Sun)$	0.5

5e	The mass to radius ratio is: $\frac{M}{R} = \frac{q^4}{12\pi^2 \varepsilon_0^2 G h^2} = 2.4\times 10^{21}$ kg m^{-1}, from where we derive that $M \geq 1.7\times 10^{29}$ kg = 0.09 $M(Sun)$	0.5

6) Fusing helium nuclei in older stars.

6a	For helium we have that $\frac{4q^2}{4\pi \varepsilon_0 m_{He} v_{rms}^2(He)} = \frac{h}{2^{1/2} m_{He} v_{rms}(He)}$; from where we get $v_{rms}(He) = \frac{2^{1/2} q^2}{\pi \varepsilon_0 h} = 2.0\times 10^6$ m s^{-1}. We now use: $T(He) = \frac{v_{rms}^2(He) m_{He}}{3k} = 6.5\times 10^8$ K. This value is of the order of magnitude of the estimates of stellar models.	0.5

Solution - Image of a charge

Solution of Task 1

Task 1a)

As the metallic sphere is grounded, its potential vanishes, V=0.

Task1b)

Let us consider an arbitrary point B on the surface of the sphere as depicted in Fig. 1.

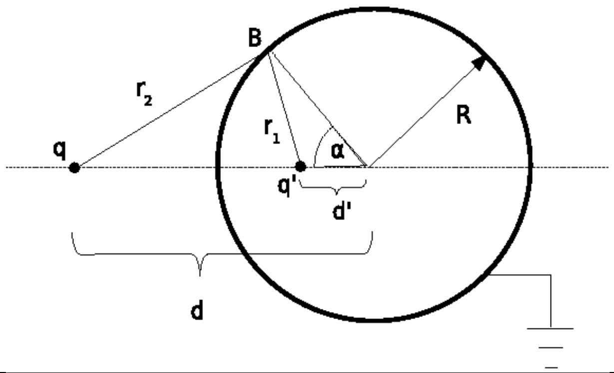


Fig 1. The potential at point *B* is zero.

The distance of point *B* from the charge q' is

$$r_1 = \sqrt{R^2 + d'^2 - 2Rd' \cos\alpha} \quad (1)$$

whereas the distance of the point B from the charge q is given with the expression

$$r_2 = \sqrt{R^2 + d^2 - 2Rd \cos\alpha} \quad (2)$$

The electric potential at the point B is

$$V = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r_2} + \frac{q'}{r_1}\right) \quad (3)$$

This potential must vanish,

$$\frac{q}{r_2} + \frac{q'}{r_1} = 0 \quad (4)$$

i.e. its numerical value is 0 V.

Combining (1), (2) and (3) we obtain

$$R^2 + d^2 - 2Rd\cos\alpha = \left(\frac{q}{q'}\right)^2 (R^2 + d'^2 - 2Rd'\cos\alpha)$$	(5)

As the surface of the sphere must be equipotential, the condition (5) must be satisfied for every angle α what leads to the following results

$$d^2 + R^2 = \left(\frac{q}{q'}\right)^2 (R^2 + d'^2)$$	(6)

and

$$dR = \left(\frac{q}{q'}\right)^2 (d'R)$$	(7)

By solving of (6) and (7) we obtain the expression for the distance d' of the charge q' from the center of the sphere

$$d' = \frac{R^2}{d}$$	(8)

and the size of the charge q'

$$q' = -q\frac{R}{d}$$	(9)

Task 1c)

Finally, the magnitude of force acting on the charge q is

$$F = \frac{1}{4\pi\varepsilon_0}\frac{q^2 Rd}{\left(d^2 - R^2\right)^2}$$	(10)

The force is apparently **attractive.**

Solution of Task 2

Task 2a)

The electric field at the point A amounts to

$$\vec{E}_A = \left(\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2} - \frac{1}{4\pi\varepsilon_0}\frac{q\frac{R}{d}}{\left(r - d + \frac{R^2}{d}\right)^2}\right)\hat{r} \tag{11}$$

Task 2b)

For very large distances r we can apply approximate formula $(1+a)^{-2} \approx 1-2a$ to the expression (11) what leads us to

$$\vec{E}_A = \frac{1}{4\pi\varepsilon_0}\frac{\left(1-\frac{R}{d}\right)q}{r^2}\hat{r} - \frac{1}{4\pi\varepsilon_0}\frac{2q\frac{R}{d}\left(d-\frac{R^2}{d}\right)}{r^3}\hat{r} \tag{12}$$

In general a grounded metallic sphere cannot completely screen a point charge q at a distance d (even in the sense that its electric field would decrease with distance faster than $1/r^2$) and the dominant dependence of the electric field on the distance r is as in standard Coulomb law.

Task 2c)

In the limit $d \to R$ the electric field at the point A vanishes and the grounded metallic sphere screens the point charge completely.

Solution of Task 3

Task 3a)

Let us consider a configuration as in Fig. 2.

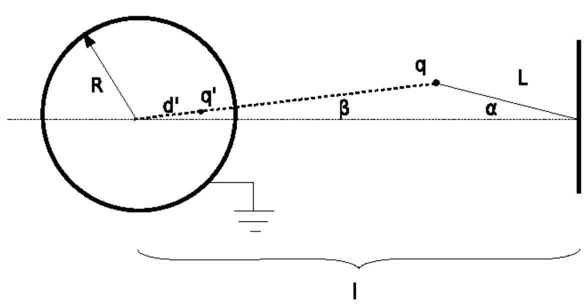


Fig 2. The pendulum formed by a charge near a grounded metallic sphere.

The distance of the charge q from the center of the sphere is

$$d = \sqrt{l^2 + L^2 - 2lL\cos\alpha}$$	(13)

The magnitude of the electric force acting on the charge q is

$$F = \frac{1}{4\pi\varepsilon_0}\frac{qq'}{(d-d')^2} = \frac{1}{4\pi\varepsilon_0}\frac{q^2Rd}{(d^2-R^2)^2}$$	(14)

From which we have

$$F = \frac{1}{4\pi\varepsilon_0}\frac{q^2R\sqrt{l^2+L^2-2lL\cos\alpha}}{\left(l^2+L^2-2lL\cos\alpha-R^2\right)^2}$$	(15)

Task 3b)

The direction of the vector of the electric force (17) is described in Fig. 3.

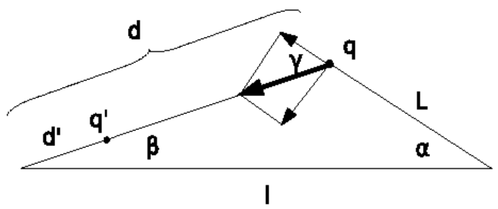


Fig 3. The direction of the force *F*.

The angles α and β are related as

$$L\sin\alpha = d\sin\beta$$	(16)

whereas for the angle γ the relation γ=α+β is valid. The component of the force perpendicular to the thread is F sin γ, that is ,

$$F_\perp = \frac{1}{4\pi\varepsilon_0}\frac{q^2R\sqrt{l^2+L^2-2lL\cos\alpha}}{\left(l^2+L^2-2lL\cos\alpha-R^2\right)^2}\sin(\alpha+\beta)$$ where $$\beta = \arcsin\left(\frac{L}{\sqrt{L^2+l^2-2Ll\cos\alpha}}\sin\alpha\right)$$	(17)

Task 3c)

The equation of motion of the mathematical pendulum is

$mL\ddot{\alpha} = -F_\perp$	(18)

As we are interested in small oscillations, the angle α is small, i.e. for its value in radians we have α much smaller than 1. For a small value of argument of trigonometric functions we have approximate relations sin x ≈ x and cos x ≈ $1-x^2/2$. So for small oscillations of the pendulum we have $\beta \approx \alpha L/(l-L)$ and $\gamma \approx l\alpha/(l-L)$.

Combining these relations with (13) we obtain

$mL\frac{d^2\alpha}{dt^2} + \frac{1}{4\pi\varepsilon_0}\frac{q^2 Rd}{\left(d^2-R^2\right)^2}\left(1+\frac{L}{d}\right)\alpha = 0$	(19)

Where $d = l - L$ what leads to

$\omega = \frac{q}{d^2-R^2}\sqrt{\frac{Rd}{4\pi\varepsilon_0}\frac{1}{mL}\left(1+\frac{L}{d}\right)} = \frac{q}{(l-L)^2-R^2}\sqrt{\frac{Rl}{4\pi\varepsilon_0}\frac{1}{mL}}$	(20)

Solution of Task 4

First we present a solution based on the definition of the electrostatic energy of a collection of charges.

Task 4a)

The total energy of the system can be separated into the electrostatic energy of interaction of the external charge with the induced charges on the sphere, $E_{el,1}$, and the electrostatic energy of mutual interaction of charges on the sphere, $E_{el,2}$, i.e.

$E_{el} = E_{el,1} + E_{el,2}$	(21)

Let there be N charges induced on the sphere. These charges q_j are located at points $\vec{r}_j, j = 1,\ldots,N$ on the sphere. We use the definition of the image charge, i.e., the potential on the surface of the sphere from the image charge is identical to the potential arising from the induced charges:

$\frac{q'}{\left\|\vec{r}-\vec{d}'\right\|}=\sum_{j=1}^{N}\frac{q_j}{\left\|\vec{r}_j-\vec{r}\right\|},$	(22)

where $\vec{r}$ is a vector on the sphere and $\vec{d}'$ denotes the vector position of the image charge. When $\vec{r}$ coincides with some $\vec{r}_i$, then we just have

$\frac{q'}{\left\|\vec{r}_i-\vec{d}'\right\|}=\sum_{\substack{j=1\\ j\neq i}}^{N}\frac{q_j}{\left\|\vec{r}_j-\vec{r}_i\right\|}.$	(23)

From the requirement that the potential on the surface of the sphere vanishes we have

$\frac{q'}{\left\|\vec{r}-\vec{d}'\right\|}+\frac{q}{\left\|\vec{r}-\vec{d}\right\|}=0,$	(24)

where $\vec{d}$ denotes the vector position of the charge $\vec{q}$ ($\vec{r}$ is on the sphere).

For the interaction of the external charge with the induced charges on the sphere we have

$E_{el,1}=\frac{q}{4\pi\varepsilon_0}\sum_{i=1}^{N}\frac{q_i}{\left\|\vec{r}_i-\vec{d}\right\|}=\frac{1}{4\pi\varepsilon_0}\frac{qq'}{\left\|\vec{d}'-\vec{d}\right\|}=\frac{1}{4\pi\varepsilon_0}\frac{qq'}{d-d'}=-\frac{1}{4\pi\varepsilon_0}\frac{q^2R}{d^2-R^2}$	(25)

Here the first equality is the definition of this energy as the sum of interactions of the charge *q* with each of the induced charges on the surface of the sphere. The second equality follows from (21).

In fact, the interaction energy $E_{el,1}$ follows directly from the definition of an image charge.

Task 4b)

The energy of mutual interactions of induced charges on the surface of the sphere is given with

$$E_{el,2}=\frac{1}{2}\frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\frac{q_i q_j}{\left|\vec{r}_i-\vec{r}_j\right|}$$
$$=\frac{1}{2}\frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}q_i\frac{q'}{\left|\vec{r}_i-\vec{d}'\right|}=$$
$$=-\frac{1}{2}\frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}q_i\frac{q}{\left|\vec{r}_i-\vec{d}\right|}=$$
$$=-\frac{1}{2}\frac{1}{4\pi\varepsilon_0}\frac{qq'}{\left|\vec{d}'-\vec{d}\right|}=-\frac{1}{2}\frac{1}{4\pi\varepsilon_0}\frac{qq'}{d-d'}=\frac{1}{2}\frac{1}{4\pi\varepsilon_0}\frac{q^2R}{d^2-R^2} \qquad (26)$$

Here the second line is obtained using (22). From the second line we obtain the third line applying (23), whereas from the third line we obtain the fourth using (22) again.

Task 4c)

Combining expressions (19) and (20) with the quantitative results for the image charge we finally obtain the total energy of electrostatic interaction

$$E_{el}(d)=-\frac{1}{2}\frac{1}{4\pi\varepsilon_0}\frac{q^2R}{d^2-R^2} \qquad (27)$$

An alternative solution follows from the definition of work. By knowing the integral

$$\int_d^{\infty}\frac{xdx}{\left(x^2-R^2\right)^2}=\frac{1}{2}\frac{1}{d^2-R^2} \qquad (28)$$

We can obtain the total energy in the system by calculating the work needed to bring the charge *q* from infinity to the distance *d* from the center of the sphere:

$$E_{el}(d)=-\int_{\infty}^{d}F(\vec{x})d\vec{x}=\int_{d}^{\infty}F(\vec{x})d\vec{x}=$$
$$=\int_{d}^{\infty}(-)\frac{1}{4\pi\varepsilon_0}\frac{q^2Rx}{\left(x^2-R^2\right)^2}dx=$$
$$=-\frac{1}{2}\frac{1}{4\pi\varepsilon_0}\frac{q^2R}{d^2-R^2} \qquad (29)$$

This solves Task 4c).

The electrostatic energy between the charge *q* and the sphere must be equal to the energy between the charges *q* and *q'* according to the definition of the image charge:

$E_{el,1} = \frac{1}{4\pi\varepsilon_0}\frac{qq'}{(d-d')} = -\frac{1}{4\pi\varepsilon_0}\frac{q^2R}{d^2-R^2}$	(30)

This solves Task 4a).

From this we immediately have that the electrostatic energy among the charges on the sphere is:

$E_{el,2} = \frac{1}{2}\frac{1}{4\pi\varepsilon_0}\frac{q^2R}{d^2-R^2}$.	(31)

This solves Task 4b).

1. Image of a charge in a metallic object

Introduction – Method of images

A point charge q is placed in the vicinity of a grounded metallic sphere of radius R [see Fig. 1(a)], and consequently a surface charge distribution is induced on the sphere. To calculate the electric field and potential from the distribution of the surface charge is a formidable task. However, the calculation can be considerably simplified by using the so called method of images. In this method, the electric field and potential produced by the charge distributed on the sphere can be represented as an electric field and potential of a single point charge q' placed inside the sphere (you do not have to prove it). Note: **The electric field of this image charge q' reproduces the electric field and the potential only outside the sphere (including its surface)**.

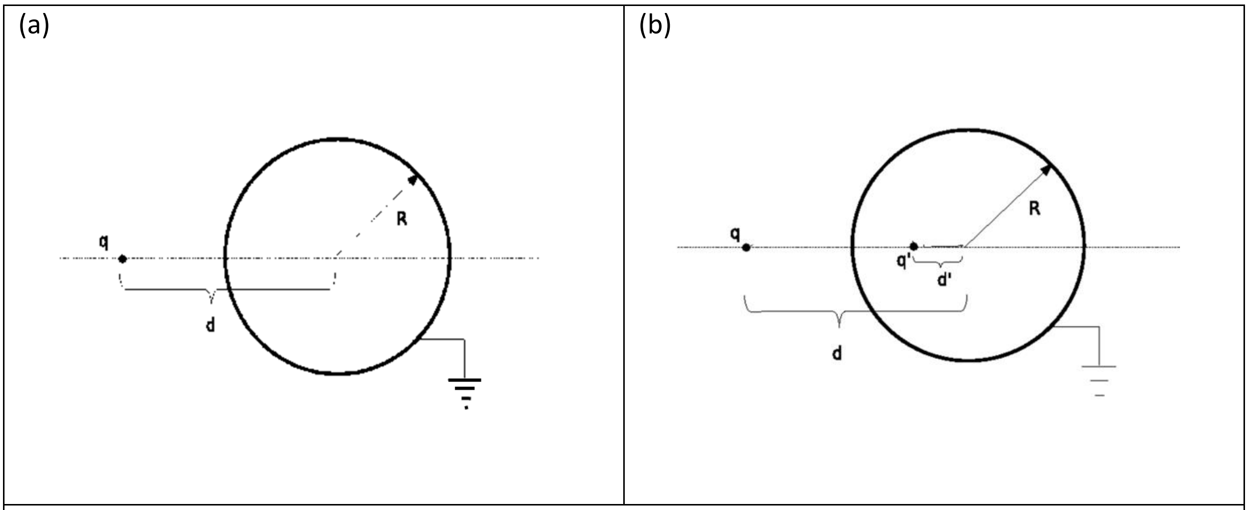


Fig 1. (a) A point charge q in the vicinity of a grounded metallic sphere. (b) The electric field of the charge induced on the sphere can be represented as electric field of an image charge q'.

Task 1 – The image charge

The symmetry of the problem dictates that the charge q' should be placed on the line connecting the point charge q and the center of the sphere [see Fig. 1(b)].

a) What is the value of the potential on the sphere? *(0.3 points)*
b) Express q' and the distance d' of the charge q' from the center of the sphere, in terms of q, d, and R. *(1.9 points)*
c) Find the magnitude of force acting on charge q. Is the force repulsive? *(0.5 points)*

Task 2 – Shielding of an electrostatic field

Consider a point charge *q* placed at a distance *d* from the center of a grounded metallic sphere of radius *R*. We are interested in how the grounded metallic sphere affects the electric field at point *A* on the opposite side of the sphere (see Fig. 2). Point *A* is on the line connecting charge *q* and the center of the sphere; its distance from the point charge *q* is *r*.

a) Find the vector of the electric field at point *A*. *(0.6 points)*

b) For a very large distance $r >> d$, find the expression for the electric field by using the approximation $(1+a)^{-2} \approx 1-2a$, where $a << 1$. *(0.6 points)*

c) In which limit of *d* does the grounded metallic sphere screen the field of the charge *q* completely, such that the electric field at point *A* is exactly zero? *(0.3 points)*

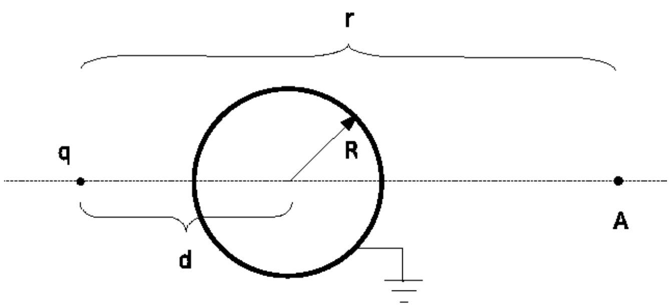


Fig 2. The electric field at point *A* is partially screened by the grounded sphere.

Task 3 – Small oscillations in the electric field of the grounded metallic sphere

A point charge *q* with mass *m* is suspended on a thread of length *L* which is attached to a wall, in the vicinity of the grounded metallic sphere. In your considerations, ignore all electrostatic effects of the wall. The point charge makes a mathematical pendulum (see Fig. 3). The point at which the thread is attached to the wall is at a distance l from the center of the sphere. Assume that the effects of gravity are negligible.

a) Find the magnitude of the electric force acting on the point charge *q* for a given angle α and indicate the direction in a clear diagram *(0.8 points)*

b) Determine the component of this force acting in the direction perpendicular to the thread in terms of l, *L, R, q and* α. *(0.8 points)*

c) Find the frequency for small oscillations of the pendulum. *(1.0 points)*

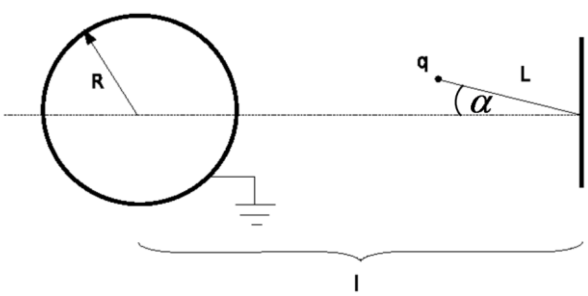


Fig 3. A point charge in the vicinity of a grounded sphere oscillates as a pendulum.

Task 4 – The electrostatic energy of the system

For a distribution of electric charges it is important to know the electrostatic energy of the system. In our problem (see Fig. 1a), there is an electrostatic interaction between the external charge *q* and the induced charges on the sphere, and there is an electrostatic interaction among the induced charges on the sphere themselves. In terms of the charge q, radius of the sphere R and the distance d determine the following electrostatic energies:

a) the electrostatic energy of the interaction between charge q and the induced charges on the sphere; *(1.0 points)*
b) the electrostatic energy of the interaction among the induced charges on the sphere; *(1.2 points)*
c) the total electrostatic energy of the interaction in the system. *(1.0 points)*

Hint: There are several ways of solving this problem:

(1) In one of them, you can use the following integral,

$$\int_d^\infty \frac{xdx}{\left(x^2 - R^2\right)^2} = \frac{1}{2}\frac{1}{d^2 - R^2}.$$

(2) In another one, you can use the fact that for a collection of N charges q_i located at points $\vec{r}_i, i = 1,\ldots,N$, the electrostatic energy is a sum over all pairs of charges: $V = \frac{1}{2}\sum_{i=1}^{N}\sum_{\substack{j=1 \\ i\neq j}}^{N}\frac{1}{4\pi\varepsilon_0}\frac{q_i q_j}{\left|\vec{r}_i - \vec{r}_j\right|}$.

Solution - Chimney physics

This problem was inspired and posed by using the following two references:

- W.W. Christie, *Chimney design and theory*, D. Van Nostrand Company, New York, 1902.
- J. Schlaich, R. Bergermann, W. Schiel, G. Weinrebe, *Design of Commercial Solar Updraft Tower Systems — Utilization of Solar Induced Convective Flows for Power Generation*, Journal of Solar Energy Engineering 127, 117 (2005).

Solution of Task 1

a) What is the minimal height of the chimney needed in order that the chimney functions efficiently, so that it can release all of the produced gas in the atmosphere?

Let $p(z)$ denote the pressure of air at height *z*; then, according to one of the assumptions $p(z) = p(0) - \rho_{Air} gz$, where $p(0)$ is the atmospheric pressure at zero altitude.

Throughout the chimney the Bernoulli law applies, that is, we can write

$\frac{1}{2}\rho_{Smoke} v(z)^2 + \rho_{Smoke} gz + p_{Smoke}(z) = const.,$	(1)

where $p_{Smoke}(z)$ is the pressure of smoke at height *z*, ρ_{Smoke} is its density, and $v(z)$ denotes the velocity of smoke; here we have used the assumption that the density of smoke does not vary throughout the chimney. Now we apply this equation at two points, (i) in the furnace, that is at point $z = -\varepsilon$, where ε is a negligibly small positive number, and (ii) at the top of the chimney where $z = h$ to obtain:

$\frac{1}{2}\rho_{Smoke} v(h)^2 + \rho_{Smoke} gh + p_{Smoke}(h) \approx p_{Smoke}(-\varepsilon)$	(2)

On the right hand side we have used the assumption that the velocity of gases in the furnace is negligible (and also $-\rho_{Smoke} g\varepsilon \approx 0$).

We are interested in the minimal height at which the chimney will operate. The pressure of smoke at the top of the chimney has to be equal or larger than the pressure of air at altitude h; for minimal height of the chimney we have $p_{Smoke}(h) \approx p(h)$. In the furnace we can use $p_{Smoke}(-\varepsilon) \approx p(0)$. The Bernoulli law applied in the furnace and at the top of the chimney [Eq. (2)] now reads

$\frac{1}{2}\rho_{Smoke} v(h)^2 + \rho_{Smoke} gh + p(h) \approx p(0).$	(3)

From this we get

$v(h) = \sqrt{2gh\left(\frac{\rho_{Air}}{\rho_{Smoke}} - 1\right)}.$	(4)

The chimney will be efficient if all of its products are released in the atmosphere, i.e.,

$v(h) \geq \frac{B}{A}$,	(5)

from which we have

$h \geq \frac{B^2}{A^2}\frac{1}{2g}\frac{1}{\frac{\rho_{Air}}{\rho_{Smoke}}-1}$.	(6)

We can treat the smoke in the furnace as an ideal gas (which is at atmospheric pressure $p(0)$ and temperature T_{Smoke}). If the air was at the same temperature and pressure it would have the same density according to our assumptions. We can use this to relate the ratio ρ_{Air}/ρ_{Smoke} to T_{Smoke}/T_{Air} that is,

$\frac{\rho_{Air}}{\rho_{Smoke}} = \frac{T_{Smoke}}{T_{Air}}$, and finally	(7)
$h \geq \frac{B^2}{A^2}\frac{1}{2g}\frac{T_{Air}}{T_{Smoke}-T_{Air}} = \frac{B^2}{A^2}\frac{1}{2g}\frac{T_{Air}}{\Delta T}$.	(8)

For minimal height of the chimney we use the equality sign.

b) How high should the chimney in warm regions be?

$\frac{h(30)}{h(-30)} = \frac{\frac{T(30)}{T_{Smoke}-T(30)}}{\frac{T(-30)}{T_{Smoke}-T(-30)}}; h(30) = 145m$.	(9)

c) How does the velocity of the gases vary along the height of the chimney?

The velocity is constant,

$v = \sqrt{2gh\left(\frac{\rho_{Air}}{\rho_{Smoke}}-1\right)} = \sqrt{2gh\left(\frac{T_{Smoke}}{T_{Air}}-1\right)} = \sqrt{2gh\frac{\Delta T}{T_{Air}}}$.	(10)

This can be seen from the equation of continuity $Av = const.$ (ρ_{Smoke} is constant). It has a sudden jump from approximately zero velocity to this constant value when the gases enter the chimney from the furnace. In fact, since the chimney operates at minimal height this constant is equal to B, that is $v = B/A$.

d) At some height *z*, from the Bernoulli equation one gets

$p_{smoke}(z) = p(0) - (\rho_{Air} - \rho_{Smoke})gh - \rho_{Smoke}gz$.	(11)

Thus the pressure of smoke suddenly changes as it enters the chimney from the furnace and acquires velocity.

Solution of Task 2

a) The kinetic energy of the hot air released in a time interval Δt is

$E_{kin} = \frac{1}{2}(Av\Delta t\rho_{Hot})v^2 = Av\Delta t\rho_{Hot}gh\frac{\Delta T}{T_{Atm}}$,	(12)

Where the index "Hot" refer to the hot air heated by the Sun. If we denote the mass of the air that exits the chimney in unit time with $w = Av\rho_{Hot}$, then the power which corresponds to kinetic energy above is

$P_{kin} = wgh\frac{\Delta T}{T_{Air}}$.	(13)

This is the maximal power that can be obtained from the kinetic energy of the gas flow.

The Sun power used to heat the air is

$P_{Sun} = GS = wc\Delta T$.	(14)

The efficiency is evidently

$\eta = \frac{P_{kin}}{P_{Sun}} = \frac{gh}{cT_{Atm}}$.	(15)

b) The change is apparently linear.

Solution of Task 3

a) The efficiency is

$\eta = \frac{gh}{cT_{Atm}} = 0.0064 = 0.64\%$.	(16)

b) The power is

$P = GS\eta = G(D/2)^2\pi\eta = 45$ kW.	(17)

c) If there are 8 sunny hours per day we get 360kWh.

Solution of Task 4

The result can be obtained by expressing the mass flow of air w as

$w = Av\rho_{Hot} = A\sqrt{2gh\frac{\Delta T}{T_{Air}}}\rho_{Hot}$	(18)
$w = \frac{GS}{c\Delta T}$	(19)

which yields

$\Delta T = (\frac{G^2S^2T_{Atm}}{A^2c^2\rho_{Hot}^2 2gh})^{1/3} \approx 9.1$K.	(20)

From this we get

$w = 760\,\text{kg/s}.$	(21)

2. Chimney physics

Introduction

Gaseous products of burning are released into the atmosphere of temperature T_{Air} through a high chimney of cross-section A and height h (see Fig. 1). The solid matter is burned in the furnace which is at temperature T_{Smoke}. The volume of gases produced per unit time in the furnace is B.

Assume that:

- the velocity of the gases in the furnace is negligibly small
- the density of the gases (smoke) does not differ from that of the air at the same temperature and pressure; while in furnace, the gases can be treated as ideal
- the pressure of the air changes with height in accordance with the hydrostatic law; the change of the density of the air with height is negligible
- the flow of gases fulfills the Bernoulli equation which states that the following quantity is conserved in all points of the flow:
  $\frac{1}{2}\rho v^2(z) + \rho g z + p(z) = const,$
  where ρ is the density of the gas, $v(z)$ is its velocity, $p(z)$ is pressure, and z is the height
- the change of the density of the gas is negligible throughout the chimney

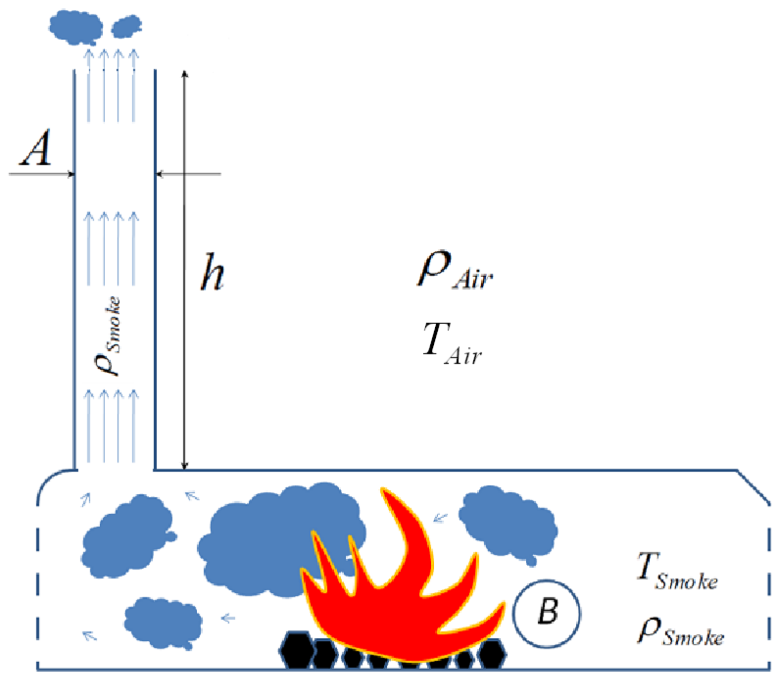


Fig 1. Sketch of a chimney of height h with a furnace at temperature T_{Smoke}.

Task 1

a) What is the minimal height of the chimney needed in order that the chimney functions efficiently, so that it can release all of the produced gas into the atmosphere? Express your result in terms of B, A, T_{Air}, g=9.81m/s^2, $\Delta T = T_{Smoke} - T_{Air}$. **Important: in all subsequent tasks assume that this minimal height is *the* height of the chimney.** *(3.5 points)*

b) Assume that two chimneys are built to serve exactly the same purpose. Their cross sections are identical, but are designed to work in different parts of the world: one in cold regions, designed to work at an average atmospheric temperature of -30 ^{o}C and the other in warm regions, designed to work at an average atmospheric temperature of 30 ^{o}C. The temperature of the furnace is 400 ^{o}C. It was calculated that the height of the chimney designed to work in cold regions is 100 m. How high is the other chimney? *(0.5 points)*

c) How does the velocity of the gases vary along the height of the chimney? Make a sketch/diagram assuming that the chimney cross-section does not change along the height. Indicate the point where the gases enter the chimney. *(0.6 points)*

d) How does the pressure of the gases vary along the height of the chimney? *(0.5 points)*

Solar power plant

The flow of gases in a chimney can be used to construct a particular kind of solar power plant (solar chimney). The idea is illustrated in Fig. 2. The Sun heats the air underneath the collector of area S with an open periphery to allow the undisturbed inflow of air (see Fig. 2). As the heated air rises through the chimney (thin solid arrows), new cold air enters the collector from its surrounding (thick dotted arrows) enabling a continuous flow of air through the power plant. The flow of air through the chimney powers a turbine, resulting in the production of electrical energy. The energy of solar radiation per unit time per unit of horizontal area of the collector is G. Assume that all that energy can be used to heat the air in the collector (the mass heat capacity of the air is c, and one can neglect its dependence on the air temperature). We define the efficiency of the solar chimney as the ratio of the kinetic energy of the gas flow and the solar energy absorbed in heating of the air prior to its entry into the chimney.

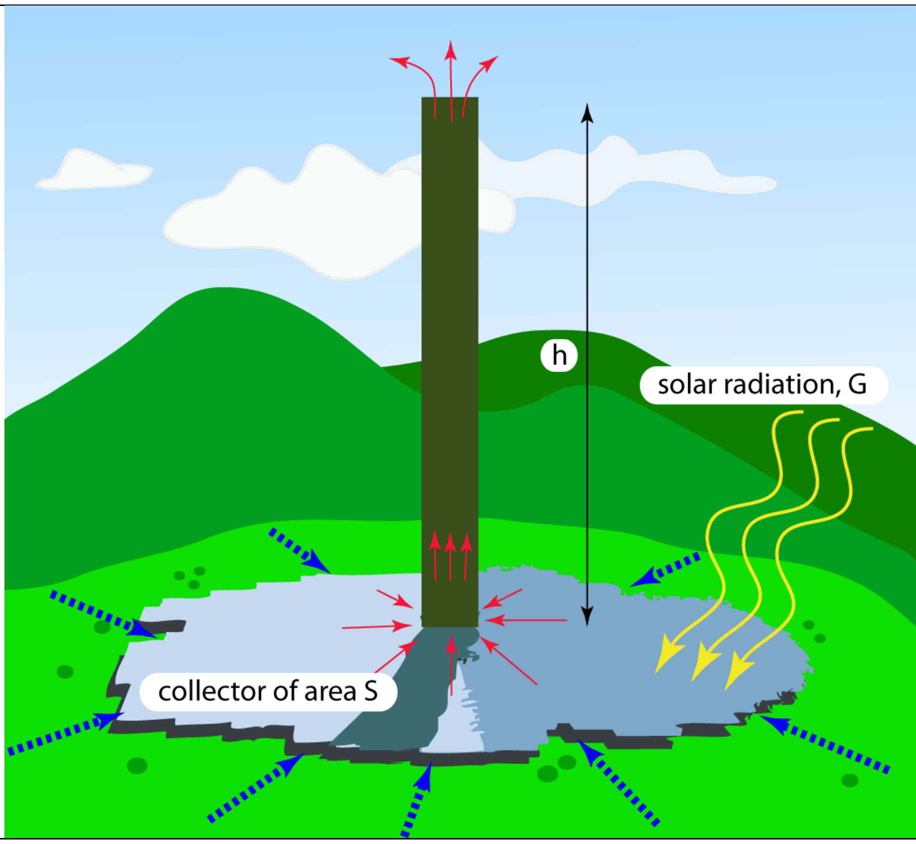


Fig 2. Sketch of a solar power plant.

Task 2

a) What is the efficiency of the solar chimney power plant? *(2.0 points)*

b) Make a diagram showing how the efficiency of the chimney changes with its height. *(0.4 points)*

Manzanares prototype

The prototype chimney built in Manzanares, Spain, had a height of 195 m, and a radius 5 m. The collector is circular with diameter of 244 m. The specific heat of the air under typical operational conditions of the prototype solar chimney is 1012 J/kg K, the density of the hot air is about 0.9 kg/m^3, and the typical temperature of the atmosphere T_{Air} = 295 K. In Manzanares, the solar power per unit of horizontal surface is typically 150 W/m^2 during a sunny day.

Task 3

a) What is the efficiency of the prototype power plant? Write down the numerical estimate. *(0.3 points)*

b) How much power could be produced in the prototype power plant? *(0.4 points)*

c) How much energy could the power plant produce during a typical sunny day? *(0.3 points)*

Task 4

a) How large is the rise in the air temperature as it enters the chimney (warm air) from the surrounding (cold air)? Write the general formula and evaluate it for the prototype chimney. *(1.0 points)*

b) What is the mass flow rate of air through the system? *(0.5 points)*

Solution - model of an atomic nucleus

Solution of Task 1

a) In the SC-system, in each of 8 corners of a given cube there is one unit (atom, nucleon, etc.), but it is shared by 8 neighboring cubes – this gives a total of one nucleon per cube. If nucleons are touching, as we assume in our simplified model, then $a = 2r_N$ is the cube edge length a. The volume of one nucleon is then

$V_N = \frac{4}{3} r_N^3 \pi = \frac{4}{3}\left(\frac{a}{2}\right)^3 \pi = \frac{4a^3}{3 \cdot 8}\pi = \frac{\pi}{6} a^3$	(1)

from which we obtain

$f = \frac{V_N}{a^3} = \frac{\pi}{6} \approx 0.52$	(2)

b) The mass density of the nucleus is:

$\rho_m = f \frac{m_N}{V_N} = 0.52 \cdot \frac{1.67 \cdot 10^{-27}}{4/3 \cdot (0.85 \cdot 10^{-15})^3 \pi} \approx 3.40 \cdot 10^{17} \frac{\mathrm{kg}}{\mathrm{m}^3}.$	(4)

Taking into account the approximation that the number of protons and neutrons is

c) approximately equal, for charge density we get:

$\rho_c = \frac{f}{2} \frac{e}{V_N} = \frac{0.52}{2} \cdot \frac{1.6 \cdot 10^{-19}}{4/3 \cdot (0.85 \cdot 10^{-15})^3 \pi} \approx 1.63 \cdot 10^{25} \frac{\mathrm{C}}{\mathrm{m}^3}$	(5)

The number of nucleons in a given nucleus is *A*. The total volume occupied by the nucleus is:

$V = \frac{AV_N}{f},$	(6)

which gives the following relation between radii of nucleus and the number of nucleons:

$R = r_N \left(\frac{A}{f}\right)^{1/3} = \frac{r_N}{f^{1/3}} A^{1/3} = \frac{0.85}{0.52^{1/3}} A^{1/3} = 1.06\,\mathrm{fm} \cdot A^{1/3}.$	(7)

The numerical constant (1.06 fm) in the equation above will be denoted as r_0 in the sequel.

Solution of Task 2

First one needs to estimate the number of surface nucleons. The surface nucleons are in a spherical shell of width $2r_N$ at the surface. The volume of this shell is

$V_{surface} = \frac{4}{3}R^3\pi - \frac{4}{3}(R-2r_N)^3\pi =$ $= \frac{4}{3}R^3\pi - \frac{4}{3}R^3\pi + \frac{4}{3}\pi 3R^2 2r_N - \frac{4}{3}\pi 3R4r_N^2 + \frac{4}{3}\pi 8r_N^3$ $= 8\pi Rr_N(R-2r_N) + \frac{4}{3}\pi 8r_N^3 =$ $= 8\pi(R^2 r_N - 2Rr_N^2 + \frac{4}{3}r_N^3)$	(8)

The number of surface nucleons is:

$A_{surface} = f\frac{V_{surface}}{V_N} = f\frac{8\pi(R^2 r_N - 2Rr_N^2 + \frac{4}{3}r_N^3)}{\frac{4}{3}r_N^3\pi} =$ $= f6\left(\left(\frac{R}{r_N}\right)^2 - 2\left(\frac{R}{r_N}\right) + \frac{4}{3}\right) =$ $= f6\left(\left(\frac{A}{f}\right)^{2/3} - 2\left(\frac{A}{f}\right)^{1/3} + \frac{4}{3}\right) =$ $= 6f^{1/3}A^{2/3} - 12f^{2/3}A^{1/3} + 8f =$ $= 6^{2/3}\pi^{1/3}A^{2/3} - 2\cdot 6^{1/3}\pi^{2/3}A^{1/3} + \frac{4}{3}\pi \approx$ $\approx 4.84A^{2/3} - 7.80A^{1/3} + 4.19.$	(9)

The binding energy is now:

$E_b = (A - A_{surface})a_V + A_{surface}\frac{a_V}{2} =$ $= Aa_V - A_{surface}\frac{a_V}{2} =$ $= Aa_V - (3f^{1/3}A^{2/3} - 6f^{2/3}A^{1/3} + 4f)a_V =$ $= Aa_V - 3f^{1/3}A^{2/3}a_V + 6f^{2/3}A^{1/3}a_V - 4fa_V =$ $= (15.8A - 38.20A^{2/3} + 61.58A^{1/3} - 33.09)\mathrm{MeV}$	(10)

Solution of Task 3 - Electrostatic (Coulomb) effects on the binding energy

a) Replacing Q_0 with Ze gives the electrostatic energy of the nucleus as:

$U_c = \frac{3(Ze)^2}{20\pi\varepsilon_0 R} = \frac{3Z^2e^2}{20\pi\varepsilon_0 R}$	(12)

The fact that each proton is not acting upon itself is taken into account by replacing Z^2 with $Z(Z-1)$:

$U_c = \dfrac{3Z(Z-1)e^2}{20\pi\varepsilon_0 R}$	(13)

b) In the formula for the electrostatic energy we should replace R with $r_N f^{-1/3}A^{1/3}$ to obtain

$\Delta E_b = -\dfrac{3e^2 f^{1/3}}{20\pi\varepsilon_0 r_N}\dfrac{Z(Z-1)}{A^{1/3}} = -\dfrac{Z(Z-1)}{A^{1/3}}\cdot 1.31\times 10^{-13}\ \mathrm{J}$ $= -\dfrac{Z(Z-1)}{A^{1/3}}\cdot 0.815\ \mathrm{MeV} \approx -0.204\mathrm{A}^{5/3}\mathrm{MeV} + 0.409\mathrm{A}^{2/3}\ \mathrm{MeV}$	(14)

where *Z≈A/2* has been used. The Coulomb repulsion reduces the binding energy, hence the negative sign before the first (main) term. The complete formula for binding energy now gives:

$E_b = Aa_V - 3f^{1/3}A^{2/3}a_V + 6f^{2/3}A^{1/3}a_V - 4fa_V - \dfrac{3e^2 f^{1/3}}{20\pi\varepsilon_0 r_N}\left(\dfrac{A^{5/3}}{4} - \dfrac{A^{2/3}}{2}\right)$	(15)

Solution of Task 4 - Fission of heavy nuclei

a) The kinetic energy comes from the difference of binding energies (2 small nuclei – the original large one) and the Coulomb energy between two smaller nuclei (with *Z/2=A/4* nucleons each):

$E_{kin}(d) = 2E_b\left(\dfrac{A}{2}\right) - E_b(A) - \dfrac{1}{4\pi\varepsilon_0}\dfrac{A^2e^2}{4\cdot 4\cdot d} =$ $= -3f^{1/3}A^{2/3}a_V(2^{1/3}-1) + 6f^{2/3}A^{1/3}a_V(2^{2/3}-1)$ $-4fa_V - \dfrac{3e^2 f^{1/3}}{20\pi\varepsilon_0 r_N}\left[\dfrac{A^{5/3}}{4}\left(2^{-2/3}-1\right) - \dfrac{A^{2/3}}{2}\left(2^{1/3}-1\right)\right]$ $-\dfrac{1}{4\pi\varepsilon_0}\dfrac{A^2e^2}{16d}$	(16)

(notice that the first term, Aa_V, cancels out).

b) The kinetic energy when $d = 2R(A/2)$ is given with:

$E_{kin} = 2E_b\left(\dfrac{A}{2}\right) - E_b(A) - \dfrac{1}{4\pi\varepsilon_0}\dfrac{2^{1/3}A^2e^2}{16\cdot 2r_N A^{1/3}f^{-1/3}} =$ $= -3f^{1/3}A^{2/3}a_V(2^{1/3}-1) + 6f^{2/3}A^{1/3}a_V(2^{2/3}-1)$ $-4fa_V - \dfrac{e^2 f^{1/3}}{\pi\varepsilon_0 r_N}\left[\dfrac{3}{80}(2^{-2/3}-1) + \dfrac{2^{1/3}}{128}\right]A^{5/3} - \dfrac{e^2 f^{1/3}}{\pi\varepsilon_0 r_N}\left[\dfrac{3}{40}(2^{1/3}-1)\right]A^{2/3} =$ $= (0.02203A^{5/3} - 10.0365A^{2/3} + 36.175A^{1/3} - 33.091)\mathrm{MeV}$	(17)

Numerically one gets:
A=100 … E_{kin}= -33.95 MeV,
A=150 … E_{kin}= -30.93 MeV,

A=200 … E_{kin}= -14.10 MeV,
A=250 … E_{kin}= +15.06 MeV.

In our model, fission is possible when $E_{kin}(d = 2R(A/2)) \geq 0$. From the numerical evaluations given above, one sees that this happens approximately halfway between A=200 and A=250 – a rough estimate would be A≈225. Precise numerical evaluation of the equation:

$$E_{kin} = (0.02203A^{5/3} - 10.0365A^{2/3} + 36.175A^{1/3} - 33.091)\mathrm{MeV} \geq 0 \quad (18)$$

gives that for $A \geq 227$ fission is possible.

Solution of Task 5 – Transfer reactions

Task 5a) This part can be solved by using either non-relativistic or relativistic kinematics.

Non-relativistic solution

First one has to find the amount of mass transferred to energy in the reaction (or the energy equivalent, so-called Q-value):

$$\begin{aligned}\Delta m &= (\text{total mass})_{\text{after reaction}} - (\text{total mass})_{\text{before reaction}} = \\ &= (57.93535 + 12.00000)\ \text{a.m.u.} - (53.93962 + 15.99491)\ \text{a.m.u.} = \\ &= 0.00082\ \text{a.m.u.} = \\ &= 1.3616 \cdot 10^{-30}\ \text{kg.}\end{aligned} \quad (19)$$

Using the Einstein formula for equivalence of mass and energy, we get:

$$\begin{aligned}Q &= (\text{total kinetic energy})_{\text{after reaction}} - (\text{total kinetic energy})_{\text{before reaction}} = \\ &= -\Delta m \cdot c^2 = \\ &= -1.3616 \cdot 10^{-30} \cdot 299792458^2 = -1.2237 \cdot 10^{-13}\ \text{J}\end{aligned} \quad (20)$$

Taking into account that 1 MeV is equal to $1.602 \cdot 10^{-13}$ J, we get:

$$Q = -1.2237 \cdot 10^{-13} / 1.602 \cdot 10^{-13} = -0.761\,\mathrm{MeV} \quad (21)$$

This exercise is now solved using the laws of conservation of energy and momentum. The latter gives (we are interested only for the case when ^{12}C and ^{16}O are having the same direction so we don't need to use vectors):

$$m(^{16}\mathrm{O})v(^{16}\mathrm{O}) = m(^{12}\mathrm{C})v(^{12}\mathrm{C}) + m(^{58}\mathrm{Ni})v(^{58}\mathrm{Ni}) \quad (22)$$

while the conservation of energy gives:

$$E_k(^{16}\mathrm{O}) + Q = E_k(^{12}\mathrm{C}) + E_k(^{58}\mathrm{Ni}) + E_x(^{58}\mathrm{Ni}) \quad (23)$$

where $E_x(^{58}\text{Ni})$ is the excitation energy of ^{58}Ni, and Q is calculated in the first part of this task. But since ^{12}C and ^{16}O have the same velocity, conservation of momentum reduced to:

$$\left[m\left(^{16}\text{O}\right)-m\left(^{12}\text{C}\right)\right]v\left(^{16}\text{O}\right)=m\left(^{58}\text{Ni}\right)v\left(^{58}\text{Ni}\right) \tag{24}$$

Now we can easily find the kinetic energy of ^{58}Ni:

$$\begin{aligned} E_k\left(^{58}\text{Ni}\right) &= \frac{m\left(^{58}\text{Ni}\right)v^2\left(^{58}\text{Ni}\right)}{2}=\frac{\left[m\left(^{58}\text{Ni}\right)v\left(^{58}\text{Ni}\right)\right]^2}{2m\left(^{58}\text{Ni}\right)}= \\ &= \frac{\left[\left[m\left(^{16}\text{O}\right)-m\left(^{12}\text{C}\right)\right]v\left(^{16}\text{O}\right)\right]^2}{2m\left(^{58}\text{Ni}\right)}= \\ &= E_k\left(^{16}\text{O}\right)\frac{\left[m\left(^{16}\text{O}\right)-m\left(^{12}\text{C}\right)\right]^2}{m\left(^{58}\text{Ni}\right)m\left(^{16}\text{O}\right)} \end{aligned} \tag{25}$$

and finally the excitation energy of ^{58}Ni:

$$\begin{aligned} E_x\left(^{58}\text{Ni}\right) &= E_k\left(^{16}\text{O}\right)+Q-E_k\left(^{12}\text{C}\right)-E_k\left(^{58}\text{Ni}\right)= \\ &= E_k\left(^{16}\text{O}\right)+Q-\frac{m\left(^{12}\text{C}\right)v^2\left(^{16}\text{O}\right)}{2}-E_k\left(^{16}\text{O}\right)\frac{\left[m\left(^{16}\text{O}\right)-m\left(^{12}\text{C}\right)\right]^2}{m\left(^{58}\text{Ni}\right)m\left(^{16}\text{O}\right)}= \\ &= Q+E_k\left(^{16}\text{O}\right)-E_k\left(^{16}\text{O}\right)\cdot\frac{m\left(^{12}\text{C}\right)}{m\left(^{16}\text{O}\right)}-E_k\left(^{16}\text{O}\right)\frac{\left[m\left(^{16}\text{O}\right)-m\left(^{12}\text{C}\right)\right]^2}{m\left(^{58}\text{Ni}\right)m\left(^{16}\text{O}\right)}= \\ &= Q+E_k\left(^{16}\text{O}\right)\left[1-\frac{m\left(^{12}\text{C}\right)}{m\left(^{16}\text{O}\right)}-\frac{\left[m\left(^{16}\text{O}\right)-m\left(^{12}\text{C}\right)\right]^2}{m\left(^{58}\text{Ni}\right)m\left(^{16}\text{O}\right)}\right]= \\ &= Q+E_k\left(^{16}\text{O}\right)\frac{\left[m\left(^{16}\text{O}\right)-m\left(^{12}\text{C}\right)\right]\cdot\left[m\left(^{58}\text{Ni}\right)-m\left(^{16}\text{O}\right)+m\left(^{12}\text{C}\right)\right]}{m\left(^{58}\text{Ni}\right)m\left(^{16}\text{O}\right)} \end{aligned} \tag{26}$$

Note that the first bracket in numerator is approximately equal to the mass of transferred particle (the ^{4}He nucleus), while the second one is approximately equal to the mass of target nucleus ^{54}Fe. Inserting the numbers we get:

$$\begin{aligned} E_x\left(^{58}\text{Ni}\right) &= -0.761+50\cdot\frac{(15.99491-12.)(57.93535-15.99491+12.)}{57.93535\cdot 15.99491}= \\ &= 10.866\ \text{MeV} \end{aligned} \tag{27}$$

<u>Relativistic solution</u>

In the relativistic version, solution is found starting from the following pair of equations (the first one is the law of conservation of energy and the second one the law of conservation of momentum):

$$m\left(^{54}\text{Fe}\right)\cdot c^2+\frac{m\left(^{16}\text{O}\right)\cdot c^2}{\sqrt{1-v^2\left(^{16}\text{O}\right)/c^2}}=\frac{m\left(^{12}\text{C}\right)\cdot c^2}{\sqrt{1-v^2\left(^{12}\text{C}\right)/c^2}}+\frac{m^*\left(^{58}\text{Ni}\right)\cdot c^2}{\sqrt{1-v^2\left(^{58}\text{Ni}\right)/c^2}} \tag{28}$$

$$\frac{m\left({}^{16}\mathrm{O}\right)\cdot v\left({}^{16}\mathrm{O}\right)}{\sqrt{1-v^2\left({}^{16}\mathrm{O}\right)/c^2}}=\frac{m\left({}^{12}\mathrm{C}\right)\cdot v\left({}^{12}\mathrm{C}\right)}{\sqrt{1-v^2\left({}^{12}\mathrm{C}\right)/c^2}}+\frac{m^*\left({}^{58}\mathrm{Ni}\right)\cdot v\left({}^{58}\mathrm{Ni}\right)}{\sqrt{1-v^2\left({}^{58}\mathrm{Ni}\right)/c^2}}$$

All the masses in the equations are the rest masses; the ^{58}Ni is NOT in its ground-state, but in one of its excited states (having the mass denoted with m^*). Since ^{12}C and ^{16}O have the same velocity, this set of equations reduces to:

$$m\left({}^{54}\mathrm{Fe}\right)+\frac{m\left({}^{16}\mathrm{O}\right)-m\left({}^{12}\mathrm{C}\right)}{\sqrt{1-v^2\left({}^{16}\mathrm{O}\right)/c^2}}=\frac{m^*\left({}^{58}\mathrm{Ni}\right)}{\sqrt{1-v^2\left({}^{58}\mathrm{Ni}\right)/c^2}}$$
$$\frac{\left(m\left({}^{16}\mathrm{O}\right)-m\left({}^{12}\mathrm{C}\right)\right)\cdot v\left({}^{16}\mathrm{O}\right)}{\sqrt{1-v^2\left({}^{16}\mathrm{O}\right)/c^2}}=\frac{m^*\left({}^{58}Ni\right)\cdot v\left({}^{58}\mathrm{Ni}\right)}{\sqrt{1-v^2\left({}^{58}\mathrm{Ni}\right)/c^2}} \quad (29)$$

Dividing the second equation with the first one gives:

$$v\left({}^{58}\mathrm{Ni}\right)=\frac{\left(m\left({}^{16}\mathrm{O}\right)-m\left({}^{12}\mathrm{C}\right)\right)\cdot v\left({}^{16}\mathrm{O}\right)}{\left(m\left({}^{16}\mathrm{O}\right)-m\left({}^{12}\mathrm{C}\right)\right)+m\left({}^{54}\mathrm{Fe}\right)\sqrt{1-v^2\left({}^{16}\mathrm{O}\right)/c^2}} \quad (30)$$

The velocity of projectile can be calculated from its energy:

$$E_{kin}\left({}^{16}\mathrm{O}\right)=\frac{m\left({}^{16}\mathrm{O}\right)\cdot c^2}{\sqrt{1-v^2\left({}^{16}\mathrm{O}\right)/c^2}}-m\left({}^{16}\mathrm{O}\right)\cdot c^2$$
$$\sqrt{1-v^2\left({}^{16}\mathrm{O}\right)/c^2}=\frac{m\left({}^{16}\mathrm{O}\right)\cdot c^2}{E_{kin}\left({}^{16}\mathrm{O}\right)+m\left({}^{16}\mathrm{O}\right)\cdot c^2}$$
$$v^2\left({}^{16}\mathrm{O}\right)/c^2=1-\left(\frac{m\left({}^{16}\mathrm{O}\right)\cdot c^2}{E_{kin}\left({}^{16}\mathrm{O}\right)+m\left({}^{16}\mathrm{O}\right)\cdot c^2}\right)^2$$
$$v\left({}^{16}\mathrm{O}\right)=\sqrt{1-\left(\frac{m\left({}^{16}\mathrm{O}\right)\cdot c^2}{E_{kin}\left({}^{16}\mathrm{O}\right)+m\left({}^{16}\mathrm{O}\right)\cdot c^2}\right)^2}\cdot c \quad (31)$$

For the given numbers we get:

$$v\left({}^{16}\mathrm{O}\right)=\sqrt{1-\left(\frac{15.99491\cdot 1.6605\cdot 10^{-27}\cdot\left(2.9979\cdot 10^8\right)^2}{50\cdot 1.602\cdot 10^{-13}+15.99491\cdot\left(2.9979\cdot 10^8\right)^2}\right)^2}\cdot c=$$
$$=\sqrt{1-0.99666^2}\cdot c=0.08172\cdot c=2.4498\cdot 10^7\ \text{km/s} \quad (32)$$

Now we can calculate:

$$v\left({}^{58}\mathrm{Ni}\right)=\frac{\left(15.99491-12.0\right)\cdot 2.4498\cdot 10^7\ \text{km/s}}{\left(15.99491-12.0\right)+53.93962\sqrt{1-0.08172^2}}=1.6946\cdot 10^6\ \text{km/s} \quad (33)$$

The mass of ^{58}Ni in its excited state is then:

$$m^*\left({}^{58}\mathrm{Ni}\right)=\left(m\left({}^{16}\mathrm{O}\right)-m\left({}^{12}\mathrm{C}\right)\right)\frac{\sqrt{1-v^2\left({}^{58}\mathrm{Ni}\right)/c^2}}{\sqrt{1-v^2\left({}^{16}\mathrm{O}\right)/c^2}}\cdot\frac{v\left({}^{16}\mathrm{O}\right)}{v\left({}^{58}\mathrm{Ni}\right)}=$$
$$=(15.99491-12.0)\frac{\sqrt{1-\left(1.6945\cdot 10^6/2.9979\cdot 10^8\right)^2}}{\sqrt{1-0.08172^2}}\cdot\frac{2.4498\cdot 10^7}{1.6945\cdot 10^6}\ \text{a.m.u.}=$$
$$=57.9470\ \text{a.m.u.} \tag{34}$$

The excitation energy of ^{58}Ni is then:

$$E_x=\left[m^*\left({}^{58}\mathrm{Ni}\right)-m\left({}^{58}\mathrm{Ni}\right)\right]\cdot c^2=(57.9470-57.93535)\cdot 1.6605\cdot 10^{-27}\left(2.9979\cdot 10^8\right)^2=$$
$$=2.00722\cdot 10^{-12}/1.602\cdot 10^{-13}\ \mathrm{MeV/J}=10.8636\ \mathrm{MeV} \tag{35}$$

The relativistic and non-relativistic results are equal within 2 keV so both can be considered as correct –we can conclude that at the given beam energy, relativistic effects are not important.

Task 5b) For gamma-emission from the static nucleus, laws of conservation of energy and momentum give:

$$E_x\left({}^{58}\mathrm{Ni}\right)=E_\gamma+E_{\mathrm{recoil}}$$
$$p_\gamma=p_{\mathrm{recoil}} \tag{36}$$

Gamma-ray and recoiled nucleus have, of course, opposite directions. For gamma-ray (photon), energy and momentum are related as:

$$E_\gamma=p_\gamma\cdot c \tag{37}$$

In part a) we have seen that the nucleus motion in this energy range is not relativistic, so we have:

$$E_{\mathrm{recoil}}=\frac{p_{\mathrm{recoil}}^2}{2m\left({}^{58}\mathrm{Ni}\right)}=\frac{p_\gamma^2}{2m\left({}^{58}\mathrm{Ni}\right)}=\frac{E_\gamma^2}{2m\left({}^{58}\mathrm{Ni}\right)\cdot c^2} \tag{38}$$

Inserting this into law of energy conservation Eq. (36), we get:

$$E_x\left({}^{58}\mathrm{Ni}\right)=E_\gamma+E_{\mathrm{recoil}}=E_\gamma+\frac{E_\gamma^2}{2m\left({}^{58}\mathrm{Ni}\right)\cdot c^2} \tag{39}$$

This reduces to the quadratic equation:

$$E_\gamma^2 + 2m\left({}^{58}\mathrm{Ni}\right)c^2 \cdot E_\gamma + 2m\left({}^{58}\mathrm{Ni}\right)c^2 E_x\left({}^{58}\mathrm{Ni}\right) = 0 \tag{40}$$

which gives the following solution:

$$E_\gamma = \frac{-2m\left({}^{58}\mathrm{Ni}\right)c^2 + \sqrt{4\left(m\left({}^{58}\mathrm{Ni}\right)c^2\right)^2 + 8m\left({}^{58}\mathrm{Ni}\right)c^2 E_x\left({}^{58}\mathrm{Ni}\right)}}{2} = \\ = \sqrt{\left(m\left({}^{58}\mathrm{Ni}\right)c^2\right)^2 + 2m\left({}^{58}\mathrm{Ni}\right)c^2 E_x\left({}^{58}\mathrm{Ni}\right)} - m\left({}^{58}\mathrm{Ni}\right)c^2 \tag{41}$$

Inserting numbers gives:

$$E_\gamma = 10.8633\,\mathrm{MeV} \tag{42}$$

The equation (37) can also be reduced to an approximate equation before inserting numbers:

$$E_\gamma = E_x\left(1 - \frac{E_x}{2m\left({}^{58}\mathrm{Ni}\right)c^2}\right) = 10.8633\,\mathrm{MeV} \tag{43}$$

The recoil energy is now easily found as:

$$E_{\mathrm{recoil}} = E_x\left({}^{58}\mathrm{Ni}\right) - E_\gamma = 1.1\,\mathrm{keV} \tag{44}$$

Due to the fact that nucleus emitting gamma-ray (^{58}Ni) is moving with the high velocity, the energy of gamma ray will be changed because of the Doppler effect. The relativistic Doppler effect (when source is moving towards observer/detector) is given with this formula:

$$f_{\mathrm{detector}} = f_{\gamma,\mathrm{emitted}}\sqrt{\frac{1+\beta}{1-\beta}} \tag{45}$$

and since there is a simple relation between photon energy and frequency *(E=hf)*, we get the similar expression for energy:

$$E_{\mathrm{detector}} = E_{\gamma,\mathrm{emitted}}\sqrt{\frac{1+\beta}{1-\beta}} \tag{46}$$

where $\beta = v/c$ and v is the velocity of emitter (the ^{58}Ni nucleus). Taking the calculated value of the ^{58}Ni velocity (equation 29) we get:

$$E_{\mathrm{detector}} = E_{\gamma,\mathrm{emitted}}\sqrt{\frac{1+\beta}{1-\beta}} = 10.863\sqrt{\frac{1+0.00565}{1-0.00565}} = 10.925\,\mathrm{MeV} \tag{47}$$

3. Simple model of an atomic nucleus

Introduction

Although atomic nuclei are quantum objects, a number of phenomenological laws for their basic properties (like radius or binding energy) can be deduced from simple assumptions: (i) nuclei are built from nucleons (i.e. protons and neutrons); (ii) strong nuclear interaction holding these nucleons together has a very short range (it acts only between neighboring nucleons); (iii) the number of protons (Z) in a given nucleus is approximately equal to the number of neutrons (N), i.e. $Z \approx N \approx A/2$, where A is the total number of nucleons ($A \gg 1$). **Important: Use these assumptions in Tasks 1-4 below.**

Task 1 - Atomic nucleus as closely packed system of nucleons

In a simple model, an atomic nucleus can be thought of as a ball consisting of closely packed nucleons [see Fig. 1(a)], where the nucleons are hard balls of radius $r_N = 0.85$ fm (1 fm = 10^{-15} m). The nuclear force is present only for two nucleons in contact. The volume of the nucleus V is larger than the volume of all nucleons AV_N, where $V_N = \frac{4}{3} r_N^3 \pi$. The ratio $f = AV_N / V$ is called the packing factor and gives the percentage of space filled by the nuclear matter.

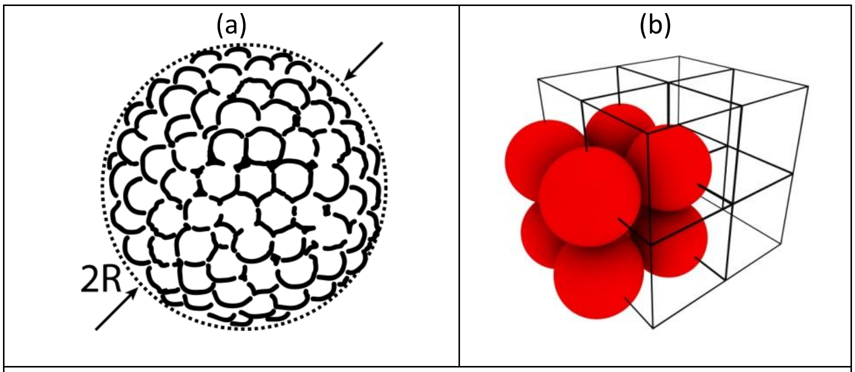


Fig. 1. (a) An atomic nucleus as a ball of closely packed nucleons. (b) The SC packing.

a) Calculate what would be the packing factor f if nucleons were arranged in a "simple cubic" (SC) crystal system, where each nucleon is centered on a lattice point of an infinite cubic lattice [see Fig. 1(b)]. *(0.3 points)*

Important: In all subsequent tasks, assume that the actual packing factor for nuclei is equal to the one from Task 1a. If you are not able to calculate it, in subsequent tasks use $f = 1/2$.

b) Estimate the average mass density ρ_m, charge density ρ_c, and the radius R for a nucleus having A nucleons. The average mass of a nucleon is $1.67 \cdot 10^{-27}$ kg. *(1.0 points)*

Task 2 - Binding energy of atomic nuclei – volume and surface terms

Binding energy of a nucleus is the energy required to disassemble it into separate nucleons and it essentially comes from the attractive nuclear force of each nucleon with its neighbors. If a given nucleon is not on the surface of the nucleus, it contributes to the total binding energy with a_V= 15.8 MeV (1 MeV = $1.602 \cdot 10^{-13}$ J). The contribution of one surface nucleon to the binding energy is approximately $a_V/2$. Express the binding energy E_b of a nucleus with A nucleons in terms of A, a_V, and f, and by including the surface correction. *(1.9 points)*

Task 3 - Electrostatic (Coulomb) effects on the binding energy

The electrostatic energy of a homogeneously charged ball (with radius R and total charge Q_0) is $U_c = \frac{3Q_0^2}{20\pi\varepsilon_0 R}$, where $\varepsilon_0 = 8.85 \cdot 10^{-12}\ \mathrm{C}^2 N^{-1} m^{-2}$.

a) Apply this formula to get the electrostatic energy of a nucleus. In a nucleus, each proton is not acting upon itself (by Coulomb force), but only upon the rest of the protons. One can take this into account by replacing $Z^2 \to Z(Z-1)$ in the obtained formula. Use this correction in subsequent tasks. *(0.4 points)*

b) Write down the complete formula for binding energy, including the main (volume) term, the surface correction term and the obtained electrostatic correction. *(0.3 points)*

Task 4 - Fission of heavy nuclei

Fission is a nuclear process in which a nucleus splits into smaller parts (lighter nuclei). Suppose that a nucleus with A nucleons splits into only two equal parts as depicted in Fig. 2.

a) Calculate the total kinetic energy of the fission products E_{kin} when the centers of two lighter nuclei are separated by the distance $d \geq 2R(A/2)$, where $R(A/2)$ is their radius. The large nucleus was initially at rest. *(1.3 points)*

b) Assume that $d = 2R(A/2)$ and evaluate the expression for E_{kin} obtained in part a) for A= 100, 150, 200 and 250 (express the results in units of MeV). Estimate the values of A for which fission is possible in the model described above? *(1.0 points)*

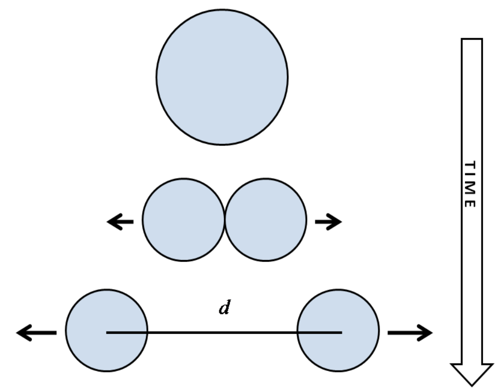


Fig. 2. A schematic description of nuclear fission in our model.

Task 5 – Transfer reactions

a) In modern physics, the energetics of nuclei and their reactions is described in terms of masses. For example, if a nucleus (with zero velocity) is in an excited state with energy E_{exc} above the ground state, its mass is $m = m_0 + E_{exc} / c^2$, where m_0 is its mass in the ground state at rest. The nuclear reaction ^{16}O+^{54}Fe→^{12}C+^{58}Ni is an example of the so-called "transfer reactions", in which a part of one nucleus ("cluster") is transferred to the other (see Fig. 3). In our example the transferred part is a ^{4}He-cluster (α -particle). The transfer reactions occur with maximum probability if the velocity of the projectile-like reaction product (in our case: ^{12}C) is equal both in magnitude and direction to the velocity of projectile (in our case: ^{16}O). The target ^{54}Fe is initially at rest. In the reaction, ^{58}Ni is excited into one of its higher-lying states. Find the excitation energy of that state (and express it units of MeV) if the kinetic energy of the projectile ^{16}O is 50 MeV. The speed of light is c= $3·10^8$ m/s. *(2.2 points)*

1.	M(^{16}O)	15.99491 a.m.u.
2.	M(^{54}Fe)	53.93962 a.m.u.
3.	M(^{12}C)	12.00000 a.m.u.
4.	M(^{58}Ni)	57.93535 a.m.u.

Table 1. The rest masses of the reactants in their ground states. 1 a.m.u.= $1.6605·10^{-27}$ kg.

b) The ^{58}Ni nucleus produced in the excited state discussed in the part a), deexcites into its ground state by emitting a gamma-photon in the direction of its motion. Consider this decay in the frame of reference in which ^{58}Ni is at rest to find the recoil energy of ^{58}Ni (*i.e.* kinetic energy which ^{58}Ni acquires after the emission of the photon). What is the photon energy in that system? What is the photon energy in the lab system of reference (i.e. what would be the energy of the photon measured in the detector which is positioned in the direction in which the ^{58}Ni nucleus moves)? *(1.6 points)*

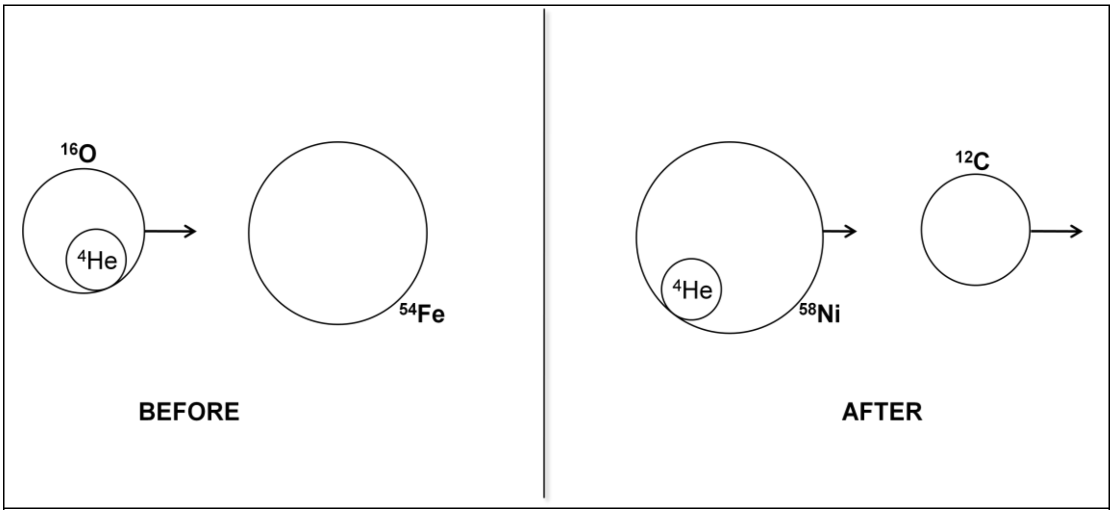


Fig. 3. The schematics of a transfer reaction.

1. A Three-body Problem and LISA

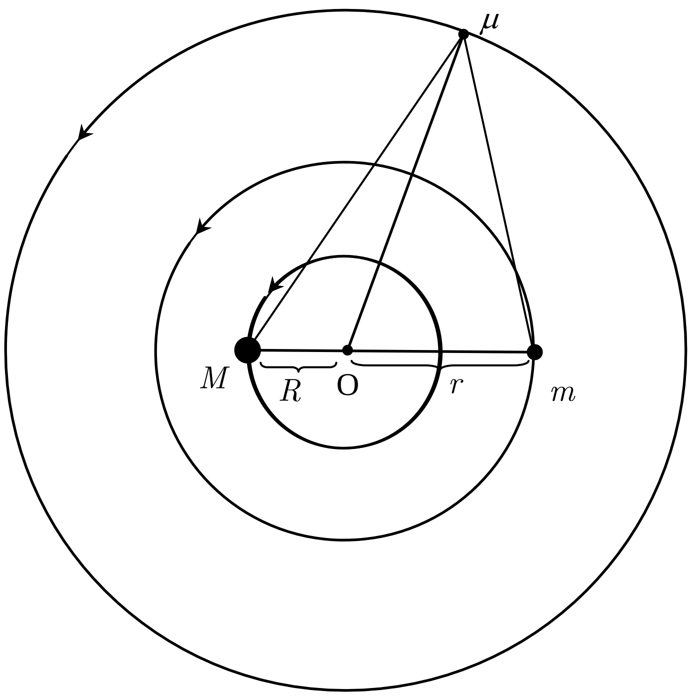


FIGURE 1 Coplanar orbits of three bodies.

1.1 Two gravitating masses M and m are moving in circular orbits of radii R and r, respectively, about their common centre of mass. Find the angular velocity ω_0 of the line joining M and m in terms of R, r, M, m and the universal gravitational constant G.

[1.5 points]

1.2 A third body of infinitesimal mass μ is placed in a coplanar circular orbit about the same centre of mass so that μ remains stationary relative to both M and m as shown in Figure 1. Assume that the infinitesimal mass is not collinear with M and m. Find the values of the following parameters in terms of R and r:

[3.5 points]

1.2.1 distance from μ to M.
1.2.2 distance from μ to m.
1.2.3 distance from μ to the centre of mass.

1.3 Consider the case $M = m$. If μ is now given a small radial perturbation (along Oμ), what is the angular frequency of oscillation of μ about the unperturbed position in terms of ω_0? Assume that the angular momentum of μ is conserved. **[3.2 points]**

The Laser Interferometry Space Antenna (LISA) is a group of three identical spacecrafts for detecting low frequency gravitational waves. Each of the spacecrafts is placed at the corners of an equilateral triangle as shown in Figure 2 and Figure 3. The sides (or 'arms') are about 5.0 million kilometres long. The LISA constellation is in an earth-like orbit around the Sun trailing the Earth by 20°. Each of them moves on a slightly inclined individual orbit around the Sun. Effectively, the three spacecrafts appear to roll about their common centre one revolution per year.

They are continuously transmitting and receiving laser signals between each other. Overall, they detect the gravitational waves by measuring tiny changes in the arm lengths using interferometric means. A collision of massive objects, such as blackholes, in nearby galaxies is an example of the sources of gravitational waves.

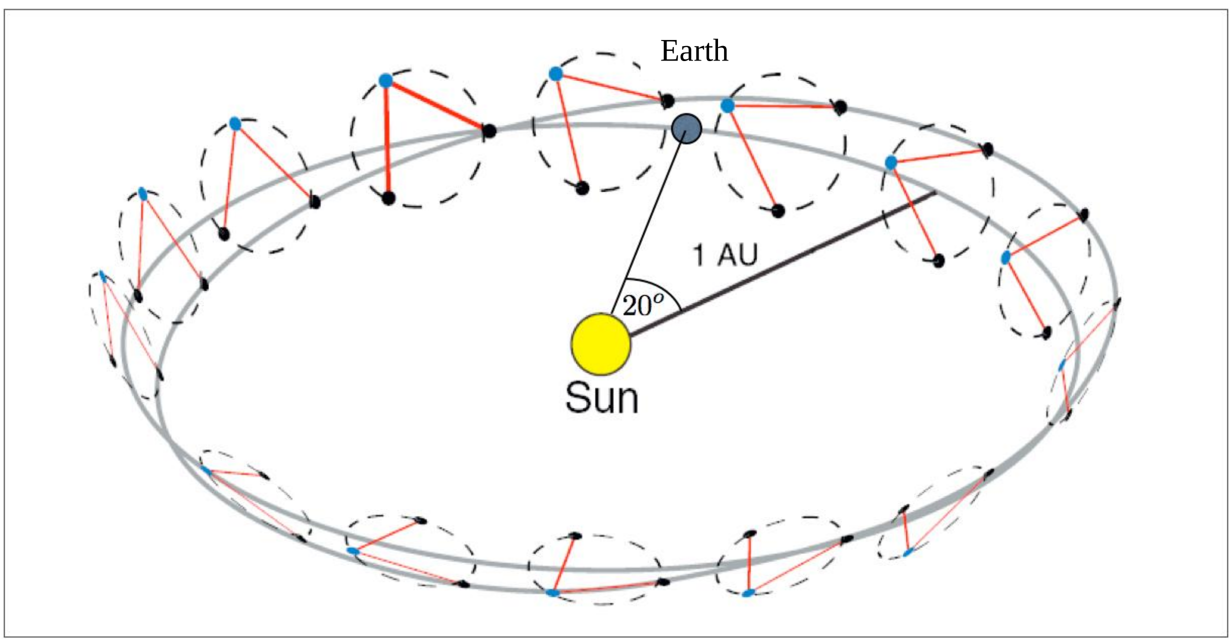


FIGURE 2 Illustration of the LISA orbit. The three spacecraft roll about their centre of mass with a period of 1 year. Initially, they trail the Earth by 20°. (Picture from D.A. Shaddock, "An Overview of the Laser Interferometer Space Antenna", *Publications of the Astronomical Society of Australia*, 2009, **26**, pp.128-132.).

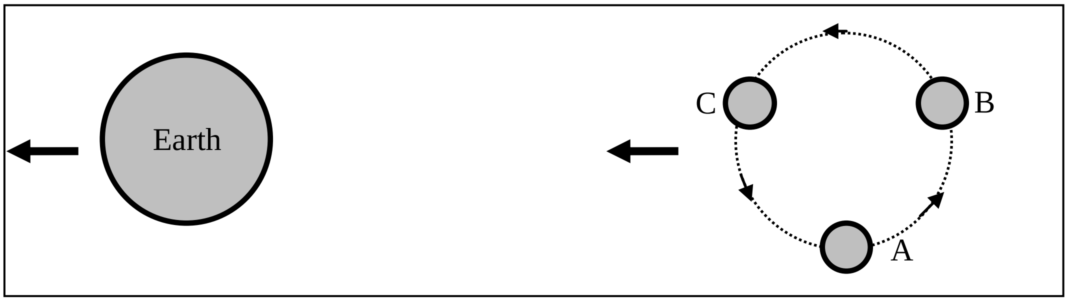


FIGURE 3 Enlarged view of the three spacecrafts trailing the Earth. A, B and C are the three spacecrafts at the corners of the equilateral triangle.

1.4 In the plane containing the three spacecrafts, what is the relative speed of one spacecraft with respect to another? **[1.8 point]**

2. An Electrified Soap Bubble

A spherical soap bubble with internal air density ρ_i, temperature T_i and radius R_0 is surrounded by air with density ρ_a, atmospheric pressure P_a and temperature T_a. The soap film has surface tension γ, density ρ_s and thickness t. The mass and the surface tension of the soap do not change with the temperature. Assume that $R_0 \gg t$.

The increase in energy, dE, that is needed to increase the surface area of a soap-air interface by dA, is given by $dE = \gamma dA$ where γ is the surface tension of the film.

2.1 Find the ratio $\dfrac{\rho_i T_i}{\rho_a T_a}$ in terms of γ, P_a and R_0. **[1.7 point]**

2.2 Find the numerical value of $\dfrac{\rho_i T_i}{\rho_a T_a} - 1$ using $\gamma = 0.0250\,\mathrm{Nm^{-1}}$, $R_0 = 1.00\ \mathrm{cm}$, and $P_a = 1.013\times 10^5\ \mathrm{Nm^{-2}}$. **[0.4 point]**

2.3 The bubble is initially formed with warmer air inside. Find the minimum numerical value of T_i such that the bubble can float in still air. Use $T_a = 300\ \mathrm{K}$, $\rho_s = 1000\ \mathrm{kgm^{-3}}$, $\rho_a = 1.30\ \mathrm{kgm^{-3}}$, $t = 100\ \mathrm{nm}$ and $g = 9.80\ \mathrm{ms^{-2}}$. **[2.0 points]**

After the bubble is formed for a while, it will be in thermal equilibrium with the surrounding. This bubble in still air will naturally fall towards the ground.

2.4 Find the minimum velocity u of an updraught (air flowing upwards) that will keep the bubble from falling at thermal equilibrium. Give your answer in terms of ρ_s, R_0, g, t and the air's coefficient of viscosity η. You may assume that the velocity is small such that Stokes's law applies, and ignore the change in the radius when the temperature lowers to the equilibrium. The drag force from Stokes' Law is $F = 6\pi\eta R_0 u$.

[1.6points]

2.5 Calculate the numerical value for u using $\eta = 1.8\times 10^{-5}\ \mathrm{kgm^{-1}\,s^{-1}}$. **[0.4 point]**

The above calculations suggest that the terms involving the surface tension γ add very little to the accuracy of the result. In all of the questions below, you can neglect the surface tension terms.

2.6 If this spherical bubble is now electrified uniformly with a total charge q, find an equation describing the new radius R_1 in terms of R_0, P_a, q and the permittivity of free space ε_0.

[2.0points]

2.7 Assume that the total charge is not too large (i.e. $\frac{q^2}{\varepsilon_0 R_0^4} << P_a$) and the bubble only experiences a small increase in its radius, find ΔR where $R_1 = R_0 + \Delta R$.

Given that $(1+x)^n \approx 1 + nx$ where $x \ll 1$. **[0.7 point]**

2.8 What must be the magnitude of this charge q in terms of $t, \rho_a, \rho_s, \varepsilon_0, R_0, P_a$ in order that the bubble will float motionlessly in still air? Calculate also the numerical value of q. The permittivity of free space $\varepsilon_0 = 8.85 \times 10^{-12}$ farad/m. **[1.2 point]**

3. To Commemorate the Centenary of Rutherford's Atomic Nucleus: the Scattering of an Ion by a Neutral Atom

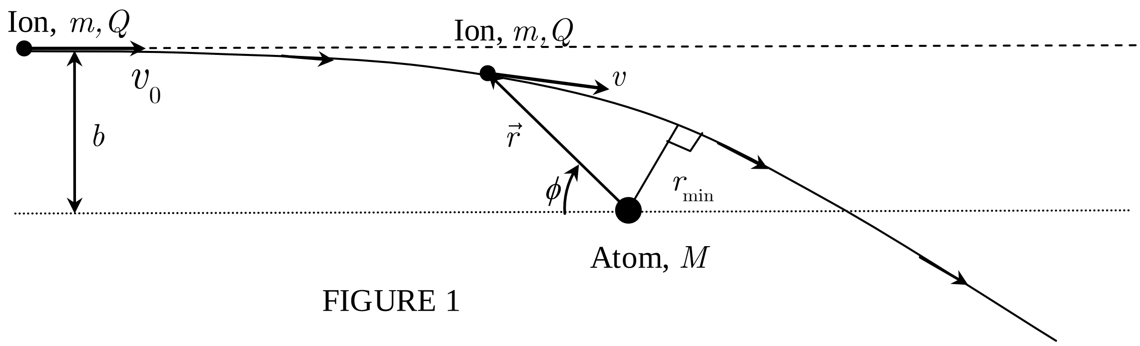


FIGURE 1

An ion of mass m, charge Q, is moving with an initial non-relativistic speed v_0 from a great distance towards the vicinity of a neutral atom of mass $M >> m$ and of electrical polarisability α. The impact parameter is b as shown in Figure 1.

The atom is instantaneously polarised by the electric field $\vec{E}$ of the in-coming (approaching) ion. The resulting electric dipole moment of the atom is $\vec{p} = \alpha \vec{E}$. Ignore any radiative losses in this problem.

3.1 Calculate the electric field intensity $\vec{E}_p$ at a distance r from an ideal electric dipole $\vec{p}$ at the origin O along the direction of $\vec{p}$ in Figure 2. **[1.2 points]**

$p = 2aq, \quad r \gg a$

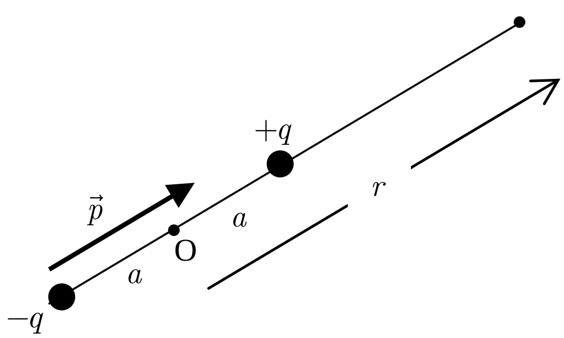


FIGURE 2

3.2 Find the expression for the force $\vec{f}$ acting on the ion due to the polarised atom. Show that this force is attractive regardless of the sign of the charge of the ion.

[3.0 points]

3.3 What is the electric potential energy of the ion-atom interaction in terms of α, Q and r?

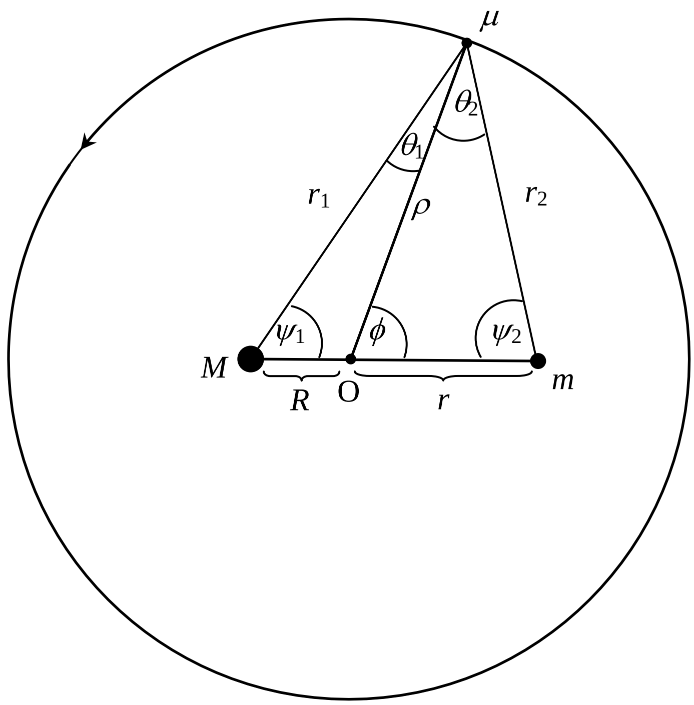
[0.9 points]

3.4 Find the expression for $r_{\min}$, the distance of the closest approach, as shown in Figure 1.

[2.4 points]

3.5 If the impact parameter b is less than a critical value b_0, the ion will descend along a spiral to the atom. In such a case, the ion will be neutralized, and the atom is, in turn, charged. This process is known as the "charge exchange" interaction. What is the cross sectional area $A = \pi b_0^2$ of this "charge exchange" collision of the atom as seen by the ion? **[2.5 points]**

I. Solution



1.1 Let O be their centre of mass. Hence

$$MR - mr = 0 \qquad \text{(1)}$$

$$m\omega_0^2 r = \frac{GMm}{(R+r)^2}$$

$$M\omega_0^2 R = \frac{GMm}{(R+r)^2} \qquad \text{(2)}$$

From Eq. (2), or using reduced mass, $\omega_0^2 = \frac{G(M+m)}{(R+r)^3}$

Hence, $\omega_0^2 = \frac{G(M+m)}{(R+r)^3} = \frac{GM}{r(R+r)^2} = \frac{Gm}{R(R+r)^2}$. $\qquad$ (3)

1.2 Since μ is infinitesimal, it has no gravitational influences on the motion of neither M nor m. For μ to remain stationary relative to both M and m we must have:

$$\frac{GM\mu}{r_1^2}\cos\theta_1 + \frac{Gm\mu}{r_2^2}\cos\theta_2 = \mu\omega_0^2\rho = \frac{G(M+m)\mu}{(R+r)^3}\rho \qquad \text{.........................} \quad (4)$$

$$\frac{GM\mu}{r_1^2}\sin\theta_1 = \frac{Gm\mu}{r_2^2}\sin\theta_2 \qquad \text{.........................} \quad (5)$$

Substituting $\frac{GM}{r_1^2}$ from Eq. (5) into Eq. (4), and using the identity

$\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2 = \sin(\theta_1+\theta_2)$, we get

$$m\frac{\sin(\theta_1+\theta_2)}{r_2^2} = \frac{(M+m)}{(R+r)^3}\rho\sin\theta_1 \qquad \text{.........................} \quad (6)$$

The distances r_2 and ρ, the angles θ_1 and θ_2 are related by two Sine Rule equations

$$\frac{\sin\psi_1}{\rho} = \frac{\sin\theta_1}{R}$$
$$\frac{\sin\psi_1}{r_2} = \frac{\sin(\theta_1+\theta_2)}{R+r} \qquad \text{.........................} \quad (7)$$

Substitute (7) into (6)

$$\frac{1}{r_2^3} = \frac{R}{(R+r)^4}\frac{(M+m)}{m} \qquad \text{.........................} \quad (10)$$

Since $\frac{m}{M+m} = \frac{R}{R+r}$,Eq. (10) gives

$$r_2 = R+r \qquad \text{.........................} \quad (11)$$

By substituting $\frac{Gm}{r_2^2}$ from Eq. (5) into Eq. (4), and repeat a similar procedure, we get

$$r_1 = R+r \qquad \text{.........................} \quad (12)$$

Alternatively, $\frac{r_1}{\sin(180^\circ-\phi)} = \frac{R}{\sin\theta_1}$ and $\frac{r_2}{\sin\phi} = \frac{r}{\sin\theta_2}$

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{R}{r}\times\frac{r_2}{r_1} = \frac{m}{M}\times\frac{r_2}{r_1}$$

Combining with Eq. (5) gives $r_1 = r_2$

Hence, it is an equilateral triangle with

$$\psi_1 = 60^\circ$$
$$\psi_2 = 60^\circ \qquad \text{.......................... (13)}$$

The distance ρ is calculated from the Cosine Rule.

$$\rho^2 = r^2 + (R+r)^2 - 2r(R+r)\cos 60^\circ$$
$$\rho = \sqrt{r^2 + rR + R^2} \qquad \text{.......................... (14)}$$

Alternative Solution to 1.2

Since μ is infinitesimal, it has no gravitational influences on the motion of neither M nor m .For μ to remain stationary relative to both M and m we must have:

$$\frac{GM\mu}{r_1^2}\cos\theta_1 + \frac{Gm\mu}{r_2^2}\cos\theta_2 = \mu\omega^2\rho = \frac{G(M+m)\mu}{(R+r)^3}\rho \qquad \text{.......................... (4)}$$

$$\frac{GM\mu}{r_1^2}\sin\theta_1 = \frac{Gm\mu}{r_2^2}\sin\theta_2 \qquad \text{.......................... (5)}$$

Note that

$$\frac{r_1}{\sin(180^\circ - \phi)} = \frac{R}{\sin\theta_1}$$

$$\frac{r_2}{\sin\phi} = \frac{r}{\sin\theta_2} \qquad \text{(see figure)}$$

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{R}{r}\times\frac{r_2}{r_1} = \frac{m}{M}\times\frac{r_2}{r_1} \qquad \text{.......................... (6)}$$

Equations (5) and (6):

$$r_1 = r_2 \qquad \text{.......................... (7)}$$

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{m}{M} \qquad \text{.......................... (8)}$$

$$\psi_1 = \psi_2 \qquad \text{.......................... (9)}$$

The equation (4) then becomes:

$$M\cos\theta_1 + m\cos\theta_2 = \frac{(M+m)}{(R+r)^3}r_1^2\rho \qquad \text{.......................... (10)}$$

Equations (8) and (10): $\sin(\theta_1+\theta_2) = \frac{M+m}{M}\frac{r_1^2\rho}{(R+r)^3}\sin\theta_2$ .... (11)

Note that from figure, $\frac{\rho}{\sin\psi_2} = \frac{r}{\sin\theta_2}$ .... (12)

Equations (11) and (12): $\sin(\theta_1+\theta_2) = \dfrac{M+m}{M}\dfrac{r_1^2 r}{(R+r)^3}\sin\psi_2$ ………………………… (13)

Also from figure,

$$(R+r)^2 = r_2^2 - 2r_1r_2\cos(\theta_1+\theta_2) + r_1^2 = 2r_1^2\left[1-\cos(\theta_1+\theta_2)\right] \quad \text{………………} \quad (14)$$

Equations (13) and (14): $\sin(\theta_1+\theta_2) = \dfrac{\sin\psi_2}{2\left[1-\cos(\theta_1+\theta_2)\right]}$ ………………………… (15)

$$\theta_1+\theta_2 = 180^\circ - \psi_1 - \psi_2 = 180^\circ - 2\psi_2 \quad \text{(see figure)}$$

$$\therefore \quad \cos\psi_2 = \frac{1}{2},\ \psi_2 = 60^\circ,\ \psi_1 = 60^\circ$$

Hence M and m from an equilateral triangle of sides $(R+r)$

Distance μ to M is $R+r$

Distance μ to m is $R+r$

Distance μ to O is $\rho = \sqrt{\left(\dfrac{R+r}{2}-R\right)^2 + \left\{(R+r)\dfrac{\sqrt{3}}{2}\right\}^2} = \sqrt{R^2+Rr+r^2}$

1.3 The energy of the mass μ is given by

$$E = -\frac{GM\mu}{r_1} - \frac{Gm\mu}{r_2} + \tfrac{1}{2}\mu\left(\left(\frac{d\rho}{dt}\right)^2 + \rho^2\omega^2\right) \quad \text{…………………………..(15)}$$

Since the perturbation is in the radial direction, angular momentum is conserved ($r_1 = r_2 = \Re$ and $m = M$),

$$E = -\frac{2GM\mu}{\Re} + \tfrac{1}{2}\mu\left(\left(\frac{d\rho}{dt}\right)^2 + \frac{\rho_0^4\omega_0^2}{\rho^2}\right) \quad \text{…………………………..(16)}$$

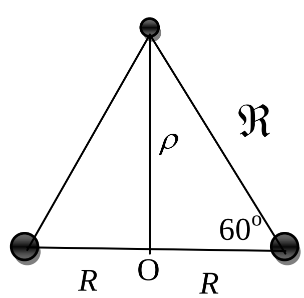
Since the energy is conserved,

$$\frac{dE}{dt} = 0$$

$$\frac{dE}{dt} = \frac{2GM\mu}{\Re^2}\frac{d\Re}{dt} + \mu\frac{d\rho}{dt}\frac{d^2\rho}{dt^2} - \mu\frac{\rho_0^4\omega_0^2}{\rho^3}\frac{d\rho}{dt} = 0 \quad \text{……………(17)}$$

$$\frac{d\Re}{dt} = \frac{d\Re}{d\rho}\frac{d\rho}{dt} = \frac{d\rho}{dt}\frac{\rho}{\Re} \quad \text{……………(18)}$$

$$\frac{dE}{dt} = \frac{2GM\mu}{\Re^3}\rho\frac{d\rho}{dt} + \mu\frac{d\rho}{dt}\frac{d^2\rho}{dt^2} - \mu\frac{\rho_0^4\omega_0^2}{\rho^3}\frac{d\rho}{dt} = 0 \quad \text{……………(19)}$$



Since $\frac{d\rho}{dt} \neq 0$, we have

$$\frac{2GM}{\Re^3}\rho + \frac{d^2\rho}{dt^2} - \frac{\rho_0{}^4\omega_0{}^2}{\rho^3} = 0 \text{ or}$$

$$\frac{d^2\rho}{dt^2} = -\frac{2GM}{\Re^3}\rho + \frac{\rho_0{}^4\omega_0{}^2}{\rho^3}. \qquad \text{..............................(20)}$$

The perturbation from $\Re_0$ and ρ_0 gives $\Re = \Re_0\left(1+\frac{\Delta\Re}{\Re_0}\right)$ and $\rho = \rho_0\left(1+\frac{\Delta\rho}{\rho_0}\right)$.

Then

$$\frac{d^2\rho}{dt^2} = \frac{d^2}{dt^2}(\rho_0 + \Delta\rho) = -\frac{2GM}{\Re_0^3\left(1+\frac{\Delta\Re}{\Re_0}\right)^3}\rho_0\left(1+\frac{\Delta\rho}{\rho_0}\right) + \frac{\rho_0{}^4\omega_0{}^2}{\rho_0^3\left(1+\frac{\Delta\rho}{\rho_0}\right)^3} \qquad \text{..................(21)}$$

Using binomial expansion $(1+\varepsilon)^n \approx 1 + n\varepsilon$,

$$\frac{d^2\Delta\rho}{dt^2} = -\frac{2GM}{\Re_0^3}\rho_0\left(1+\frac{\Delta\rho}{\rho_0}\right)\left(1-\frac{3\Delta\Re}{\Re_0}\right) + \rho_0\omega_0^2\left(1-\frac{3\Delta\rho}{\rho_0}\right). \qquad \text{..................(22)}$$

Using $\Delta\rho = \frac{\Re}{\rho}\Delta\Re$,

$$\frac{d^2\Delta\rho}{dt^2} = -\frac{2GM}{\Re_0^3}\rho_0\left(1+\frac{\Delta\rho}{\rho_0} - \frac{3\rho_0\Delta\rho}{\Re_0^2}\right) + \rho_0\omega_0^2\left(1-\frac{3\Delta\rho}{\rho_0}\right). \qquad \text{..................(23)}$$

Since $\omega_0^2 = \frac{2GM}{\Re_0^3}$,

$$\frac{d^2\Delta\rho}{dt^2} = -\omega_0^2\rho_0\left(1+\frac{\Delta\rho}{\rho_0} - \frac{3\rho_0\Delta\rho}{\Re_0^2}\right) + \omega_0^2\rho_0\left(1-\frac{3\Delta\rho}{\rho_0}\right) \qquad \text{..................(24)}$$

$$\frac{d^2\Delta\rho}{dt^2} = -\omega_0^2\rho_0\left(\frac{4\Delta\rho}{\rho_0} - \frac{3\rho_0\Delta\rho}{\Re_0^2}\right) \qquad \text{..................(25)}$$

$$\frac{d^2\Delta\rho}{dt^2} = -\omega_0^2\Delta\rho\left(4 - \frac{3\rho_0^2}{\Re_0^2}\right) \qquad \text{..................(26)}$$

From the figure, $\rho_0 = \Re_0 \cos 30°$ or $\frac{\rho_0{}^2}{\Re_0{}^2} = \frac{3}{4}$,

$$\frac{d^2\Delta\rho}{dt^2} = -\omega_0^2\Delta\rho\left(4-\frac{9}{4}\right) = -\frac{7}{4}\omega_0^2\Delta\rho. \qquad \text{..................(27)}$$

Angular frequency of oscillation is $\frac{\sqrt{7}}{2}\omega_0$.

Alternative solution:

$M = m$ gives $R = r$ and $\omega_0^2 = \frac{G(M+M)}{(R+R)^3} = \frac{GM}{4R^3}$. The unperturbed radial distance of μ is $\sqrt{3}R$, so the perturbed radial distance can be represented by $\sqrt{3}R+\zeta$ where $\zeta << \sqrt{3}R$ as shown in the following figure.

Using Newton's 2nd law, $-\frac{2GM\mu}{\{R^2+(\sqrt{3}R+\zeta)^2\}^{3/2}}(\sqrt{3}R+\zeta) = \mu\frac{d^2}{dt^2}(\sqrt{3}R+\zeta) - \mu\omega^2(\sqrt{3}R+\zeta)$.

(1)

The conservation of angular momentum gives $\mu\omega_0(\sqrt{3}R)^2 = \mu\omega(\sqrt{3}R+\zeta)^2$.

(2)

Manipulate (1) and (2) algebraically, applying $\zeta^2 \approx 0$ and binomial approximation.

$$-\frac{2GM}{\{R^2+(\sqrt{3}R+\zeta)^2\}^{3/2}}(\sqrt{3}R+\zeta) = \frac{d^2\zeta}{dt^2} - \frac{\omega_0^2\sqrt{3}R}{(1+\zeta/\sqrt{3}R)^3}$$

$$-\frac{2GM}{\{4R^2+2\sqrt{3}\zeta R\}^{3/2}}(\sqrt{3}R+\zeta) \approx \frac{d^2\zeta}{dt^2} - \frac{\omega_0^2\sqrt{3}R}{(1+\zeta/\sqrt{3}R)^3}$$

$$-\frac{GM}{4R^3}\sqrt{3}R\frac{(1+\zeta/\sqrt{3}R)}{(1+\sqrt{3}\zeta/2R)^{3/2}} = \frac{d^2\zeta}{dt^2} - \frac{\omega_0^2\sqrt{3}R}{(1+\zeta/\sqrt{3}R)^3}$$

$$-\omega_0^2\sqrt{3}R\left(1-\frac{3\sqrt{3}\zeta}{4R}\right)\left(1+\frac{\zeta}{\sqrt{3}R}\right) \approx \frac{d^2\zeta}{dt^2} - \omega_0^2\sqrt{3}R\left(1-\frac{3\zeta}{\sqrt{3}R}\right)$$

$$\frac{d^2}{dt^2}\zeta = -\left(\frac{7}{4}\omega_0^2\right)\zeta$$

1.4 Relative velocity

Let v = speed of each spacecraft as it moves in circle around the centre O.
The relative velocities are denoted by the subscripts A, B and C.
For example, v_{BA} is the velocity of B as observed by A.

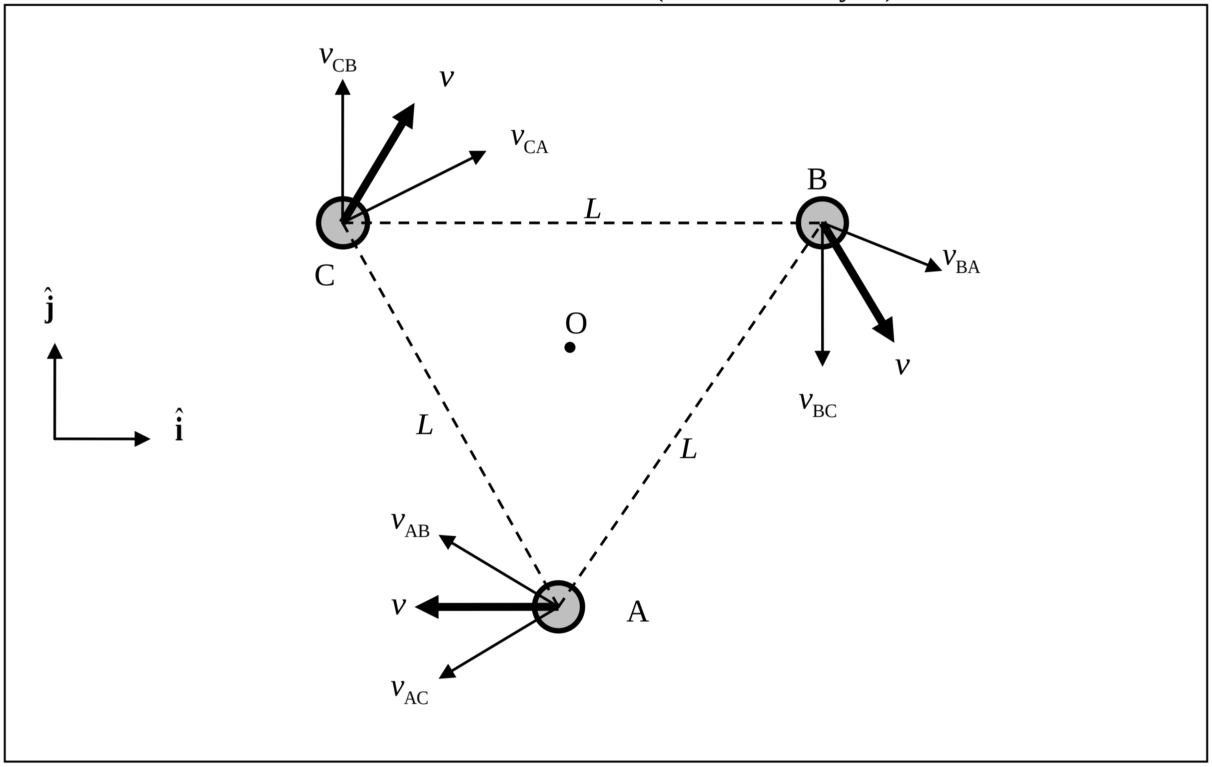
The period of circular motion is 1 year $T = 365\times24\times60\times60$ s. ………… (28)

The angular frequency $\omega = \frac{2\pi}{T}$

The speed $v = \omega\frac{L}{2\cos30°} = 575$ m/s ………… (29)

The speed is much less than the speed light → Galilean transformation.

In Cartesian coordinates, the velocities of B and C (as observed by O) are



For B, $\vec{v}_B = v\cos 60°\hat{\mathbf{i}} - v\sin 60°\hat{\mathbf{j}}$

For C, $\vec{v}_C = v\cos 60°\hat{\mathbf{i}} + v\sin 60°\hat{\mathbf{j}}$

Hence $\vec{v}_{\mathrm{BC}} = -2v\sin 60°\hat{\mathbf{j}} = -\sqrt{3}v\hat{\mathbf{j}}$

The speed of B as observed by C is $\sqrt{3}v \approx 996$ m/s ………… (30)

Notice that the relative velocities for each pair are anti-parallel.

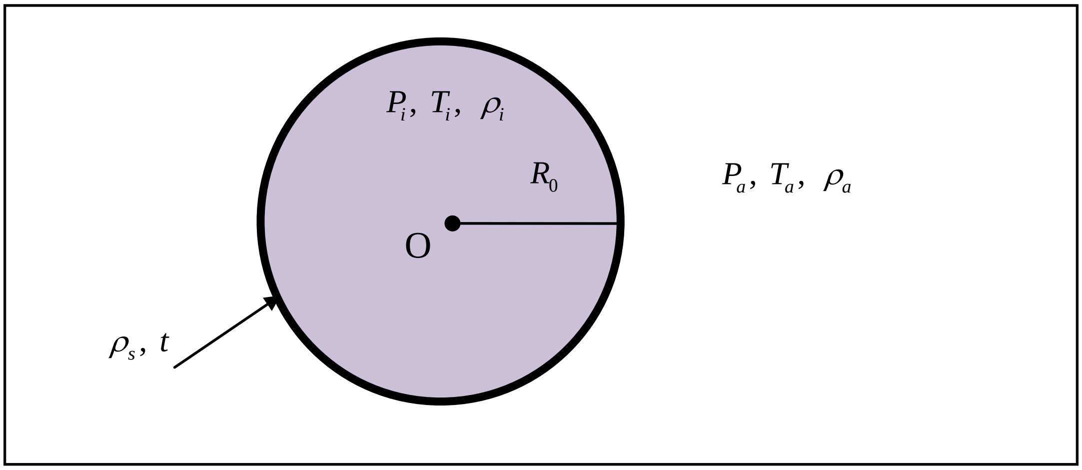
Alternative solution for 1.4

One can obtain v_{BC} by considering the rotation about the axis at one of the spacecrafts.

$$v_{\mathrm{BC}} = \omega L = \frac{2\pi}{365\times 24\times 60\times 60\ \mathrm{s}}(5\times 10^6\ \mathrm{km}) \approx 996\ \mathrm{m/s}$$

2. SOLUTION

2.1. The bubble is surrounded by air.



Cutting the sphere in half and using the projected area to balance the forces give

$$P_i\pi R_0^2 = P_a\pi R_0^2 + 2\left(2\pi R_0\gamma\right)$$
$$P_i = P_a + \frac{4\gamma}{R_0} \qquad \ldots (1)$$

The pressure and density are related by the ideal gas law:

$PV = nRT$ or $P = \dfrac{\rho RT}{M}$, where M = the molar mass of air. … (2)

Apply the ideal gas law to the air inside and outside the bubble, we get

$$\rho_i T_i = P_i\frac{M}{R}$$
$$\rho_a T_a = P_a\frac{M}{R},$$

$$\frac{\rho_i T_i}{\rho_a T_a} = \frac{P_i}{P_a} = \left[1 + \frac{4\gamma}{R_0 P_a}\right] \qquad \ldots (3)$$

2.2. Using $\gamma = 0.025\ \mathrm{Nm^{-1}}$, $R_0 = 1.0$ cm and $P_a = 1.013\times10^5\ \mathrm{Nm^{-2}}$, the numerical value of the ratio is

$$\frac{\rho_i T_i}{\rho_a T_a} = 1 + \frac{4\gamma}{R_0 P_a} = 1 + 0.0001 \qquad \ldots (4)$$

(The effect of the surface tension is very small.)

2.3. Let W = total weight of the bubble, F = buoyant force due to air around the bubble

$$\begin{aligned} W &= (\text{mass of film+mass of air})\,g \\ &= \left(4\pi R_0^2\,\rho_s t + \frac{4}{3}\pi R_0^3 \rho_i\right) g \qquad \ldots (5) \\ &= 4\pi R_0^2\,\rho_s t g + \frac{4}{3}\pi R_0^3 \frac{\rho_a T_a}{T_i}\left[1 + \frac{4\gamma}{R_0 P_a}\right] g \end{aligned}$$

The buoyant force due to air around the bubble is

$$B = \frac{4}{3}\pi R_0^3 \rho_a g \qquad \ldots (6)$$

If the bubble floats in still air,

$$B \geq W$$

$$\frac{4}{3}\pi R_0^3 \rho_a g \geq 4\pi R_0^2\,\rho_s t g + \frac{4}{3}\pi R_0^3 \frac{\rho_a T_a}{T_i}\left[1 + \frac{4\gamma}{R_0 P_a}\right] g \qquad \ldots (7)$$

Rearranging to give

$$\begin{aligned} T_i &\geq \frac{R_0 \rho_a T_a}{R_0 \rho_a - 3\rho_s t}\left[1 + \frac{4\gamma}{R_0 P_a}\right] \qquad \ldots (8) \\ &\geq 307.1\ \mathrm{K} \end{aligned}$$

The air inside must be about $7.1^\circ\mathrm{C}$ warmer.

2.4. Ignore the radius change → Radius remains $R_0 = 1.0\ \text{cm}$

(The radius actually decreases by 0.8% when the temperature decreases from 307.1 K to 300 K. The film itself also becomes slightly thicker.)

The drag force from Stokes' Law is $F = 6\pi\eta R_0 u$ … (9)

If the bubble floats in the updraught,

$$F \geq W - B$$

$$6\pi\eta R_0 u \geq \left(4\pi R_0^2\, \rho_s t + \frac{4}{3}\pi R_0^3 \rho_i\right) g - \frac{4}{3}\pi R_0^3 \rho_a g \qquad \ldots (10)$$

When the bubble is in thermal equilibrium $T_i = T_a$.

$$6\pi\eta R_0 u \geq \left(4\pi R_0^2\, \rho_s t + \frac{4}{3}\pi R_0^3 \rho_a \left[1 + \frac{4\gamma}{R_0 P_a}\right]\right) g - \frac{4}{3}\pi R_0^3 \rho_a g$$

Rearranging to give

$$u \geq \frac{4R_0\, \rho_s t g}{6\eta} + \frac{\frac{4}{3} R_0^2 \rho_a g \left(\frac{4\gamma}{R_0 P_a}\right)}{6\eta} \qquad \ldots (11)$$

2.5. The numerical value is $u \geq 0.36\ \text{m/s}$.

The 2nd term is about 3 orders of magnitude lower than the 1st term.

From now on, ignore the surface tension terms.

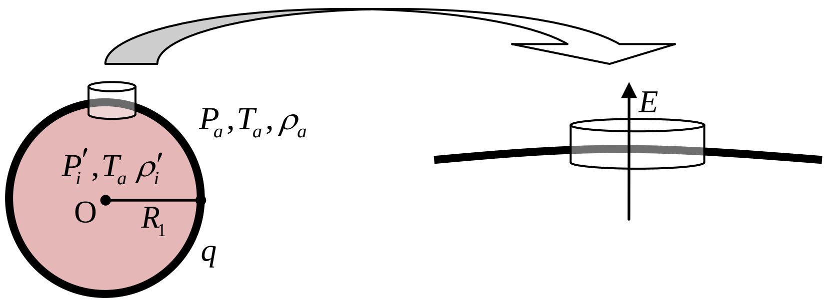
2.6. When the bubble is electrified, the electrical repulsion will cause the bubble to expand in size and thereby raise the buoyant force.

The force/area is (e-field on the surface × charge/area)

There are two alternatives to calculate the electric field ON the surface of the soap film.

A. From Gauss's Law

Consider a very thin pill box on the soap surface.



E = electric field on the film surface that results from all other parts of the soap film, excluding the surface inside the pill box itself.

E_q = total field just outside the pill box = $\dfrac{q}{4\pi\varepsilon_0 R_1^2} = \dfrac{\sigma}{\varepsilon_0}$

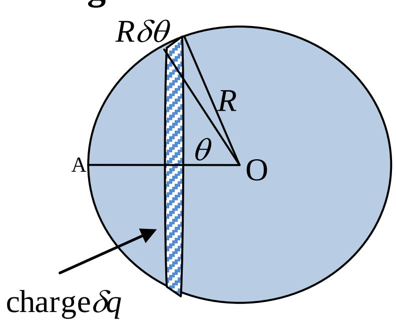
= E + electric field from surface charge σ

= $E + E_\sigma$

Using Gauss's Law on the pill box, we have $E_\sigma = \dfrac{\sigma}{2\varepsilon_0}$ perpendicular to the film as a result of symmetry.

Therefore, $E = E_q - E_\sigma = \dfrac{\sigma}{\varepsilon_0} - \dfrac{\sigma}{2\varepsilon_0} = \dfrac{\sigma}{2\varepsilon_0} = \dfrac{1}{2\varepsilon_0}\dfrac{q}{4\pi R_1^2}$ … (12)

B. From direct integration



$$\delta q = \left(\frac{q}{4\pi R^2}\right) 2\pi R \sin\theta . R\delta\theta$$

To find the magnitude of the electrical repulsion we must first find the electric field intensity E at a point on (not outside) the surface itself.

Field at A in the direction $\overrightarrow{\mathrm{OA}}$ is

$$\delta E_A = \frac{1}{4\pi\varepsilon_0}\frac{\left(q/4\pi R_1^2\right)2\pi R_1^2 \sin\theta\delta\theta}{\left(2R_1\sin\frac{\theta}{2}\right)^2}\sin\frac{\theta}{2} = \frac{\left(q/4\pi R_1^2\right)}{2\varepsilon_0}\cos\frac{\theta}{2}\delta\left(\frac{\theta}{2}\right)$$

$$E_A = \frac{\left(q/4\pi R_1^2\right)}{2\varepsilon_0}\int_{\theta=0}^{\theta=180^\circ}\cos\frac{\theta}{2}d\left(\frac{\theta}{2}\right) = \frac{\left(q/4\pi R_1^2\right)}{2\varepsilon_0} \quad \ldots (13)$$

The repulsive force per unit area of the surface of bubble is

$$\left(\frac{q}{4\pi R_1^2}\right)E = \frac{\left(q/4\pi R_1^2\right)^2}{2\varepsilon_0} \quad \ldots (14)$$

Let P_i' and ρ_i' be the new pressure and density when the bubble is electrified.

This electric repulsive force will augment the gaseous pressure P_i'.

P_i' is related to the original P_i through the gas law.

$$P_i'\frac{4}{3}\pi R_1^3 = P_i\frac{4}{3}\pi R_0^3$$

$$P_i' = \left(\frac{R_0}{R_1}\right)^3 P_i = \left(\frac{R_0}{R_1}\right)^3 P_a \quad \ldots (15)$$

<u>In the last equation, the surface tension term has been ignored.</u>

From balancing the forces on the half-sphere projected area, we have (again ignoring the surface tension term)

$$P_i' + \frac{\left(q/4\pi R_1^2\right)^2}{2\varepsilon_0} = P_a$$

$$P_a\left(\frac{R_0}{R_1}\right)^3 + \frac{\left(q/4\pi R_1^2\right)^2}{2\varepsilon_0} = P_a \quad \ldots (16)$$

Rearranging to get

$$\left(\frac{R_1}{R_0}\right)^4 - \left(\frac{R_1}{R_0}\right) - \frac{q^2}{32\pi^2\varepsilon_0 R_0^4 P_a} = 0 \quad \dots (17)$$

Note that (17) yields $\frac{R_1}{R_0} = 1$ when $q = 0$, as expected.

2.7. <u>Approximate solution</u> for R_1 when $\frac{q^2}{32\pi^2\varepsilon_0 R_0^4 P_a} << 1$

Write $R_1 = R_0 + \Delta R,\ \Delta R << R_0$

Therefore, $\frac{R_1}{R_0} = 1 + \frac{\Delta R}{R_0},\ \left(\frac{R_1}{R_0}\right)^4 \approx 1 + 4\frac{\Delta R}{R_0}$ … (18)

Eq. (17) gives:

$$\Delta R \approx \frac{q^2}{96\pi^2\varepsilon_0 R_0^3 P_a} \quad \dots (19)$$

$$R_1 \approx R_0 + \frac{q^2}{96\pi^2\varepsilon_0 R_0^3 P_a} \approx R_0\left(1 + \frac{q^2}{96\pi^2\varepsilon_0 R_0^4 P_a}\right) \quad \dots (20)$$

2.8. The bubble will float if

$$B \geq W$$

$$\frac{4}{3}\pi R_1^3 \rho_a g \geq 4\pi R_0^2\ \rho_s t g + \frac{4}{3}\pi R_0^3 \rho_i g \quad \dots (21)$$

Initially, $T_i = T_a \Rightarrow \rho_i = \rho_a$ for $\gamma \to 0$ and $R_1 = R_0\left(1 + \frac{\Delta R}{R_0}\right)$

$$\frac{4}{3}\pi R_0^3\left(1+\frac{\Delta R}{R_0}\right)^3\rho_a g \geq 4\pi R_0^2\,\rho_s t g+\frac{4}{3}\pi R_0^3\rho_a g$$

$$\frac{4}{3}\pi\left(3\Delta R\right)\rho_a g \geq 4\pi R_0^2\,\rho_s t g \qquad \ldots (22)$$

$$\frac{4}{3}\pi\frac{3q^2}{96\pi^2\varepsilon_0 R_0 P_a}\rho_a g \geq 4\pi R_0^2\,\rho_s t g$$

$$q^2 \geq \frac{96\pi^2 R_0^3\,\rho_s t\varepsilon_0 P_a}{\rho_a}$$

$$q \approx 256\times10^{-9}\ \mathrm{C} \approx 256\,\mathrm{nC}$$

Note that if the surface tension term is retained, we get

$$R_1 \approx \left(1+\frac{q^2/96\pi^2\varepsilon_0 R_0^4 P_a}{\left[1+\frac{2}{3}\left(\frac{4\gamma}{R_0 P_a}\right)\right]}\right)R_0$$

QUESTION 3: SOLUTION

1. Using Coulomb's Law, we write the electric field at a distance r is given by

$$E_p = \frac{q}{4\pi\varepsilon_0 (r-a)^2} - \frac{q}{4\pi\varepsilon_0 (r+a)^2}$$

$$E_p = \frac{q}{4\pi\varepsilon_0 r^2}\left(\frac{1}{\left(1-\frac{a}{r}\right)^2} - \frac{1}{\left(1+\frac{a}{r}\right)^2}\right) \qquad \text{..................(1)}$$

Using binomial expansion for small a,

$$E_p = \frac{q}{4\pi\varepsilon_0 r^2}\left(1+\frac{2a}{r}-1+\frac{2a}{r}\right)$$

$$= +\frac{4qa}{4\pi\varepsilon_0 r^3} = +\frac{qa}{\pi\varepsilon_0 r^3} \qquad \text{................(2)}$$

$$= \frac{2p}{4\pi\varepsilon_0 r^3}$$

2. The electric field seen by the atom from the ion is

$$\vec{E}_{ion} = -\frac{Q}{4\pi\varepsilon_0 r^2}\hat{r} \qquad \text{................ (3)}$$

The induced dipole moment is then simply

$$\vec{p} = \alpha \vec{E}_{ion} = -\frac{\alpha Q}{4\pi\varepsilon_0 r^2}\hat{r} \qquad \text{................ (4)}$$

From eq. (2)

$$\vec{E}_p = \frac{2p}{4\pi\varepsilon_0 r^3}\hat{r}$$

The electric field intensity $\vec{E}_p$ at the position of an ion at that instant is, using eq. (4),

$$\vec{E}_p = \frac{1}{4\pi\varepsilon_0 r^3}\left[-\frac{2\alpha Q}{4\pi\varepsilon_0 r^2}\hat{r}\right] = -\frac{\alpha Q}{8\pi^2\varepsilon_0^2 r^5}\hat{r}$$

The force acting on the ion is

$$\vec{f} = Q\vec{E}_p = -\frac{\alpha Q^2}{8\pi^2\varepsilon_0^2 r^5}\hat{r} \qquad \text{................ (5)}$$

The "-'' sign implies that this force is attractive and Q^2 implies that the force is attractive regardless of the sign of Q.

3. The potential energy of the ion-atom is given by $U = \int_r^\infty \vec{f}.d\vec{r}$ ……………………………………(6)

Using this, $U = \int_r^\infty \vec{f}.d\vec{r} = -\dfrac{\alpha Q^2}{32\pi^2\varepsilon_0^2 r^4}$ …………………………………………………………………(7)

[Remark: Students might use the term $-\vec{p}\cdot\vec{E}$ which changes only the factor in front.]

4. At the position $r_{\min}$ we have, according to the Principle of Conservation of Angular Momentum,

$$mv_{\max}r_{\min} = mv_0 b$$

$$v_{\max} = v_0\frac{b}{r_{\min}} \qquad \text{……………….. (8)}$$

And according to the Principle of Conservation of Energy:

$$\frac{1}{2}mv_{\max}^2 + \frac{-\alpha Q^2}{32\pi^2\varepsilon_0^2 r^4} = \frac{1}{2}mv_0^2 \qquad \text{……………….. (9)}$$

Eqs.(12) & (13):

$$\left(\frac{b}{r_{\min}}\right)^2 - \frac{\alpha Q^2 \Big/ \frac{1}{2}mv_0^2}{32\pi^2\varepsilon_0^2 b^4}\left(\frac{b}{r_{\min}}\right)^4 = 1$$

$$\left(\frac{r_{\min}}{b}\right)^4 - \left(\frac{r_{\min}}{b}\right)^2 + \frac{\alpha Q^2}{16\pi^2\varepsilon_0^2 mv_0^2 b^4} = 0 \qquad \text{……………….. (10)}$$

The roots of eq. (14) are:

$$r_{\min} = \frac{b}{\sqrt{2}}\left[1 \pm \sqrt{1 - \frac{\alpha Q^2}{4\pi^2\varepsilon_0^2 mv_0^2 b^4}}\right]^{\frac{1}{2}} \qquad \text{……………….. (11)}$$

[Note that the equation (14) implies that $r_{\min}$ cannot be zero, unless b is itself zero.]

Since the expression has to be valid at $Q = 0$, which gives

$$r_{\min} = \frac{b}{\sqrt{2}}[1\pm 1]^{\frac{1}{2}}$$

We have to choose "+" sign to make $r_{\min} = b$

Hence,

$$r_{\min} = \frac{b}{\sqrt{2}}\left[1 + \sqrt{1 - \frac{\alpha Q^2}{4\pi^2\varepsilon_0^2 mv_0^2 b^4}}\right]^{\frac{1}{2}} \qquad \text{…………………………………………..(12)}$$

5. A spiral trajectory occurs when (16) is imaginary (because there is no minimum distance of approach).

$r_{\min}$ is real under the condition:

$$1 \geq \frac{\alpha Q^2}{4\pi^2 \varepsilon_0^2 m v_0^2 b^4}$$

$$b \geq b_0 = \left(\frac{\alpha Q^2}{4\pi^2 \varepsilon_0^2 m v_0^2} \right)^{\frac{1}{4}} \qquad \text{................. (13)}$$

For $b < b_0 = \left(\frac{\alpha Q^2}{4\pi^2 \varepsilon_0^2 m v_0^2} \right)^{\frac{1}{4}}$ the ion will collide with the atom.

Hence the atom, as seen by the ion, has a cross-sectional area A,

$$A = \pi b_0^2 = \pi \left(\frac{\alpha Q^2}{4\pi^2 \varepsilon_0^2 m v_0^2} \right)^{\frac{1}{2}} \qquad \text{................. (14)}$$

The 43rd International Physics Olympiad — Theoretical Competition

Tartu, Estonia — Tuesday, July 17th 2012

- The examination lasts for 5 hours. There are 3 problems worth a total of 30 points. Please note that the point values of the three theoretical problems are not equal.

- **You must not open the envelope with the problems before the sound signal indicating the beginning of competition (three short signals).**

- **You are not allowed to leave your working place without permission.** If you need any assistance (broken calculator, need to visit a restroom, etc), please raise the corresponding flag ("HELP" or "TOILET" with a long handle at your seat) above your seat box walls and keep it raised until an organizer arrives.

- **Your answers must be expressed in terms of those quantities, which are highlighted** in the problem text, and can contain also fundamental constants, if needed. So, if it is written that "the box height is a and the width - b" then a can be used in the answer, and b cannot be used (unless it is highlighted somewhere else, see below). Those quantities which are highlighted in the text of a subquestion can be used only in the answer to that subquestion; the quantities which are highlighted in the introductory text of the Problem (or a Part of a Problem), i.e. outside the scope of any subquestion, can be used for all the answers of that Problem (or of that Problem Part).

- Use only the front side of the sheets of paper.

- For each problem, there are **dedicated Solution Sheets** (see header for the number and pictogramme). Write your solutions onto the appropriate Solution Sheets. For each Problem, the Solution Sheets are numbered; use the sheets according to the enumeration. **Always mark which Problem Part and Question you are dealing with.** Copy the final answers into the appropriate boxes of the **Answer Sheets**. There are also **Draft** papers; use these for writing things which you don't want to be graded. If you have written something what you don't want to be graded onto the Solution Sheets (such as initial and incorrect solutions), cross these out.

- If you need more paper for a certain problem, please raise the flag "HELP" and tell an organizer the problem number; you are given two Solution sheets (you can do this more than once).

- **You should use as little text as possible:** try to explain your solution mainly with equations, numbers, symbols and diagrams.

- The first single sound signal tells you that there are 30 min of solving time left; the second double sound signal means that 5 min is left; the third triple sound signal marks the end of solving time. **After the third sound signal you must stop writing immediately.** Put all the papers into the envelope at your desk. **You are not allowed to take any sheet of paper out of the room.** If you have finished solving before the final sound signal, please raise your flag.

Problem T1. Focus on sketches *(13 points)*

Part A. Ballistics *(4.5 points)*

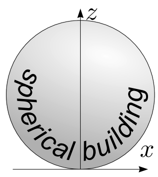
A ball, thrown with an initial speed v_0, moves in a homogeneous gravitational field in the x-z plane, where the x-axis is horizontal, and the z-axis is vertical and antiparallel to the free fall acceleration g. Neglect the effect of air drag.

i. (0.8 pts) By adjusting the launching angle for a ball thrown with a fixed initial speed v_0 from the origin, targets can be hit within the region given by

$$z \leq z_0 - kx^2.$$

You can use this fact without proving it. Find the constants z_0 and k.

ii. (1.2 pts) The launching point can now be freely selected on the ground level $z = 0$, and the launching angle can be adjusted as needed. The aim is to hit the topmost point of a spherical building of radius R (see fig.) with the minimal initial speed v_0. Bouncing off the roof prior to hitting the target is not allowed. Sketch qualitatively the shape of the optimal trajectory of the ball (use the designated box on the answer sheet). Note that the marks are given only for the sketch.



iii. (2.5 pts) What is the minimal launching speed $v_{\min}$ needed to hit the topmost point of a spherical building of radius R?

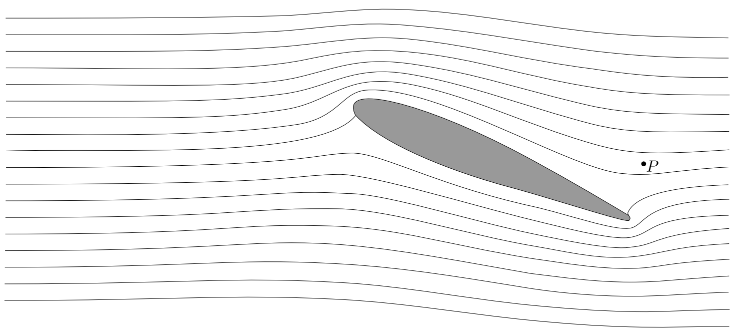
La Geode, Parc de la Villette, Paris. Photo: katchooo/flickr.com

Part B. Air flow around a wing *(4 points)*

For this part of the problem, the following information may be useful. For a flow of liquid or gas in a tube along a streamline, $p + \rho g h + \frac{1}{2}\rho v^2 = \text{const.}$, assuming that the velocity v is much less than the speed of sound. Here ρ is the density, h is the height, g is free fall acceleration and p is hydrostatic pressure. Streamlines are defined as the trajectories of fluid particles (assuming that the flow pattern is stationary). Note that the term $\frac{1}{2}\rho v^2$ is called the dynamic pressure.

In the fig. shown below, a cross-section of an aircraft wing is depicted together with streamlines of the air flow around the wing, as seen in the wing's reference frame. Assume that (a) the air flow is purely two-dimensional (i.e. that the velocity vectors of air lie in the plane of the figure); (b) the streamline pattern is independent of the aircraft speed; (c) there is no wind; (d) the dynamic pressure is much smaller than the atmospheric pressure, $p_0 = 1.0 \times 10^5\,\text{Pa}$.

You can use a ruler to take **measurements from the fig. on the answer sheet**.



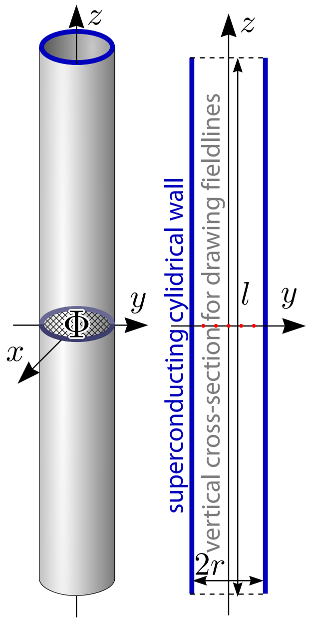
i. (0.8 pts) If the aircraft's ground speed is $v_0 = 100\,\text{m/s}$, what is the speed of the air, v_P, at the point P (marked in the fig.) with respect to the ground?

ii. (1.2 pts) In the case of high relative humidity, as the ground speed of the aircraft increases over a critical value v_{crit}, a stream of water droplets is created behind the wing. The droplets emerge at a certain point Q. Mark the point Q in the fig. on the answer sheet. Explain qualitatively (using formulae and as little text as possible) how you determined the position of Q.

iii. (2.0 pts) Estimate the critical speed v_{crit} using the following data: relative humidity of the air is $r = 90\%$, specific heat capacity of air at constant pressure $c_p = 1.00 \times 10^3\,\text{J/kg}\cdot\text{K}$, pressure of saturated water vapour: $p_{sa} = 2.31\,\text{kPa}$ at the temperature of the unperturbed air $T_a = 293\,\text{K}$ and $p_{sb} = 2.46\,\text{kPa}$ at $T_b = 294\,\text{K}$. Depending on your approximations, you may also need the specific heat capacity of air at constant volume $c_V = 0.717 \times 10^3\,\text{J/kg}\cdot\text{K}$. Note that the relative humidity is defined as the ratio of the vapour pressure to the saturated vapour pressure at the given temperature. Saturated vapour pressure is defined as the vapour pressure by which vapour is in equilibrium with the liquid.

Part C. Magnetic straws *(4.5 points)*

Consider a cylindrical tube made of a superconducting material. The length of the tube is l and the inner radius is r with $l \gg r$. The centre of the tube coincides with the origin, and its axis coincides with the z-axis.

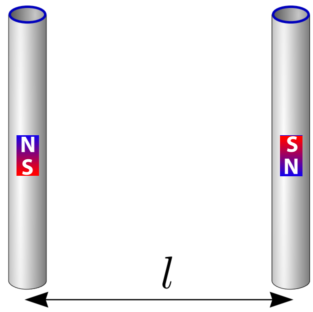


There is a magnetic flux Φ through the central cross-section of the tube, $z = 0$, $x^2 + y^2 < r^2$. A superconductor is a material which expels any magnetic field (the field is zero inside the material).

i. (0.8 pts) Sketch five such magnetic field lines, which pass through the five red dots marked on the axial cross-section of the tube, on the designated diagram on the answer sheet.

ii. (1.2 pts) Find the tension force T along the z-axis in the middle of the tube (i.e. the force by which two halves of the tube, $z > 0$ and $z < 0$, interact with each other).

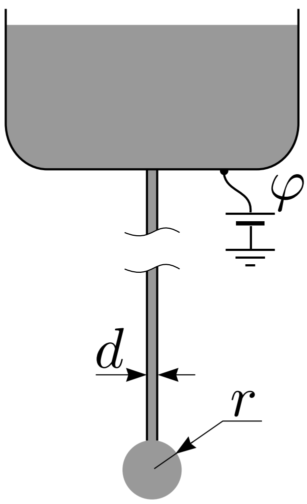
iii. (2.5 pts) Consider another tube, identical and parallel to the first one.



The second tube has the same magnetic field but in the opposite direction and its centre is placed at $y = l, x = z = 0$ (so that the tubes form opposite sides of a square). Determine the magnetic interaction force F between the two tubes.

Problem T2. Kelvin water dropper *(8 points)*

The following facts about the surface tension may turn out to be useful for this problem. For the molecules of a liquid, the positions at the liquid-air interface are less favourable as compared with the positions in the bulk of the liquid. This interface is described by the so-called surface energy, $U = \sigma S$, where S is the surface area of the interface and σ is the surface tension coefficient of the liquid. Moreover, two fragments of the liquid surface pull each other with a force $F = \sigma l$, where l is the length of a straight line separating the fragments.



A long metallic pipe with internal diameter d is pointing directly downwards. Water is slowly dripping from a nozzle at its lower end, see fig. Water can be considered to be electrically conducting; its surface tension is σ and its density is ρ. A droplet of radius r hangs below the nozzle. The radius grows slowly in time until the droplet separates from the nozzle due to the free fall acceleration g. Always assume that $d \ll r$.

Part A. Single pipe *(4 points)*

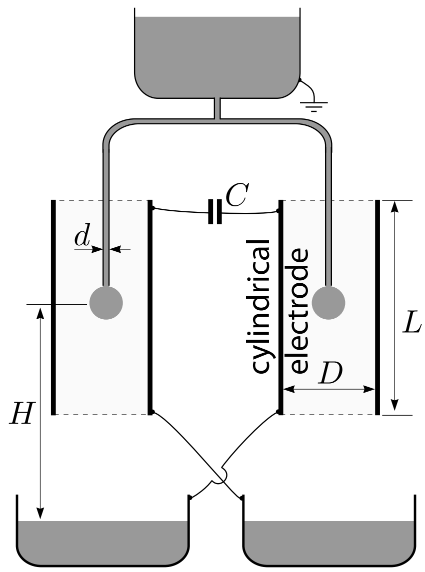
i. (1.2 pts) Find the radius $r_{\max}$ of a drop just before it separates from the nozzle.

ii. (1.2 pts) Relative to the far-away surroundings, the pipe's electrostatic potential is φ. Find the charge Q of a drop when its radius is r.

iii. (1.6 pts) Consider the situation in which r is kept constant and φ is slowly increased. The droplet becomes unstable and breaks into pieces if the hydrostatic pressure inside the droplet becomes smaller than the atmospheric pressure. Find the critical potential $\varphi_{\max}$ at which this will happen.

Part B. Two pipes *(4 points)*

An apparatus called the "Kelvin water dropper" consists of two pipes, each identical to the one described in Part A, connected via a T-junction, see fig. The ends of both pipes are at the centres of two cylindrical electrodes (with height L and diameter D with $L \gg D \gg r$). For both tubes, the dripping rate is n droplets per unit time. Droplets fall from height H into conductive bowls underneath the nozzles, cross-connected to the electrodes as shown in the diagram. The electrodes are connected via a capacitance C. There is no net charge on the system of bowls and electrodes. Note that the top water container is earthed as shown. The first droplet to fall will have some microscopic charge which will cause an imbalance between the two sides and a small charge separation across the capacitor.



i. (1.2 pts) Express the absolute value of the charge Q_0 of the drops as they separate from the tubes, and at the instant when the capacitor's charge is q. Express Q_0 in terms of $r_{\max}$ (from Part A-i) and neglect the effect described in Part A-iii.

ii. (1.5 pts) Find the dependence of q on time t by approximating it with a continuous function $q(t)$ and assuming that $q(0) = q_0$.

iii. (1.3 pts) The dropper's functioning can be hindered by the effect shown in Part A-iii. In addition, a limit $U_{\max}$ to the achievable potential between the electrodes is set by the electrostatic push between a droplet and the bowl beneath it. Find $U_{\max}$.

Problem T3. Protostar formation *(9 points)*

Let us model the formation of a star as follows. A spherical cloud of sparse interstellar gas, initially at rest, starts to collapse due to its own gravity. The initial radius of the ball is r_0 and the mass is m. The temperature of the surroundings (much sparser than the gas) and the initial temperature of the gas is uniformly T_0. The gas may be assumed to be ideal. The average molar mass of the gas is μ and its adiabatic index is $\gamma > \frac{4}{3}$. Assume that $G\frac{m\mu}{r_0} \gg RT_0$, where R is the gas constant and G is the gravitational constant.

i. (0.8 pts) During much of the collapse, the gas is so transparent that any heat generated is immediately radiated away, i.e. the ball stays in thermodynamic equilibrium with its surroundings. What is the number of times, n, by which the pressure increases when the radius is halved to $r_1 = 0.5r_0$? Assume that the gas density remains uniform.

ii. (1 pt) Estimate the time t_2 needed for the radius to shrink from r_0 to $r_2 = 0.95r_0$. Neglect the change of the gravity field at the position of a falling gas particle.

iii. (2.5 pts) Assuming that the pressure remains negligible, find the time $t_{r\to 0}$ needed for the ball to collapse from r_0 down to a much smaller radius, using Kepler's Laws.

iv. (1.7 pts) At some radius $r_3 \ll r_0$, the gas becomes dense enough to be opaque to the heat radiation. Calculate the amount of heat Q radiated away during the collapse from the radius r_0 down to r_3.

v. (1 pt) For radii smaller than r_3 you may neglect heat loss due to radiation. Determine how the temperature T of the ball depends on its radius for $r < r_3$.

vi. (2 pts) Eventually we cannot neglect the effect of the pressure on the dynamics of the gas and the collapse stops at $r = r_4$ (with $r_4 \ll r_3$). However, the radiation loss can still be neglected and the temperature is not yet high enough to ignite nuclear fusion. The pressure of such a protostar is not uniform anymore, but rough estimates with inaccurate numerical prefactors can still be done. *Estimate* the final radius r_4 and the respective temperature T_4.

Problem T1. Focus on sketches *(13 points)*

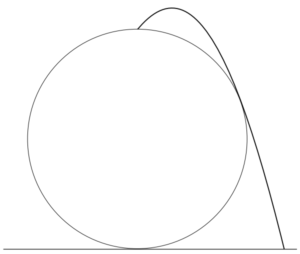
Part A. Ballistics *(4.5 points)*

i. (0.8 pts) When the stone is thrown vertically upwards, it can reach the point $x = 0$, $z = v_0^2/2g$ (as it follows from the energy conservation law). Comparing this with the inequality $z \le z_0 - kx^2$ we conclude that

$$z_0 = v_0^2/2g. \qquad [0.3 \text{ pts}]$$

Let us consider the asymptotics $z \to -\infty$; the trajectory of the stone is a parabola, and at this limit, the horizontal displacement (for the given z) is very sensitive with respect to the curvature of the parabola: the flatter the parabola, the larger the displacement. The parabola has the flattest shape when the stone is thrown horizontally, $x = v_0 t$ and $z = -gt^2/2$, i.e. its trajectory is given by $z = -gx^2/2v_0^2$. Now, let us recall that $z \le z_0 - kx^2$, i.e. $-gx^2/2v_0^2 \le z_0 - kx^2 \Rightarrow k \le g/2v_0^2$. Note that $k < g/2v_0^2$ would imply that there is a gap between the parabolic region $z \le z_0 - kx^2$ and the given trajectory $z = -gx^2/2v_0^2$. This trajectory is supposed to be optimal for hitting targets far below ($z \to -\infty$), so there should be no such a gap, and hence, we can exclude the option $k < g/2v_0^2$. This leaves us with

$$k = g/2v_0^2. \qquad [0.5 \text{ pts}]$$



ii. (1.2 pts) Let us note that the stone trajectory is reversible and due to the energy conservation law, one can equivalently ask, what is the minimal initial speed needed for a stone to be thrown from the topmost point of the spherical building down to the ground without hitting the roof, and what is the respective trajectory. It is easy to understand that the trajectory either needs to touch the roof, or start horizontally from the topmost point with the curvature radius equal to R. Indeed, if neither were the case, it would be possible to keep the same throwing angle and just reduce the speed a little bit — the stone would still reach the ground without hitting the roof. Further, if it were tangent at the topmost point, the trajectory wouldn't touch nor intersect the roof anywhere else, because the curvature of the parabola has maximum at its topmost point. Then, it would be possible to keep the initial speed constant, and increase slightly the throwing angle (from horizontal to slightly upwards): the new trajectory wouldn't be neither tangent at the top nor touch the roof at any other point; now we can reduce the initial speed as we argued previously. So we conclude that the optimal trajectory needs to touch the roof somewhere, as shown in Fig.

iii. (2.5 pts) The brute force approach would be writing down the condition that the optimal trajectory intersects with the building at two points and touches at one. This would be described by a fourth order algebraic equation and therefore, it is not realistic to accomplish such a solution within a reasonable time frame.

Note that the interior of the building needs to lie inside the region where the targets can be hit with a stone thrown from the top with initial speed $v_{\min}$. Indeed, if we can throw over the building, we can hit anything inside by lowering the throwing angle. On the other hand, the boundary of the targetable region needs to touch the building. Indeed, if there were a gap, it would be possible to hit a target just above the point where the optimal trajectory touches the building; the trajectory through that target wouldn't touch the building anywhere, hence we arrive at a contradiction.

So, with v_0 corresponding to the optimal trajectory, the targetable region touches the building; due to symmetry, overall there are two touching points (for smaller speeds, there would be four, and for larger speeds, there would be none). With the origin at the top of the building, the intersection points are defined by the following system of equations:

$$x^2 + z^2 + 2zR = 0, \quad z = \frac{v_0^2}{2g} - \frac{gx^2}{2v_0^2}.$$

Upon eliminating z, this becomes a biquadratic equation for x:

$$x^4\left(\frac{g}{2v_0^2}\right)^2 + x^2\left(\frac{1}{2} - \frac{gR}{v_0^2}\right) + \left(\frac{v_0^2}{4g} + R\right)\frac{v_0^2}{g} = 0.$$

Hence the speed by which the real-valued solutions disappear can be found from the condition that the discriminant vanishes:

$$\left(\frac{1}{2} - \frac{gR}{v_0^2}\right)^2 = \frac{1}{4} + \frac{gR}{v_0^2} \implies \frac{gR}{v_0^2} = 2.$$

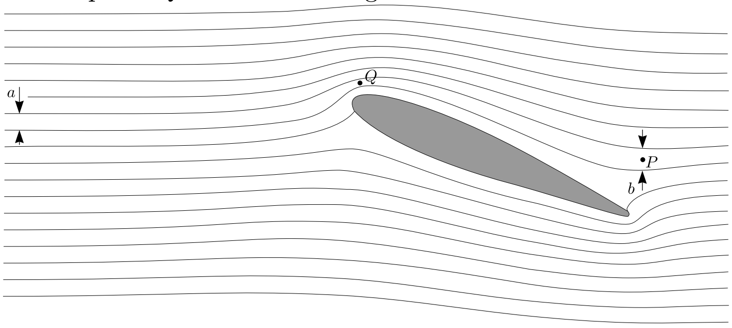
Bearing in mind that due to the energy conservation law, at the ground level the squared speed is increased by $4gR$. Thus we finally obtain

$$v_{\min} = \sqrt{v_0^2 + 4gR} = 3\sqrt{\frac{gR}{2}}.$$

Part B. Mist *(4 points)*

i. (0.8 pts) In the plane's reference frame, along the channel between two streamlines the volume flux of air (volume flow rate) is constant due to continuity. The volume flux is the product of speed and channel's cross-section area, which, due to the two-dimensional geometry, is proportional to the channel width and can be measured from the Fig. Due to the absence of wind, the unperturbed air's speed in the plane's frame is just v_0. So, upon measuring the dimensions $a = 10\,\text{mm}$ and $b = 13\,\text{mm}$ (see Fig), we can write $v_0 a = ub$ and hence $u = v_0\frac{a}{b}$. Since at point P, the streamlines are horizontal where all the velocities are parallel, the vector addition is reduced to the scalar addition: the air's ground speed $v_P = v_0 - u = v_0(1 - \frac{a}{b}) = 23\,\text{m/s}$.

ii. (1.2 pts) Although the dynamic pressure $\frac{1}{2}\rho v^2$ is relatively small, it gives rise to some adiabatic expansion and compression. In expanding regions the temperature will drop and hence, the pressure of saturated vapours will also drop. If the dew point is reached, a stream of droplets will appear. This process will start in a point where the adiabatic expansion is maximal, i.e. where the hydrostatic pressure is minimal and consequently, as it follows from the Bernoulli's law $p + \frac{1}{2}\rho v^2 = \text{const}$, the dynamic pressure is maximal: in the place where the air speed in wing's frame is maximal and the streamline distance minimal. Such a point Q is marked in Fig.



iii. (2 pts) First we need to calculate the dew point for the air of given water content (since the relative pressure change will be small, we can ignore the dependence of the dew point on pressure). The water vapour pressure is $p_w = p_{sa} r = 2.08\,\text{kPa}$. The relative change of the pressure of the saturated vapour is small, so we can linearize its temperature dependence:

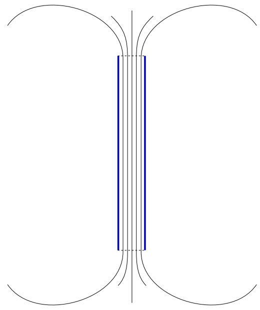
$$\frac{p_{sa} - p_w}{T_a - T} = \frac{p_{sb} - p_{sa}}{T_b - T_a} \implies T_a - T = (T_b - T_a)\frac{(1-r)p_{sa}}{p_{sb} - p_{sa}};$$

numerically $T \approx 291.5\,\text{K}$. Further we need to relate the air speed to the temperature. To this end we need to use the energy conservation law. A convenient ready-to-use form of it is provided by the Bernoulli's law. Applying this law will give a good approximation of the reality, but strictly speaking, it needs to be modified to take into account the compressibility of air and the associated expansion/contraction work. Consider one mole of air, which has the mass μ and the volume $V = RT/p$. Apparently the process is fast and the air parcels are large, so that heat transfer across the air parcels is negligible. Additionally, the process is subsonic; all together we can conclude that the process is adiabatic. Consider a segment of a tube formed by the streamlines. Let us denote the physical quantities at its one end by index 1, and at the other end — by index 2. Then, while one mole of gas flows into the tube at one end, as much flows out at the other end. The inflow carries in kinetic energy $\frac{1}{2}\mu v_1^2$, and the outflow carries out $\frac{1}{2}\mu v_2^2$. The inflowing gas receives work due to the pushing gas equal to $p_1 V_1 = RT_1$, the outflowing gas performs work $p_2 V_2 = RT_2$. Let's define molar heat capacities $C_V = \mu c_V$ and $C_p = \mu c_p$. The inflow carries in heat energy $C_V R T_1$, and the outflow carries out $C_V R T_2$. All together, the energy balance can be written as $\frac{1}{2}\mu v^2 + C_p T = \text{const}$. From this we can easily express $\Delta \frac{v^2}{2} = \frac{1}{C} v_{\text{crit}}^2 (\frac{a^2}{c^2} - 1) = c_p \Delta T$, where c is the streamline distance at the point Q, and further

$$v_{\text{crit}} = c\sqrt{\frac{2c_p \Delta T}{a^2 - c^2}} \approx 23\,\text{m/s},$$

where we have used $c \approx 4.5\,\text{mm}$ and $\Delta T = 1.5\,\text{K}$. Note that in reality, the required speed is probably somewhat higher, because for a fast condensation, a considerable over-saturation is needed. However, within an order of magnitude, this estimate remains valid.

Part C. Magnetic straws *(4.5 points)*



i. (0.8 pts) Due to the superconducting walls, the magnetic field lines cannot cross the walls, so the flux is constant along the tube. For a closed contour inside the tube, there should be no circulation of the magnetic field, hence the field lines cannot be curved, and the field needs to be homogeneous. The field lines close from outside the tube, similarly to a solenoid.

ii. (1.2 pts) Let us consider the change of the magnetic energy when the tube is stretched (virtually) by a small amount Δl. Note that the magnetic flux trough the tube is conserved: any change of flux would imply a non-zero electromotive force $\frac{d\Phi}{dt}$, and for a zero resistivity, an infinite current. So, the induction $B = \frac{\Phi}{\pi r^2}$. The energy density of the magnetic field is $\frac{B^2}{2\mu_0}$. Thus, the change of the magnetic energy is calculated as

$$\Delta W = \frac{B^2}{2\mu_0}\pi r^2 \Delta l = \frac{\Phi^2}{2\mu_0 \pi r^2}\Delta l.$$

This energy increase is achieved owing to the work done by the stretching force, $\Delta W = T\Delta l$. Hence, the force

$$T = \frac{\Phi^2}{2\mu_0 \pi r^2}.$$

iii. (2.5 pts) Let us analyse, what would be the change of the magnetic energy when one of the straws is displaced to a small distance. The magnetic field inside the tubes will remain constant due to the conservation of magnetic flux, but outside, the magnetic field will be changed. The magnetic field outside the straws is defined by the following condition: there is no circulation of $\vec{B}$ (because there are no currents outside the straws); there are no sources of the field lines, other than the endpoints of the straws; each of the endpoints of the straws is a source of streamlines with a fixed magnetic flux $\pm\Phi$. These are exactly the same condition as those which define the electric field of four charges $\pm Q$. We know that if the distance between charges is much larger than the geometrical size of a charge, the charges can be considered as point charges (the electric field near the charges remains almost constant, so that the respective contribution to the change of the overall electric field energy is negligible). Therefore we can conclude that the endpoints of the straws can be considered as magnetic point charges. In order to calculate the force between two magnetic charges (magnetic monopoles), we need to establish the correspondence between magnetic and electric quantities.

For two electric charges Q separated by a distance a, the force is $F = \frac{1}{4\pi\varepsilon_0}\frac{Q^2}{a^2}$, and at the position of one charge, the electric field of the other charge has energy density $w = \frac{1}{32\pi^2\varepsilon_0}\frac{Q^2}{a^4}$; hence we can write $F = 8\pi w a^2$. This is a universal expression for the force (for the case when the field lines have the same shape as in the case of two opposite and equal by modulus electric charges) relying only on the energy density, and not related to the nature of the field; so we can apply it to the magnetic field. Indeed, the force can be calculated as a derivative of the full field energy with respect to a virtual displacement of a field line source (electric or magnetic charge); if the energy densities of two fields are respectively equal at one point, they are equal everywhere, and so are equal the full field energies. As it follows from the Gauss law, for a point source of a fixed magnetic flux Φ at a distance a, the induction $B = \frac{1}{4\pi}\frac{\Phi}{a^2}$. So, the energy density $w = \frac{B^2}{2\mu_0} = \frac{1}{32\pi^2\mu_0}\frac{\Phi^2}{a^4}$, hence

$$F = \frac{1}{4\pi\mu_0}\frac{\Phi^2}{a^2}.$$

For the two straws, we have four magnetic charges. The longitudinal (along a straw axis) forces cancel out (the diagonally positioned pairs of same-sign-charges push in opposite directions). The normal force is a superposition of the attraction due to the two pairs of opposite charges, $F_1 = \frac{1}{4\pi\mu_0}\frac{\Phi^2}{l^2}$, and the repulsive forces of diagonal pairs, $F_2 = \frac{\sqrt{2}}{8\pi\mu_0}\frac{\Phi^2}{2l^2}$. The net attractive force will be

$$F = 2(F_1 - F_2) = \frac{4-\sqrt{2}}{8\pi\mu_0}\frac{\Phi^2}{l^2}.$$

Problem T2. Kelvin water dropper *(8 points)*

Part A. Single pipe *(4 points)*

i. (1.2 pts) Let us write the force balance for the droplet. Since $d \ll r$, we can neglect the force $\frac{\pi}{4}\Delta p d^2$ due to the excess pressure Δp inside the tube. So, the gravity force $\frac{4}{3}\pi r_{\max}^3 \rho g$ is balanced by the capillary force. When the droplet separates from the tube, the water surface forms in the vicinity of the nozzle a "neck", which has vertical tangent. In the horizontal cross-section of that "neck", the capillary force is vertical and can be calculated as $\pi\sigma d$. So,

$$r_{\max} = \sqrt[3]{\frac{3\sigma d}{4\rho g}}.$$

ii. (1.2 pts) Since $d \ll r$, we can neglect the change of the droplet's capacitance due to the tube. On the one hand, the droplet's potential is φ; on the other hand, it is $\frac{1}{4\pi\varepsilon_0}\frac{Q}{r}$. So,

$$Q = 4\pi\varepsilon_0\varphi r.$$

iii. (1.6 pts) Excess pressure inside the droplet is caused by the capillary pressure $2\sigma/r$ (increases the inside pressure), and by the electrostatic pressure $\frac{1}{2}\varepsilon_0 E^2 = \frac{1}{2}\varepsilon_0\varphi^2/r^2$ (decreases the pressure). So, the sign of the excess pressure will change, if $\frac{1}{2}\varepsilon_0\varphi_{\max}^2/r^2 = 2\sigma/r$, hence

$$\varphi_{\max} = 2\sqrt{\sigma r/\varepsilon_0}.$$

The expression for the electrostatic pressure used above can be derived as follows. The electrostatic force acting on a surface charge of density σ and surface area S is given by $F = \sigma S \cdot \bar{E}$, where $\bar{E}$ is the field at the site without the field created by the surface charge element itself. Note that this force is perpendicular to the surface, so F/S can be interpreted as a pressure. The surface charge gives rise to a field drop on the surface equal to $\Delta E = \sigma/\varepsilon_0$ (which follows from the Gauss law); inside the droplet, there is no field due to the conductivity of the droplet: $\bar{E} - \frac{1}{2}\Delta E = 0$; outside the droplet, there is field $E = \bar{E} + \frac{1}{2}\Delta E$, therefore $\bar{E} = \frac{1}{2}E = \frac{1}{2}\Delta E$. Bringing everything together, we obtain the expression used above.

Note that alternatively, this expression can be derived by considering a virtual displacement of a capacitor's surface and comparing the pressure work $p\Delta V$ with the change of the electrostatic field energy $\frac{1}{2}\varepsilon_0 E^2 \Delta V$.

Finally, the answer to the question can be also derived from the requirement that the mechanical work dA done for an infinitesimal droplet inflation needs to be zero. From the energy conservation law, $dW + dW_{\text{el}} = \sigma\, d(4\pi r^2) + \frac{1}{2}\varphi_{\max}^2\, dC_d$, where the droplet's capacitance $C_d = 4\pi\varepsilon_0 r$; the electrical work $dW_{\text{el}} = \varphi_{\max} dq = 4\pi\varepsilon_0\varphi_{\max}^2 dr$. Putting $dW = 0$ we obtain an equation for $\varphi_{\max}$, which recovers the earlier result.

Part B. Two pipes *(4 points)*

i. (1.2 pts) This is basically the same as Part A-ii, except that the surroundings' potential is that of the surrounding electrode, $-U/2$ (where $U = q/C$ is the capacitor's voltage) and droplet has the ground potential (0). As it is not defined which electrode is the positive one, opposite sign of the potential may be chosen, if done consistently. Note that since the cylindrical electrode is long, it shields effectively the environment's (ground, wall, etc) potential. So, relative to its surroundings, the droplet's potential is $U/2$. Using the result of Part A we obtain

$$Q = 2\pi\varepsilon_0 U r_{\max} = 2\pi\varepsilon_0 q r_{\max}/C.$$

ii. (1.5 pts) The sign of the droplet's charge is the same as that of the capacitor's opposite plate (which is connected to the farther electrode). So, when the droplet falls into the bowl, it will increase the capacitor's charge by Q:

$$dq = 2\pi\varepsilon_0 U r_{\max} dN = 2\pi\varepsilon_0 r_{\max} n dt \frac{q}{C},$$

where $dN = ndt$ is the number of droplets which fall during the time dt This is a simple linear differential equation which is solved easily to obtain

$$q = q_0 e^{\gamma t}, \quad \gamma = \frac{2\pi\varepsilon_0 r_{\max} n}{C} = \frac{\pi\varepsilon_0 n}{C}\sqrt[3]{\frac{6\sigma d}{\rho g}}.$$

iii. (1.3 pts) The droplets can reach the bowls if their mechanical energy mgH (where m is the droplet's mass) is large enough to overcome the electrostatic push: The droplet starts at the point where the electric potential is 0, which is the sum of the potential $U/2$, due to the electrode, and of its self-generated potential $-U/2$. Its motion is not affected by the self-generated field, so it needs to fall from the potential $U/2$ down to the potential $-U/2$, resulting in the change of the electrostatic energy equal to $UQ \le mgH$, where $Q = 2\pi\varepsilon_0 U r_{\max}$ (see above). So,

$$U_{\max} = \frac{mgH}{2\pi\varepsilon_0 U_{\max} r_{\max}},$$

$$\therefore U_{\max} = \sqrt{\frac{H\sigma d}{2\varepsilon_0 r_{\max}}} = \sqrt[6]{\frac{H^3 g\sigma^2\rho d^2}{6\varepsilon_0^3}}.$$

Problem T3. Protostar formation *(9 points)*

i. (0.8 pts)

$$T = \text{const} \implies pV = \text{const}$$
$$V \propto r^3$$
$$\therefore p \propto r^{-3} \implies \frac{p(r_1)}{p(r_0)} = 2^3 = 8.$$

ii. (1 pt) During the period considered the pressure is negligible. Therefore the gas is in free fall. By Gauss' theorem and symmetry, the gravitational field at any point in the ball is equivalent to the one generated when all the mass closer to the center is compressed into the center. Moreover, while the ball has not yet shrunk much, the field strength on its surface does not change much either. The acceleration of the outermost layer stays approximately constant. Thus,

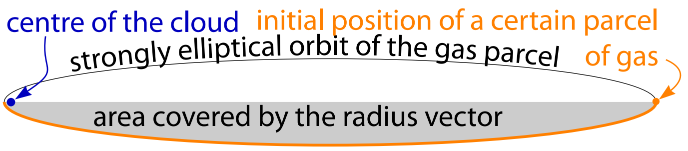
$$t \approx \sqrt{\frac{2(r_0 - r_2)}{g}}$$

where

$$g \approx \frac{Gm}{r_0^2},$$
$$\therefore t \approx \sqrt{\frac{2r_0^2(r_0 - r_2)}{Gm}} = \sqrt{\frac{0.1 r_0^3}{Gm}}.$$

iii. (2.5 pts) Gravitationally the outer layer of the ball is influenced by the rest just as the rest were compressed into a point mass. Therefore we have Keplerian motion: the fall of any part of the outer layer consists in a halfperiod of an ultra-elliptical orbit. The ellipse is degenerate into a line; its foci are at the ends of the line; one focus is at the center of the ball (by Kepler's 1st law) and the other one is at r_0, see figure (instead of a degenerate ellipse, a strongly elliptical ellipse is depicted). The period of the orbit is determined by the longer semiaxis of the ellipse (by Kepler's 3rd law). The longer semiaxis is $r_0/2$ and we are interested in half a period. Thus, the answer is equal to the halfperiod of a circular orbit of radius $r_0/2$:

$$\left(\frac{2\pi}{2t_{r\to 0}}\right)^2 \frac{r_0}{2} = \frac{Gm}{(r_0/2)^2} \implies t_{r\to 0} = \pi\sqrt{\frac{r_0^3}{8Gm}}.$$



Alternatively, one may write the energy conservation law $\frac{\dot{r}^2}{2} - \frac{Gm}{r} = E$ (that in turn is obtainable from Newton's II law $\ddot{r} = -\frac{Gm}{r^2}$) with $E = -\frac{Gm}{r_0}$, separate the variables ($\frac{dr}{dt} = -\sqrt{2E + \frac{2Gm}{r}}$) and write the integral $t = -\int \frac{dr}{\sqrt{2E+\frac{2Gm}{r}}}$. This integral is probably not calculable during the limitted time given during the Olympiad, but a possible approach can be sketched as follows. Substituting $\sqrt{2E + \frac{2Gm}{r}} = \xi$ and $\sqrt{2E} = v$, one gets

$$\frac{t_\infty}{4Gm} = \int_0^\infty \frac{d\xi}{(v^2 - \xi^2)^2}$$
$$= \frac{1}{4v^3}\int_0^\infty \left[\frac{v}{(v-\xi)^2} + \frac{v}{(v+\xi)^2} + \frac{1}{v-\xi} + \frac{1}{v+\xi}\right] d\xi.$$

Here (after shifting the variable) one can use $\int \frac{d\xi}{\xi} = \ln\xi$ and $\int \frac{d\xi}{\xi^2} = -\frac{1}{\xi}$, finally getting the same answer as by Kepler's laws.

iv. (1.7 pts) By Clapeyron–Mendeleyev law,

$$p = \frac{mRT_0}{\mu V}.$$

Work done by gravity to compress the ball is

$$W = -\int p\,dV = -\frac{mRT_0}{\mu}\int_{\frac{4}{3}\pi r_0^3}^{\frac{4}{3}\pi r_3^3} \frac{dV}{V} = \frac{3mRT_0}{\mu}\ln\frac{r_0}{r_3}.$$

The temperature stays constant, so the internal energy does not change; hence, according to the 1st law of thermodynamics, the compression work W is the heat radiated.

v. (1 pt) The collapse continues adiabatically.

$$pV^\gamma = \text{const} \implies TV^{\gamma-1} = \text{const}.$$
$$\therefore T \propto V^{1-\gamma} \propto r^{3-3\gamma}$$
$$\therefore T = T_0\left(\frac{r_3}{r}\right)^{3\gamma-3}.$$

vi. (2 pts) During the collapse, the gravitational energy is converted into heat. Since $r_3 \gg r_4$, The released gravitational energy can be estimated as $\Delta\Pi = -Gm^2(r_4^{-1} - r_3^{-1}) \approx -Gm^2/r_4$ (exact calculation by integration adds a prefactor $\frac{3}{5}$); the terminal heat energy is estimated as $\Delta Q = c_V\frac{m}{\mu}(T_4 - T_0) \approx c_V\frac{m}{\mu}T_4$ (the approximation $T_4 \gg T_0$ follows from the result of the previous question, when combined with $r_3 \gg r_4$). So, $\Delta Q = \frac{R}{\gamma-1}\frac{m}{\mu}T_4 \approx \frac{m}{\mu}RT_4$. For the temperature T_4, we can use the result of the previous question, $T_4 = T_0\left(\frac{r_3}{r_4}\right)^{3\gamma-3}$. Since initial full energy was approximately zero, $\Delta Q + \Delta\Pi \approx 0$, we obtain

$$\frac{Gm^2}{r_4} \approx \frac{m}{\mu}RT_0\left(\frac{r_3}{r_4}\right)^{3\gamma-3} \implies r_4 \approx r_3\left(\frac{RT_0 r_3}{\mu m G}\right)^{\frac{1}{3\gamma-4}}.$$

Therefore,

$$T_4 \approx T_0\left(\frac{RT_0 r_3}{\mu m G}\right)^{\frac{3\gamma-3}{4-3\gamma}}.$$

Alternatively, one can obtain the result by approximately equating the hydrostatic pressure $\rho r_4 \frac{Gm}{r_4^2}$ to the gas pressure $p_4 = \frac{\rho}{\mu}RT_4$; the result will be exactly the same as given above.

ANSWER SHEET

Problem 1

Problem T1. Focus on sketches *(13 points)*

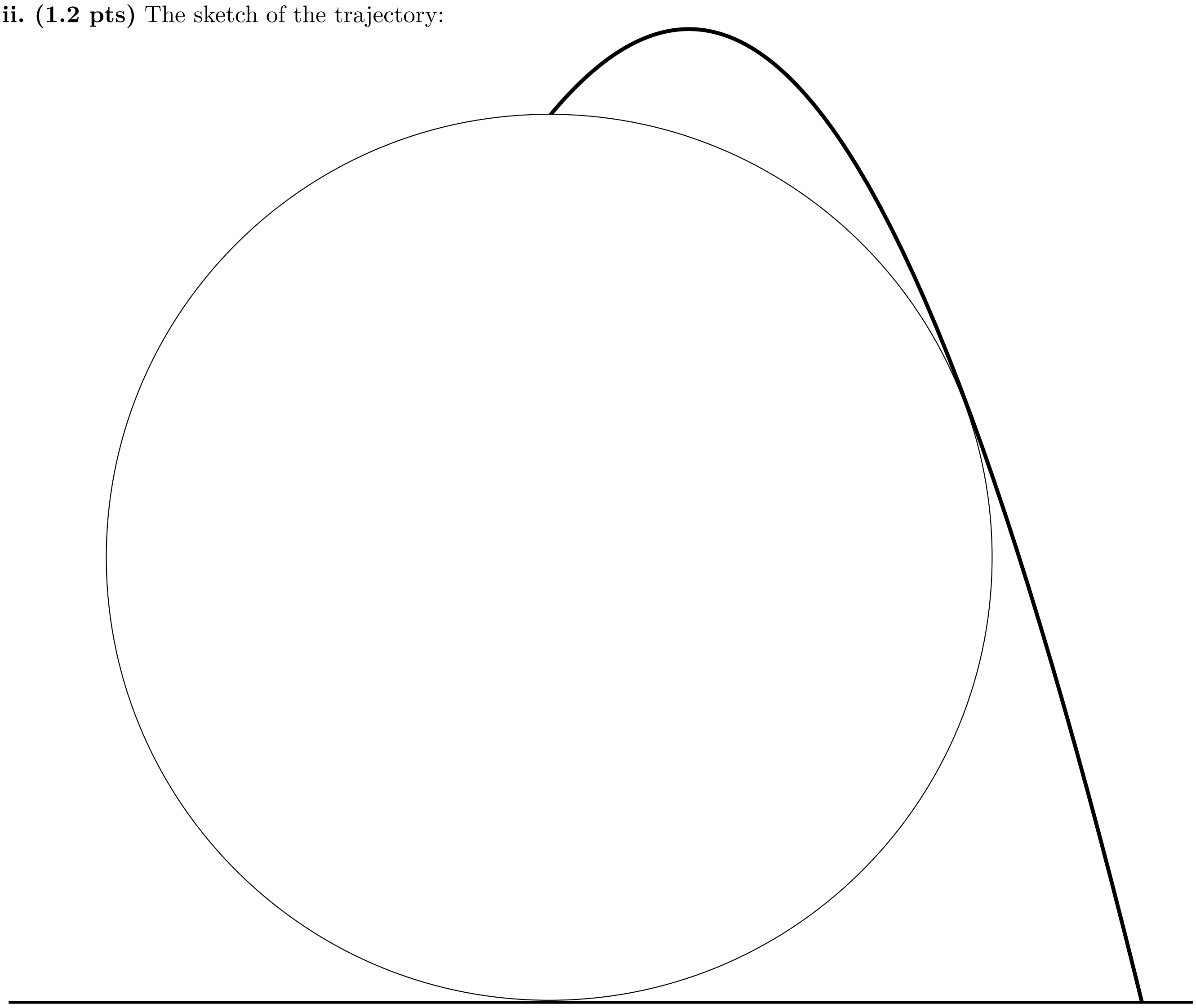
Part A. Ballistics *(4.5 points)*

i. (0.8 pts)

$$z_0 = v_0^2/2g$$

$$k = g/2v_0^2$$

ii. (1.2 pts) The sketch of the trajectory:



iii. (2.5 pts)

$$v_{\min} = 3\sqrt{\frac{gR}{2}}$$

ANSWER SHEET

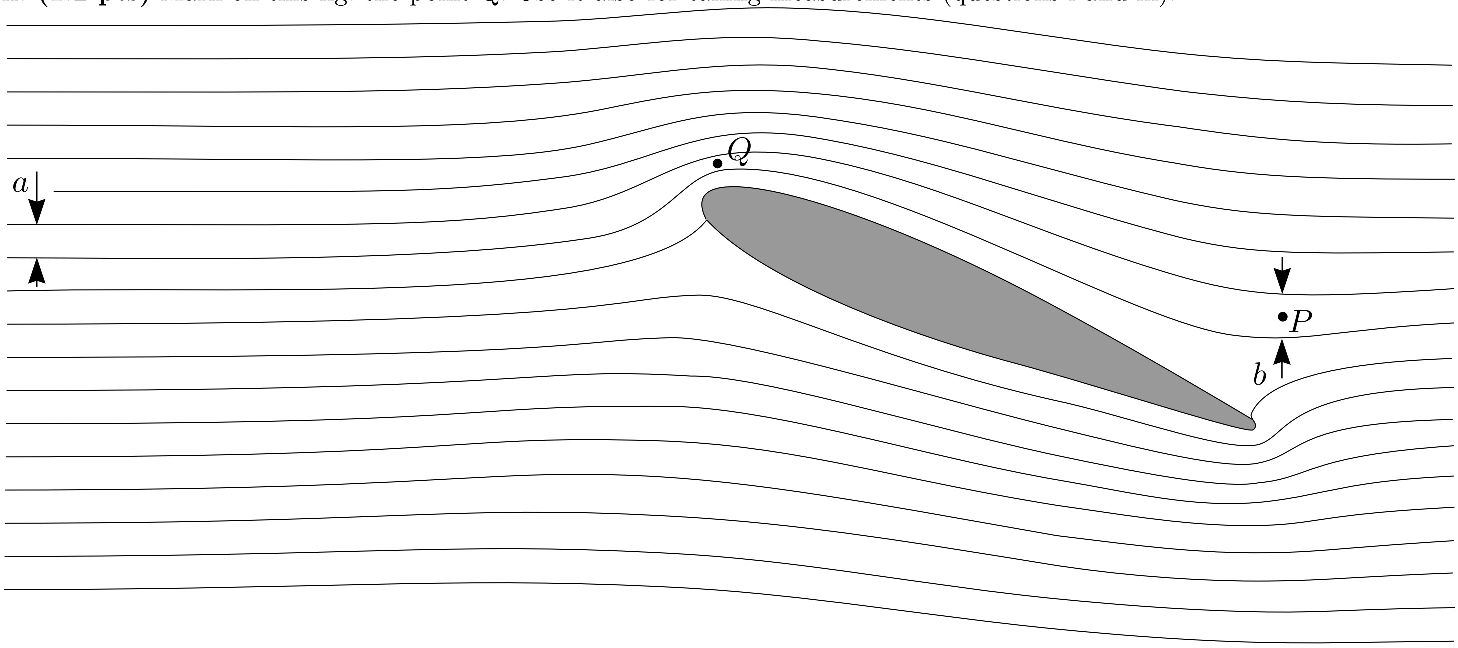
Problem 1

Part B. Air flow around a wing *(4 points)*

i. (0.8 pts)

$v_P = 23\,\mathrm{m/s}$

ii. (1.2 pts) Mark on this fig. the point Q. Use it also for taking measurements (questions i and iii).



Formulae motivating the choice of point Q:

$$av = \text{const}$$

$$p + \frac{1}{2}\rho v^2 = \text{const}$$

$$p^{1-\gamma}T^{\gamma} = \text{const}$$

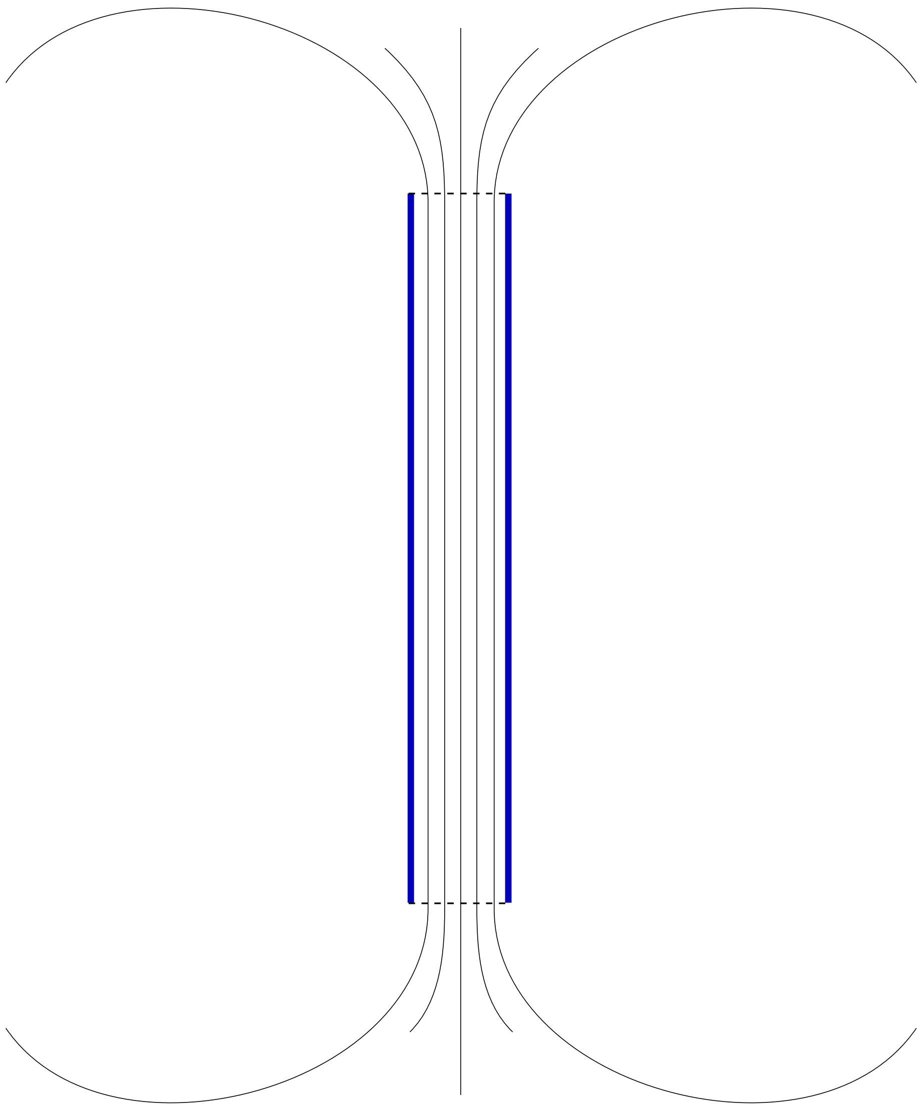
iii. (2.0 pts)

Formula: $v_{\text{crit}} = c\sqrt{\dfrac{2c_p\Delta T}{a^2 - c^2}}$

Numerical: $v_{\text{crit}} \approx 23\,\mathrm{m/s}$

Part C. Magnetic straws *(4.5 points)*

i. (0.8 pts)
Sketch here five magnetic field lines.



ii. (1.2 pts)

$$T = \frac{\Phi^2}{2\mu_0 \pi r^2}$$

iii. (2.5 pts)

$$F = \frac{4-\sqrt{2}}{8\pi\mu_0}\frac{\Phi^2}{l^2}$$

ANSWER SHEET

Problem 2

Problem T2. Kelvin water dropper *(8 points)*

Part A. Single pipe *(4 points)*

i. (1.2 pts)

$$r_{\max} = \sqrt[3]{\frac{3\sigma d}{4\rho g}}$$

ii. (1.2 pts)

$$Q = 4\pi\varepsilon_0 \varphi r$$

iii. (1.6 pts)

$$\varphi_{\max} = 2\sqrt{\sigma r/\varepsilon_0}$$

Part B. Two pipes *(4 points)*

i. (1.2 pts)

$$Q_0 = 2\pi\varepsilon_0 q r_{\max}/C$$

ii. (1.5 pts)

$$q(t) = q_0 e^{\gamma t}, \quad \gamma = \frac{\pi\varepsilon_0 n}{C}\sqrt[3]{\frac{6\sigma d}{\rho g}}.$$

iii. (1.3 pts)

$$U_{\max} = \sqrt[6]{\frac{H^3 g \sigma^2 \rho d^2}{6\varepsilon_0^3}}$$

Problem T3. Protostar formation *(9 points)*

i. (0.8 pts)

$$n = 8$$

ii. (1 pt)

$$t_2 \approx \sqrt{\frac{0.1 r_0^3}{Gm}}$$

iii. (2.5 pts)

$$t_{r\to 0} = \pi\sqrt{\frac{r_0^3}{8Gm}}$$

iv. (1.7 pts)

$$Q = \frac{3mRT_0}{\mu}\ln\frac{r_0}{r_3}$$

v. (1 pt)

$$T(r) = T_0\left(\frac{r_3}{r}\right)^{3\gamma-3}$$

vi. (2 pts)

$$r_4 \approx r_3\left(\frac{RT_0 r_3}{\mu m G}\right)^{\frac{1}{3\gamma-4}}$$

$$T_4 \approx T_0\left(\frac{RT_0 r_3}{\mu m G}\right)^{\frac{3\gamma-3}{4-3\gamma}}$$

After falling a time t in the atmosphere, an outer shell of Maribo of thickness x will have been heated to a temperature significantly larger than T_0. This thickness can be estimated by dimensional analysis as the simple product of powers of the thermodynamic parameters: $x \approx t^{\alpha} \rho_{\mathrm{sm}}^{\beta} c_{\mathrm{sm}}^{\gamma} k_{\mathrm{sm}}^{\delta}$.

1.3a	Determine by dimensional (unit) analysis the value of the four powers α, β, γ, and δ.	0.6
1.3b	Calculate the thickness x after a fall time $t = 5$ s, and determine the ratio x/R_{M}.	0.4

The age of a meteorite

The chemical properties of radioactive elements may be different, so during the crystallization of the minerals in a given meteorite, some minerals will have a high content of a specific radioactive element and others a low content. This difference can be used to determine the age of a meteorite by radiometric dating of its radioactive minerals.

As a specific example, we study the isotope ^{87}Rb (element no. 37), which decays into the stable isotope ^{87}Sr (element no. 38) with a half-life of $T_{½} = 4.9 \times 10^{10}$ year, relative to the stable isotope ^{86}Sr. At the time of crystallization the ratio ^{87}Sr/^{86}Sr was identical for all minerals, while the ratio ^{87}Rb/^{86}Sr was different. As time passes on, the amount of ^{87}Rb decreases by decay, while consequently the amount of ^{87}Sr increases. As a result, the ratio ^{87}Sr/^{86}Sr will be different today. In Fig. 1.2(a), the points on the horizontal line refer to the ratio ^{87}Rb/^{86}Sr in different minerals at the time, when they are crystallized.

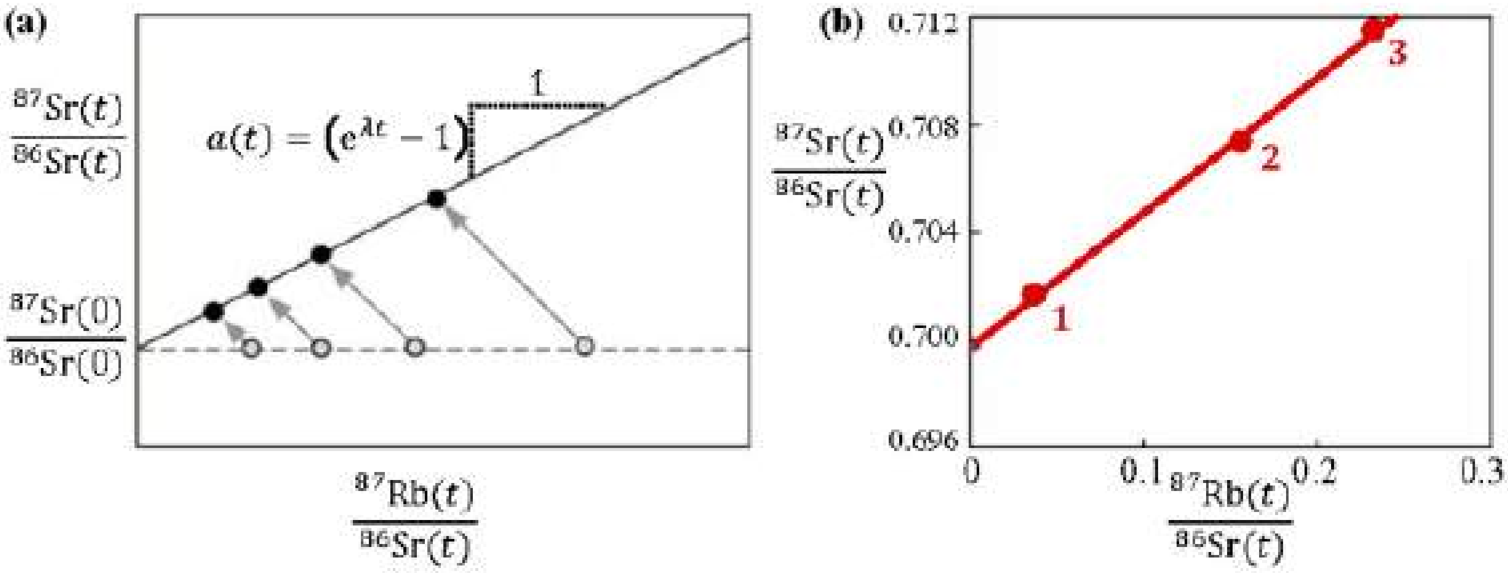


Figure 1.2 **(a)** The ratio ^{87}Sr/^{86}Sr in different minerals at the time $t = 0$ of crystallization (open circles) and at present time (filled circles). **(b)** The isochron-line for three different mineral samples taken from a meteorite at present time.

1.4a	Write down the decay scheme for the transformation of $^{87}_{37}$Rb to $^{87}_{38}$Sr.	0.3
1.4b	Show that the present-time ratio ^{87}Sr/^{86}Sr plotted versus the present-time ratio ^{87}Rb/^{86}Sr in different mineral samples from the same meteorite forms a straight line, the so-called isochron-line, with slope $a(t) = (e^{\lambda t} - 1)$. Here t is the time since the formation of the minerals, while λ is the decay constant inversely proportional to half-life $T_{½}$.	0.7
1.4c	Determine the age τ_{M} of the meteorite using the isochron-line of Fig. 1.2(b).	0.4

The incident light beam loses some time-averaged power P_{scat} by scattering on the oscillating electron cloud (re-emission). P_{scat} depends on the scattering source amplitude x_0, charge Q, angular frequency ω_{p} and properties of the light (the speed of light c and permittivity ε_0 in vacuum). In terms of these four variables, P_{scat} is given by $P_{\text{scat}} = \frac{Q^2 x_0^2 \omega_{\text{p}}^4}{12\pi\varepsilon_0 c^3}$.

2.9	By use of P_{scat}, find an expression of the equivalent scattering resistance R_{scat} (in analogy with R_{heat}) in an equivalent resistor-model, and calculate its value.	1.0

The above equivalent circuit elements are combined into an LCR series circuit model of the silver nanoparticle, which is driven by a harmonically oscillating equivalent voltage $V = V_0 \cos(\omega_{\text{p}} t)$ determined by the electric field E_0 of the incident light.

2.10a	Derive expressions for the time-averaged power losses P_{heat} and P_{scat} involving the amplitude E_0 of the electric field in the incident light at the plasmon resonance $\omega = \omega_p$.	1.2
2.10b	Calculate the numerical value of E_0, P_{heat}, and P_{scat}.	0.3

Steam generation by light

An aqueous solution of silver nanoparticles is prepared with a concentration $n_{\text{np}} = 7.3 \times 10^{15}\ \text{m}^{-3}$. It is placed inside a rectangular transparent vessel of size $h \times h \times a = 10 \times 10 \times 1.0\ \text{cm}^3$ and illuminated by light at the plasmon frequency with the same intensity $S = 1.00\ \text{MW m}^{-2}$ at normal incidence as above, see Fig. 2.1(e). The temperature of the water is $T_{\text{wa}} = 20\ °\text{C}$ and we assume, in fair agreement with observations, that in steady state all Joule heating of the nanoparticle goes to the production of steam of temperature $T_{\text{st}} = 110\ °\text{C}$, without raising the temperature of the water.

The thermodynamic efficiency η of the plasmonic steam generator is defined by the power ratio $\eta = P_{\text{st}}/P_{\text{tot}}$, where P_{st} is the power going into the production of steam in the entire vessel, while P_{tot} is the total power of the incoming light that enters the vessel.

Most of the time any given nanoparticle is surrounded by steam instead of water, and it can thus be described as being in vacuum.

2.11a	Calculate the total mass per second μ_{st} of steam produced by the plasmonic steam generator during illumination by light at the plasmon frequency and intensity S.	0.6
2.11b	Calculate the numerical value of the thermodynamic efficiency η of the plasmonic steam generator.	0.2

Figure 1.2 The unlabeled six buttons are irrelevant (they are used to calculate area and volume). The relevant buttons are:

A: On/off

B: Switch between measurement from the rear and the front of the instrument.

C: Indicator for measurement from the rear/front

D: Turn on laser/start measurement

E: Continuous measurement

F: Indicator for continuous measurement

Figure 1.3 The laser distance meter seen from the front end:

A: Receiver: Lens for the telescope focused on the laser dot

B: Emitter: Do not look into the laser beam!

1.1 Measurement with the laser distance meter

The instrument will perform a measurement when you press the button **D**, see Fig. 1.2.

1.1a	Use the LDM to measure the distance H from the top of the table to the floor. Write down the uncertainty ΔH. Show with a sketch how you perform this measurement.	0.4

(a)

(b)

Frame	Time	Azimuth	Altitude
155	1.46 s	215°	19.2°
161	2.28 s	221°	14.7°
Landing at M		230°	0.0°

(c)

Figure 1.1 (a) Azimuth is the clockwise angular position from north in the horizontal plane, and altitude is the angular position above the horizon. A series of frames recorded by the surveillance camera in Sweden, showing the motion of Maribo as a fireball on its way down through the atmosphere. **(b)** The data from two frames indicating the time, the direction (azimuth) in degrees, as seen by the camera (C), and the height above the horizon (altitude) in degrees. **(c)** Sketch of the directions of the path (magenta arrow) of Maribo relative to north (N) and of the landing site (M) in Denmark as seen by the camera (C).

Heating of Maribo during its fall in the atmosphere

When the stony meteoroid Maribo entered the atmosphere at supersonic speed it appeared as a fireball because the surrounding air was glowing. Nevertheless, only the outermost layer of Maribo was heated. Assume that Maribo is a homogenous sphere with density ρ_{sm}, specific heat capacity c_{sm}, and thermal conductivity k_{sm} (for values see the data sheet). Furthermore, when entering the atmosphere, it had the temperature $T_0 = 200\ \text{K}$. While falling through the atmosphere its surface temperature was constant $T_s = 1000\ \text{K}$ due to the air friction, thus gradually heating up the interior.

The electric field in a charge-neutral region inside a charged sphere

For the rest of the problem assume that the relative dielectric permittivity of all materials is $\varepsilon = 1$. Inside a charged sphere of homogeneous charge density ρ and radius R is created a small spherical charge-neutral region of radius R_1 by adding the opposite charge density $-\rho$, with its center displaced by $\boldsymbol{x}_\mathrm{d} = x_\mathrm{d}\,\boldsymbol{e}_x$ from the center of the R-sphere, see Fig. 2.1(b).

2.2	Show that the electric field inside the charge-neutral region is homogenous of the form $\boldsymbol{E} = A\,(\rho/\varepsilon_0)\,\boldsymbol{x}_\mathrm{d}$, and determine the pre-factor A.	1.2

The restoring force on the displaced electron cloud

In the following, we study the collective motion of the free electrons, and therefore model them as a single negatively charged sphere of homogeneous charge density $-\rho$ with a center position $\boldsymbol{x}_\mathrm{p}$, which can move along the x-axis relative to the center of the positively charged sphere (silver ions) fixed at the origin of the coordinate system, see Fig. 2.1(c). Assume that an external force $\boldsymbol{F}_\mathrm{ext}$ displaces the electron cloud to a new equilibrium position $\boldsymbol{x}_\mathrm{p} = x_\mathrm{p}\,\boldsymbol{e}_x$ with $|x_\mathrm{p}| \ll R$. Except for tiny net charges at opposite ends of the nanoparticle, most of its interior remains charge-neutral.

2.3	Express in terms of $\boldsymbol{x}_\mathrm{p}$ and n the following two quantities: The restoring force $\boldsymbol{F}$ exerted on the electron cloud and the work W_el done on the electron cloud during displacement.	1.0

The spherical silver nanoparticle in an external constant electric field

A nanoparticle is placed in vacuum and influenced by an external force $\boldsymbol{F}_\mathrm{ext}$ due to an applied static homogeneous electric field $\boldsymbol{E}_0 = -E_0\boldsymbol{e}_x$, which displaces the electron cloud the small distance $|x_\mathrm{p}|$, where $|x_\mathrm{p}| \ll R$.

2.4	Find the displacement x_p of the electron cloud in terms of E_0 and n, and determine the amount $-\Delta Q$ of electron charge displaced through the yz-plane at the center of the nanoparticle in terms of n, R and x_p.	0.6

The equivalent capacitance and inductance of the silver nanoparticle

For both a constant and a time-dependent field $\boldsymbol{E}_0$, the nanoparticle can be modeled as an equivalent electric circuit. The equivalent capacitance can be found by relating the work W_el, done on the separation of charges ΔQ, to the energy of a capacitor, carrying charge $\pm\Delta Q$. The charge separation will cause a certain equivalent voltage V_0 across the equivalent capacitor.

2.5a	Express the systems equivalent capacitance C in terms of ε_0 and R, and find its value.	0.7
2.5b	For this capacitance, determine in terms of E_0 and R the equivalent voltage V_0 that should be connected to the equivalent capacitor in order to accumulate the charge ΔQ.	0.4

For a time-dependent field $\boldsymbol{E}_0$, the electron cloud moves with velocity $\boldsymbol{v} = v\,\boldsymbol{e}_x$, Fig. 2.1(d). It has the kinetic energy W_{kin} and forms an electric current I flowing through the fixed yz-plane. The kinetic energy of the electron cloud can be attributed to the energy of an equivalent inductor of inductance L carrying the current I.

2.6a	Express both W_{kin} and I in terms of the velocity v.	0.7
2.6b	Express the equivalent inductance L in terms of particle radius R, the electron charge e and mass m_e, the electron concentration n, and calculate its value.	0.5

The plasmon resonance of the silver nanoparticle

From the above analysis it follows that the motion, arising from displacing the electron cloud from its equilibrium position and then releasing it, can be modeled by an ideal LC-circuit oscillating at resonance. This dynamical mode of the electron cloud is known as the plasmon resonance, which oscillates at the so-called angular plasmon frequency ω_{p}.

2.7a	Find an expression for the angular plasmon frequency ω_{p} of the electron cloud in terms of the electron charge e and mass m_e, the electron density n, and the permittivity ε_0.	0.5
2.7b	Calculate ω_{p} in rad/s and the wavelength λ_{p} in nm of light in vacuum having angular frequency $\omega = \omega_{\text{p}}$.	0.4

The silver nanoparticle illuminated with light at the plasmon frequency

In the rest of the problem, the nanoparticle is illuminated by monochromatic light at the angular plasmon frequency ω_{p} with the incident intensity $S = \frac{1}{2}c\varepsilon_0 E_0^2 = 1.00\ \text{MW m}^{-2}$. As the wavelength is large, $\lambda_{\text{p}} \gg R$, the nanoparticle can be considered as being placed in a homogeneous harmonically oscillating field $\boldsymbol{E}_0 = -E_0 \cos(\omega_{\text{p}} t)\,\boldsymbol{e}_x$. Driven by $\boldsymbol{E}_0$, the center $\boldsymbol{x}_{\text{p}}(t)$ of the electron cloud oscillates at the same frequency with velocity $\boldsymbol{v} = \mathrm{d}\boldsymbol{x}_{\text{p}}/\mathrm{d}t$ and constant amplitude x_0. This oscillating electron motion leads to absorption of light. The energy captured by the particle is either converted into Joule heating inside the particle or re-emitted by the particle as scattered light.

Joule heating is caused by random inelastic collisions, where any given free electron once in a while hits a silver ion and loses its total kinetic energy, which is converted into vibrations of the silver ions (heat). The average time between the collisions is $\tau \gg 1/\omega_{\text{p}}$, where for silver nanoparticle we use $\tau = 5.24 \times 10^{-15}$ s.

2.8a	Find an expression for the time-averaged Joule heating power P_{heat} in the nanoparticle as well as the time-averaged current squared $\langle I^2 \rangle$, which includes explicitly the time-averaged velocity squared $\langle v^2 \rangle$ of the electron cloud.	1.0
2.8b	Find an expression for the equivalent ohmic resistance R_{heat} in an equivalent resistor-model of the nanoparticle having the Joule heating power P_{heat} due to the electron cloud current I. Calculate the numerical value of R_{heat}.	1.0

Observations from the Greenland ice sheet show that $\delta^{18}\mathrm{O}$ in the snow varies approximately linearly with temperature, Fig. 3.2(a). Assuming that this has always been the case, $\delta^{18}\mathrm{O}$ retrieved from an ice core at depth $H_{\mathrm{m}} - z$ leads to an estimate of the temperature T near Greenland at the age $\tau(z)$.

Measurements of $\delta^{18}\mathrm{O}$ in a 3060 m long Greenlandic ice core show an abrupt change in $\delta^{18}\mathrm{O}$ at a depth of 1492 m, Fig. 3.2(b), marking the end of the last ice age. The ice age began 120,000 years ago, corresponding to a depth of 3040 m, and the current interglacial age began 11,700 years ago, corresponding to a depth of 1492 m. Assume that these two periods can be described by two different accumulation rates, c_{ia} (ice age) and c_{ig} (interglacial age), respectively. You can assume H_{m} to be constant throughout these 120,000 years.

3.7a	Determine the accumulation rates c_{ia} and c_{ig}.	0.8
3.7b	Use the data in Fig. 3.2 to find the temperature change at the transition from the ice age to the interglacial age.	0.2

Sea level rise from melting of the Greenland ice sheet

A complete melting of the Greenlandic ice sheet will cause a sea level rise in the global ocean. As a crude estimate of this sea level rise, one may simply consider a uniform rise throughout a global ocean with constant area $A_{\mathrm{O}} = 3.61 \times 10^{14}\ \mathrm{m}^2$.

3.8	Calculate the average global sea level rise, which would result from a complete melting of the Greenlandic ice sheet, given its present area of $A_{\mathrm{G}} = 1.71 \times 10^{12}\mathrm{m}^2$ and $S_{\mathrm{b}} = 100\ \mathrm{kPa}$.	0.6

The massive Greenland ice sheet exerts a gravitational pull on the surrounding ocean. If the ice sheet melts, this local high tide is lost and the sea level will drop close to Greenland, an effect which partially counteracts the sea level rise calculated above.

To estimate the magnitude of this gravitational pull on the water, the Greenlandic ice sheet is now modeled as a point mass located at the ground level and having the total mass of the Greenlandic ice sheet. Copenhagen lies at a distance of 3500 km along the Earth surface from the center of the point mass. One may consider the Earth, without the point mass, to be spherically symmetric and having a global ocean spread out over the entire surface of the Earth of area $A_{\mathrm{E}} = 5.10 \times 10^{14}\mathrm{m}^2$. All effects of rotation of the Earth may be neglected.

3.9	Within this model, determine the difference $h_{\mathrm{CPH}} - h_{\mathrm{OPP}}$ between sea levels in Copenhagen (h_{CPH}) and diametrically opposite to Greenland (h_{OPP}).	1.8

Introduction

A meteoroid is a small particle (typically smaller than 1 m) from a comet or an asteroid. A meteoroid that impacts the ground is called a meteorite.

On the night of 17 January 2009 many people near the Baltic Sea saw the glowing trail or fireball of a meteoroid falling through the atmosphere of the Earth. In Sweden a surveillance camera recorded a video of the event, see Fig. 1.1(a). From these pictures and eyewitness accounts it was possible to narrow down the impact area, and six weeks later a meteorite with the mass 0.025 kg was found in the vicinity of the town Maribo in southern Denmark. Measurements on the meteorite, now named Maribo, and its orbit in the sky showed interesting results. Its speed when entering the atmosphere had been exceptionally high. Its age, 4.567×10^9 year, shows that it had been formed shortly after the birth of the solar system. The Maribo meteorite is possibly a part of Comet Encke.

The speed of Maribo

The fireball was moving in westerly direction, heading 285° relative to north, toward the location where the meteorite was subsequently found, as sketched in Fig. 1.1. The meteorite was found at a distance 195 km from the surveillance camera in the direction 230° relative to north.

1.1	Use this and the data in Fig. 1.1 to calculate the average speed of Maribo in the time interval between frames 155 and 161. The curvature of the Earth and the gravitational force on the meteoroid can both be neglected.	1.3

Through the atmosphere and melting?

The friction from the air on a meteoroid moving in the higher atmosphere depends in a complicated way on the shape and velocity of the meteoroid, and on the temperature and density of the atmosphere. As a reasonable approximation the friction force F in the upper atmosphere is given by the expression $F = k\rho_{\text{atm}}Av^2$, where k is a constant, ρ_{atm} the density of the atmosphere, A the projected cross-section area of the meteorite, and v its speed.

The following simplifying assumptions are made to analyze the meteoroid: The object entering the atmosphere was a sphere of mass $m_{\text{M}} = 30$ kg, radius $R_{\text{M}} = 0.13$ m, temperature $T_0 = 200$ K, and speed $v_{\text{M}} = 2.91 \times 10^4$ m/s. The density of the atmosphere is constant (its value 40 km above the surface of the Earth), $\rho_{\text{atm}} = 4.1 \times 10^{-3}$ kg/m^3, and the friction coefficient is $k = 0.60$.

1.2a	Estimate how long time after entering the atmosphere it takes the meteoroid to have its speed reduced by 10 % from v_{M} to $0.90\ v_{\text{M}}$. You can neglect the gravitational force on the meteoroid and assume, that it maintains its mass and shape.	0.7
1.2b	Calculate how many times larger the kinetic energy E_{kin} of the meteoroid entering the atmosphere is than the energy E_{melt} necessary for melting it completely (see data sheet).	0.3

Data sheet: Table of physical parameters

Speed of light in vacuum	$c = 2.998 \times 10^8\ \mathrm{m\ s^{-1}}$
Planck's constant over 2π	$\hbar = 1.055 \times 10^{-34}\ \mathrm{J\ s}$
Gravitational constant	$G = 6.67 \times 10^{-11}\ \mathrm{m^3 kg^{-1} s^{-2}}$
Gravitational acceleration	$g = 9.82\ \mathrm{m\ s^{-2}}$
Elementary charge	$e = 1.602 \times 10^{-19}\ \mathrm{C}$
Electric permittivity of vacuum	$\varepsilon_0 = 8.854 \times 10^{-12}\ \mathrm{C^2\ J^{-1}\ m^{-1}}$
Electron mass	$m_e = 9.109 \times 10^{-31}\ \mathrm{kg}$
Avogadro constant	$N_\mathrm{A} = 6.022 \times 10^{23}\ \mathrm{mol^{-1}}$
Boltzmann constant	$k_\mathrm{B} = 1.381 \times 10^{-23}\ \mathrm{J\ K^{-1}}$
Stony meteorite, specific heat	$c_\mathrm{sm} = 1.2 \times 10^3\ \mathrm{J\ kg^{-1} K^{-1}}$
Stony meteorite, thermal conductivity	$k_\mathrm{sm} = 2.0\ \mathrm{W\ m^{-1}\ K^{-1}}$
Stony meteorite, density	$\rho_\mathrm{sm} = 3.3 \times 10^3\ \mathrm{kg\ m^{-3}}$
Stony meteorite, melting point	$T_\mathrm{sm} = 1.7 \times 10^3\ \mathrm{K}$
Stony meteorite, specific melting heat	$L_\mathrm{sm} = 2.6 \times 10^5\ \mathrm{J\ kg^{-1}}$
Silver, molar mass	$M_\mathrm{Ag} = 1.079 \times 10^{-1}\ \mathrm{kg\ mol^{-1}}$
Silver, density	$\rho_\mathrm{Ag} = 1.049 \times 10^4\ \mathrm{kg\ m^{-3}}$
Silver, specific heat capacity	$c_\mathrm{Ag} = 2.40 \times 10^2\ \mathrm{J\ kg^{-1} K^{-1}}$
Water, molar mass	$M_\mathrm{wa} = 1.801 \times 10^{-2}\ \mathrm{kg\ mol^{-1}}$
Water, density	$\rho_\mathrm{wa} = 0.998 \times 10^3\ \mathrm{kg\ m^{-3}}$
Water, specific heat capacity	$c_\mathrm{wa} = 4.181 \times 10^3\ \mathrm{J\ kg^{-1} K^{-1}}$
Water, heat of vaporization	$L_\mathrm{wa} = 2.260 \times 10^6\ \mathrm{J\ kg^{-1}}$
Water, boiling temperature	$T_{100} = 100\ ^\circ\mathrm{C} = 373.15\ \mathrm{K}$
Ice, density of glacier	$\rho_\mathrm{ice} = 0.917 \times 10^3\ \mathrm{kg\ m^{-3}}$
Steam, specific heat capacity	$c_\mathrm{st} = 2.080 \times 10^3\ \mathrm{J\ kg^{-1}\ K^{-1}}$
Earth, mass of the	$m_\mathrm{E} = 5.97 \times 10^{24}\ \mathrm{kg}$
Earth, radius of the	$R_\mathrm{E} = 6.38 \times 10^6\ \mathrm{m}$
Sun, mass of the	$m_\mathrm{S} = 1.99 \times 10^{30}\ \mathrm{kg}$
Sun, radius of the	$R_\mathrm{S} = 6.96 \times 10^8\ \mathrm{m}$
Average Sun-Earth distance	$a_\mathrm{E} = 1.50 \times 10^{11}\ \mathrm{m}$

Two useful formulas

In this problem you can make use of the integral:

$$\int_0^1 \sqrt{1-x}\,\mathrm{d}x = \frac{2}{3}$$

and the approximation $(1+x)^a \approx 1 + ax$, valid for $|ax| \ll 1$.

The height profile of the ice sheet

On short time scales the glacier is an incompressible hydrostatic system with fixed height profile $H(x)$.

3.1	Write down an expression for the pressure $p(x,z)$ inside the ice sheet as a function of vertical height z above the ground and distance x from the ice divide. Neglect the atmospheric pressure.	0.3

Consider a given vertical slab of the ice sheet in equilibrium, covering a small horizontal base area $\Delta x \Delta y$ between x and $x + \Delta x$, see the red dashed lines in Fig. 3.1(c). The size of Δy does not matter. The net horizontal force component ΔF on the two vertical sides of the slab, arising from the difference in height on the center-side versus the coastal-side of the slab, is balanced by a friction force $\Delta F = S_\mathrm{b}\, \Delta x \Delta y$ from the ground on the base area $\Delta x \Delta y$, where $S_\mathrm{b} = 100$ kPa.

3.2a	For a given value of x, show that in the limit $\Delta x \to 0$, $S_\mathrm{b} = kH\, \mathrm{d}H/\mathrm{d}x$, and determine k	0.9
3.2b	Determine an expression for the height profile $H(x)$ in terms of ρ_ice, g, L, S_b and distance x from the divide. The result will show, that the maximum glacier height H_m scales with the half-width L as $H_\mathrm{m} \propto L^{1/2}$.	0.8
3.2c	Determine the exponent γ with which the total volume V_ice of the ice sheet scales with the area A of the rectangular island, $V_\mathrm{ice} \propto A^\gamma$.	0.5

A dynamical ice sheet

On longer time scale, the ice is a viscous incompressible fluid, which by gravity flows from the center part to the coast. In this model, the ice maintains its height profile $H(x)$ in a steady state, where accumulation of ice due to snow fall in the central region is balanced by melting at the coast. In addition to the ice sheet geometry of Fig. 3.1(b) and (c) make the following model assumptions:

1) Ice flows in the xz-plane away from the ice divide (the y-axis).
2) The accumulation rate c (m/year) in the central region is a constant.
3) Ice can only leave the glacier by melting near the coasts at $x = \pm L$.
4) The horizontal (x-)component $v_x(x) = \mathrm{d}x/\mathrm{d}t$ of the ice-flow velocity is independent of z.
5) The vertical (z-)component $v_z(z) = \mathrm{d}z/\mathrm{d}t$ of the ice-flow velocity is independent of x.

Consider only the central region $|x| \ll L$ close to the middle of the ice sheet, where height variations of the ice sheet are very small and can be neglected altogether, i.e. $H(x) \approx H_\mathrm{m}$.

3.3	Use mass conservation to find an expression for the horizontal ice-flow velocity $v_x(x)$ in terms of c, x, and H_m.	0.6

From the assumption of incompressibility, i.e. the constant density ρ_{ice} of the ice, it follows that mass conservation implies the following restriction on the ice flow velocity components

$$\frac{dv_x}{dx} + \frac{dv_z}{dz} = 0.$$

3.4	Write down an expression for the z dependence of the vertical component $v_z(z)$ of the ice-flow velocity.	0.6

A small ice particle with the initial surface position (x_i, H_m) will, as time passes, flow as part of the ice sheet along a flow trajectory $z(x)$ in the vertical xz-plane.

3.5	Derive an expression for such a flow trajectory $z(x)$.	0.9

Age and climate indicators in the dynamical ice sheet

Based on the ice-flow velocity components $v_x(x)$ and $v_z(z)$, one can estimate the age $\tau(z)$ of the ice in a specific depth $H_m - z$ from the surface of the ice sheet.

3.6	Find an expression for the age $\tau(z)$ of the ice as a function of height z above ground, right at the ice divide $x = 0$.	1.0

An ice core drilled in the interior of the Greenland ice sheet will penetrate through layers of snow from the past, and the ice core can be analyzed to reveal past climate changes. One of the best indicators is the so-called $\delta^{18}O$, defined as

$$\delta^{18}O = \frac{R_{\text{ice}} - R_{\text{ref}}}{R_{\text{ref}}} 1000 ‰,$$

where $R = [^{18}O]/[^{16}O]$ denotes the relative abundance of the two stable isotopes ^{18}O and ^{16}O of oxygen. The reference R_{ref} is based on the isotopic composition of the oceans around Equator.

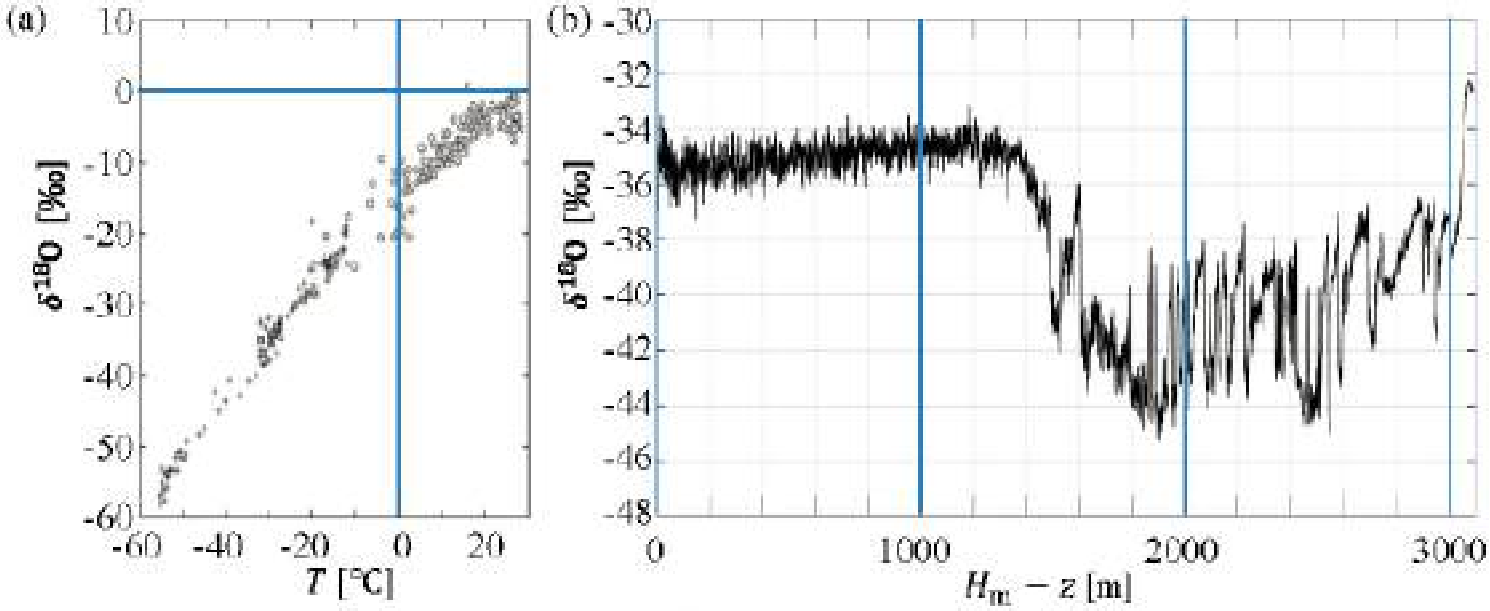


Figure 3.2 (a) Observed relationship between $\delta^{18}O$ in snow versus the mean annual surface temperature T. **(b)** Measurements of $\delta^{18}O$ versus depth $H_m - z$ from the surface, taken from an ice core drilled from surface to bedrock at a specific place along the Greenlandic ice divide where $H_m = 3060$ m.

Introduction

This problem deals with the physics of the Greenlandic ice sheet, the second largest glacier in the world, Fig. 3.1(a). As an idealization, Greenland is modeled as a rectangular island of width $2L$ and length $5L$ with the ground at sea level and completely covered by incompressible ice (constant density ρ_{ice}), see Fig. **3.1(b)**. The height profile $H(x)$ of the ice sheet does not depend on the y-coordinate and it increases from zero at the coasts $x = \pm L$ to a maximum height H_{m} along the middle north-south axis (the y-axis), known as the ice divide, see Fig. 3.1(c).

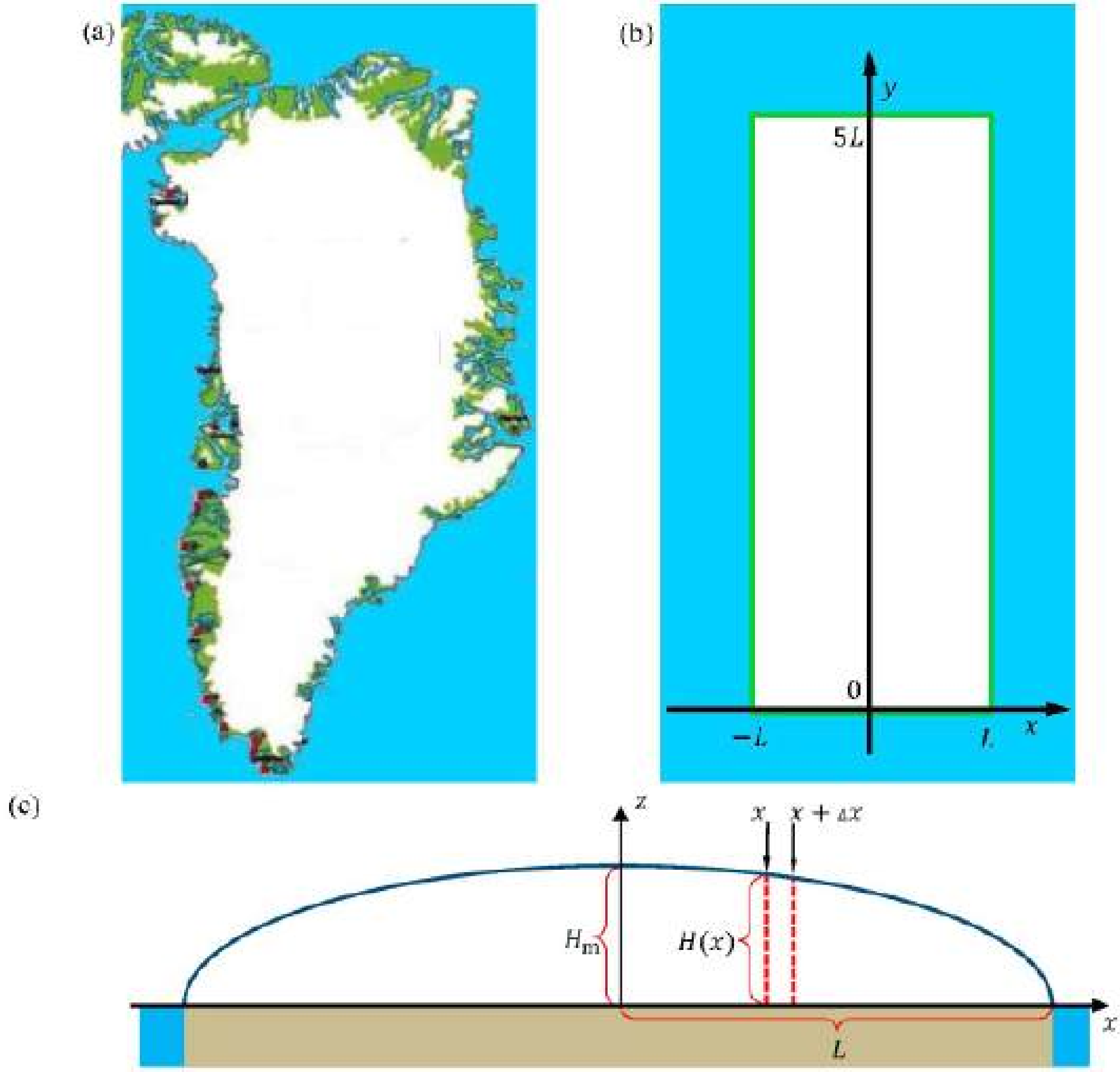


Figure 3.1 (a) A map of Greenland showing the extent of the ice sheet (white), the ice-free, coastal regions (green), and the surrounding ocean (blue). **(b)** The crude model of the Greenlandic ice sheet as covering a rectangular area in the xy-plane with side lengths $2L$ and $5L$. The ice divide, the line of maximum ice sheet height H_{m} runs along the y-axis. **(c)** A vertical cut (xz-plane) through the ice sheet showing the height profile $H(x)$ (blue line). $H(x)$ is independent of the y-coordinate for $0 < y < 5L$, while it drops abruptly to zero at $y = 0$ and $y = 5L$. The z-axis marks the position of the ice divide. For clarity, the vertical dimensions are expanded compared to the horizontal dimensions. The density ρ_{ice} of ice is constant.

Comet Encke, from which Maribo may originate

In its orbit around the Sun, the minimum and maximum distances between comet Encke and the Sun are $a_{\min} = 4.95 \times 10^{10}$ m and $a_{\max} = 6.16 \times 10^{11}$ m, respectively.

1.5	Calculate the orbital period t_{Encke} of comet Encke.	0.6

Consequences of an asteroid impact on Earth

65 million years ago Earth was hit by a huge asteroid with density $\rho_{\text{ast}} = 3.0 \times 10^3\ \text{kg m}^{-3}$, radius $R_{\text{ast}} = 5.0$ km, and final speed of $v_{\text{ast}} = 2.5 \times 10^4$ m/s. This impact resulted in the extermination of most of the life on Earth and the formation of the enormous Chicxulub Crater. Assume that an identical asteroid would hit Earth today in a completely inelastic collision, and use the fact that the moment of inertia of Earth is 0.83 times that for a homogeneous sphere of the same mass and radius. The moment of inertia of a homogeneous sphere with mass M and radius R is $\frac{2}{5}MR^2$. Neglect any changes in the orbit of the Earth.

1.6a	Let the asteroid hit the North Pole. Find the maximum change in angular orientation of the axis of Earth after the impact.	0.7
1.6b	Let the asteroid hit the Equator in a radial impact. Find the change $\Delta\tau_{\text{rat}}$ in the duration of one revolution of Earth after the impact.	0.7
1.6c	Let the asteroid hit the Equator in a tangential impact in the equatorial plane. Find the change $\Delta\tau_{\text{tan}}$ in the duration of one revolution of Earth after the impact.	0.7

Maximum impact speed

Consider a celestial body, gravitationally bound in the solar system, which impacts the surface of Earth with a speed v_{imp}. Initially the effect of the gravitational field of the Earth on the body can be neglected. Disregard the friction in the atmosphere, the effect of other celestial bodies, and the rotation of the Earth.

1.7	Calculate $v_{\text{imp}}^{\max}$, the largest possible value of v_{imp}.	1.6

Introduction

In this problem we study an efficient process of steam production that has been demonstrated to work experimentally. An aqueous solution of spherical nanometer-sized silver spheres (nanoparticles) with only about 10^{13} particles per liter is illuminated by a focused light beam. A fraction of the light is absorbed by the nanoparticles, which are heated up and generate steam locally around them without heating up the entire water solution. The steam is released from the system in the form of escaping steam bubbles. Not all details of the process are well understood at present, but the core process is known to be absorption of light through the so-called collective electron oscillations of the metallic nanoparticles. The device is known as a plasmonic steam generator.

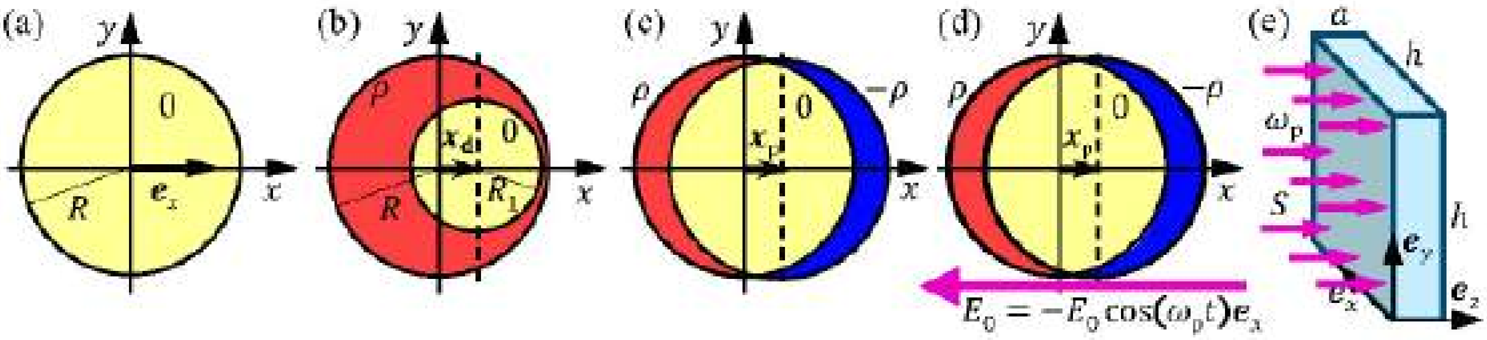


Figure 2.1 **(a)** A spherical charge-neutral nanoparticle of radius R placed at the center of the coordinate system. **(b)** A sphere with a positive homogeneous charge density ρ (red), and containing a smaller spherical charge-neutral region (0, yellow) of radius R_1, with its center displaced by $\boldsymbol{x}_\mathrm{d} = x_\mathrm{d}\,\boldsymbol{e}_x$. **(c)** The sphere with positive charge density ρ of the nanoparticle silver ions is fixed in the center of the coordinate system. The center of the spherical region with negative spherical charge density $-\rho$ (blue) of the electron cloud is displaced by $\boldsymbol{x}_\mathrm{p}$, where $x_\mathrm{p} \ll R$. **(d)** An external homogeneous electric field $\boldsymbol{E}_0 = -E_0\boldsymbol{e}_x$. For time-dependent $\boldsymbol{E}_0$, the electron cloud moves with velocity $\boldsymbol{v} = \mathrm{d}\boldsymbol{x}_\mathrm{p}/\mathrm{d}t$. **(e)** The rectangular vessel ($h \times h \times a$) containing the aqueous solution of nanoparticles illuminated by monochromatic light propagating along the z-axis with angular frequency ω_p and intensity S.

A single spherical silver nanoparticle

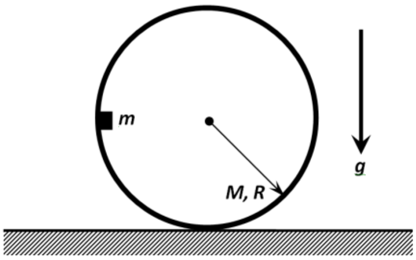
Throughout this problem we consider a spherical silver nanoparticle of radius $R = 10{,}0$ nm and with its center fixed at the origin of the coordinate system, see Fig. 2.1(a). All motions, forces and driving fields are parallel to the horizontal x-axis (with unit vector $\boldsymbol{e}_x$). The nanoparticle contains free (conduction) electrons moving within the whole nanoparticle volume without being bound to any silver atom. Each silver atom is a positive ion that has donated one such free electron.

2.1	Find the following quantities: The volume V and mass M of the nanoparticle, the number N and charge density ρ of silver ions in the particle, and for the free electrons their concentration n, their total charge Q, and their total mass m_0.	0.7

Problem 1 (9 points)

This problem consists of three independent parts.

Part A (3 points)



A small puck of mass m is carefully placed onto the inner surface of the thin hollow thin cylinder of mass M and of radius R. Initially, the cylinder rests on the horizontal plane and the puck is located at the height R above the plane as shown in the figure on the left. Find the interaction force F between the puck and the cylinder at the moment when the puck passes the lowest point of its trajectory. Assume that the friction between the puck and the inner surface of the cylinder is absent, and the cylinder moves on the plane without slipping. The free fall acceleration is g.

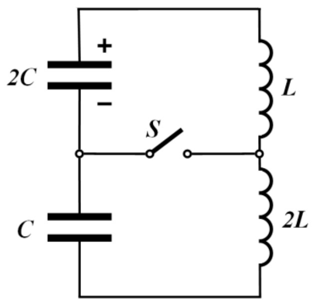
Part B (3 points)

A bubble of radius $r = 5.00$ cm, containing a diatomic ideal gas, has the soap film of thickness $h = 10.0\ \mu\text{m}$ and is placed in vacuum. The soap film has the surface tension $\sigma = 4.00 \cdot 10^{-2}\ \frac{\text{N}}{\text{m}}$ and the density $\rho = 1.10\ \frac{\text{g}}{\text{cm}^3}$. 1) Find formula for the molar heat capacity of the gas in the bubble for such a process when the gas is heated so slowly that the bubble remains in a mechanical equilibrium and evaluate it; 2) Find formula for the frequency ω of the small radial oscillations of the bubble and evaluate it under the assumption that the heat capacity of the soap film is much greater than the heat capacity of the gas in the bubble. Assume that the thermal equilibrium inside the bubble is reached much faster than the period of oscillations.

Hint: Laplace showed that there is pressure difference between inside and outside of a curved surface, caused by surface tension of the interface between liquid and gas, so that $\Delta p = \frac{2\sigma}{r}$.

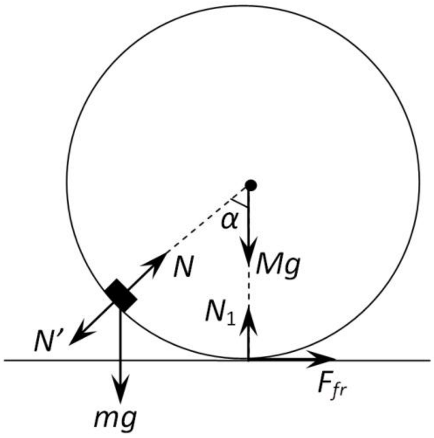
Part C (3 points)

Initially, a switch S is unshorted in the circuit shown in the figure on the right, a capacitor of capacitance $2C$ carries the electric charge q_0, a capacitor of capacitance C is uncharged, and there are no electric currents in both coils of inductance L and $2L$, respectively. The capacitor starts to discharge and at the moment when the current in the coils reaches its maximum value, the switch S is instantly shorted. Find the maximum current $I_{\max}$ through the switch S thereafter.



Problem 1
Solution
Part A

Consider the forces acting on the puck and the cylinder and depicted in the figure on the right. The puck is subject to the gravity force mg and the reaction force from the cylinder N. The cylinder is subject to the gravity force Mg, the reaction force from the plane N_1, the friction force F_{fr} and the pressure force from the puck $N' = -N$. The idea is to write the horizontal projections of the equations of motion. It is written for the puck as follows



$$ma_x = N\sin\alpha, \quad \text{(A.1)}$$

where a_x is the horizontal projection of the puck acceleration.

For the cylinder the equation of motion with the acceleration w is found as

$$Mw = N\sin\alpha - F_{fr}. \quad \text{(A.2)}$$

Since the cylinder moves along the plane without sliding its angular acceleration is obtained as

$$\varepsilon = w/R \quad \text{(A.3)}$$

Then the equation of rotational motion around the center of mass of the cylinder takes the form

$$I\varepsilon = F_{fr}R, \quad \text{(A.4)}$$

where the inertia moment of the hollow cylinder is given by

$$I = MR^2. \quad \text{(A.5)}$$

Solving (A.2)-(A.5) yields

$$2Mw = N\sin\alpha. \quad \text{(A.6)}$$

From equations (A.1) and (A.6) it is easily concluded that

$$ma_x = 2Mw. \quad \text{(A.7)}$$

Since the initial velocities of the puck and of the cylinder are both equal to zero, then, it follows from (A.7) after integrating that

$$mu = 2Mv. \quad \text{(A.8)}$$

It is obvious that the conservation law for the system is written as

$$mgR = \frac{mu^2}{2} + \frac{Mv^2}{2} + \frac{I\omega^2}{2}, \quad \text{(A.9)}$$

where the angular velocity of the cylinder is found to be

$$\omega = \frac{v}{R}, \quad \text{(A.10)}$$

since it does not slide over the plane.

Solving (A.8)-(A.10) results in velocities at the lowest point of the puck trajectory written as

$$u = 2\sqrt{\frac{MgR}{(2M+m)}}, \quad \text{(A.12)}$$

$$v = \frac{m}{M}\sqrt{\frac{MgR}{(2M+m)}}. \quad \text{(A.13)}$$

In the reference frame sliding progressively along with the cylinder axis, the puck moves in a circle of radius R and, at the lowest point of its trajectory, have the velocity

$$v_{rel} = u + v \quad \text{(A.14)}$$

and the acceleration

$$a_{\text{rel}} = \frac{v_{\text{rel}}^2}{R}. \quad \text{(A.15)}$$

At the lowest point of the puck trajectory the acceleration of the cylinder axis is equal to zero, therefore, the puck acceleration in the laboratory reference frame is also given by (A.15).

$$F - mg = \frac{mv_{rel}^2}{R}. \quad \text{(A.16)}$$

then the interaction force between the puck and the cylinder is finally found as

$$F = 3mg\left(1 + \frac{m}{3M}\right). \quad \text{(A.17)}$$

Part B

1) According to the first law of thermodynamics, the amount of heat transmitted δQ to the gas in the bubble is found as

$$\delta Q = \nu C_V dT + p dV, \tag{B.1}$$

where the molar heat capacity at arbitrary process is as follows

$$C = \frac{1}{\nu}\frac{\delta Q}{dT} = C_V + \frac{p}{\nu}\frac{dV}{dT}. \tag{B.2}$$

Here C_V stands for the molar heat capacity of the gas at constant volume, p designates its pressure, ν is the total amount of moles of gas in the bubble, V and T denote the volume and temperature of the gas, respectively.

Evaluate the derivative standing on the right hand side of (B.2). According to the Laplace formula, the gas pressure inside the bubble is defined by

$$p = \frac{4\sigma}{r}, \tag{B.3}$$

thus, the equation of any equilibrium process with the gas in the bubble is a polytrope of the form

$$p^3 V = \text{const}. \tag{B.4}$$

The equation of state of an ideal gas has the form

$$pV = \nu RT, \tag{B.5}$$

and hence equation (B.4) can be rewritten as

$$T^3 V^{-2} = \text{const}. \tag{B.6}$$

Differentiating (B.6) the derivative with respect to temperature sought is found as

$$\frac{dV}{dT} = \frac{3V}{2T}. \tag{B.7}$$

Taking into account that the molar heat capacity of a diatomic gas at constant volume is

$$C_V = \frac{5}{2}R, \tag{B.8}$$

and using (B.5) it is finally obtained that

$$C = C_V + \frac{3}{2}R = 4R = 33.2\ \frac{\text{J}}{\text{mole}\cdot\text{K}}. \tag{B.9}$$

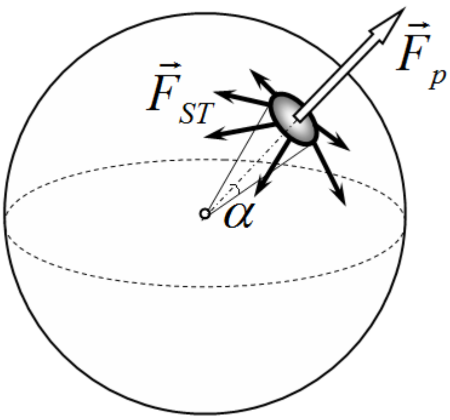
2) Since the heat capacity of the gas is much smaller than the heat capacity of the soap film, and there is heat exchange between them, the gas can be considered as isothermal since the soap film plays the role of thermostat. Consider the fragment of soap film, limited by the angle α as shown in the figure. It's area is found as

$$S = \pi(\alpha r)^2. \tag{B.10}$$

and the corresponding mass is obtained as

$$m = \rho S h. \tag{B.11}$$

Let x be an increase in the radius of the bubble, then the Newton second law for the fragment of the soap film mentioned above takes the form



$$m\ddot{x} = p' S' - F_{surf}, \tag{B.12}$$

where F_{surf} denotes the projection of the resultant surface tension force acting in the radial direction, p' stands for the gas pressure beneath the surface of the soap film and

$$S' = S\left(1 + 2\frac{x}{r}\right).$$

F_{surf} is easily found as

$$F_{surf} = F_{ST}\alpha = \sigma \cdot 2 \cdot 2\pi[(r + x)\alpha] \cdot \alpha. \tag{B.13}$$

Since the gaseous process can be considered isothermal, it is written that

$$p'V' = pV. \tag{B.14}$$

Assuming that the volume increase is quite small, (B.14) yields

$$p' = p\frac{1}{\left(1+\frac{x}{r}\right)^3} \approx p\frac{1}{\left(1+\frac{3x}{r}\right)} \approx p\left(1 - \frac{3x}{r}\right). \tag{B.15}$$

Thus, from (B.10) - (B.16) and (B.3) the equation of small oscillations of the soap film is derived as

$$\rho h\ddot{x} = -\frac{8\sigma}{r^2}x \tag{B.16}$$

with the frequency

$$\omega = \sqrt{\frac{8\sigma}{\rho h r^2}} = 108\ \mathrm{s}^{-1}. \tag{B.17}$$

Part C

The problem can be solved in different ways. Herein several possible solutions are considered.

Method 1. Direct approach

At the moment when the current in the coils is a maximum, the total voltage across the coils is equal to zero, so the capacitor voltages must be equal in magnitude and opposite in polarity. Let U be a voltage on the capacitors at the time moment just mentioned and I_0 be that maximum current. According to the law of charge conservation

$$q_0 = 2CU + CU, \tag{C1.1}$$

thus,

$$U = \frac{q_0}{3C}. \tag{C1.2}$$

Then, from the energy conservation law

$$\frac{q_0^2}{2\cdot 2C} = \frac{LI_0^2}{2} + \frac{2LI_0^2}{2} + \frac{CU^2}{2} + \frac{2CU^2}{2} \tag{C1.3}$$

the maximum current is found as

$$I_0 = \frac{q_0}{3\sqrt{2LC}}. \tag{C1.4}$$

After the key K is shortened there will be independent oscillations in both circuits with the frequency

$$\omega = \frac{1}{\sqrt{2LC}}, \tag{C1.5}$$

and their amplitudes are obtained from the corresponding energy conservation laws written as

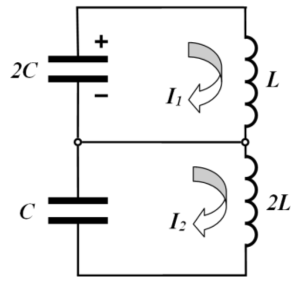
$$\frac{2CU^2}{2} + \frac{LI_0^2}{2} = \frac{LJ_1^2}{2}, \tag{C1.6}$$

$$\frac{CU^2}{2} + \frac{2LI_0^2}{2} = \frac{2LJ_2^2}{2}. \tag{C1.7}$$

Hence, the corresponding amplitudes are found as

$$J_1 = \sqrt{5}I_0, \tag{C1.8}$$

$$J_2 = \sqrt{2}I_0. \tag{C1.9}$$



Choose the positive directions of the currents in the circuits as shown in the figure on the right. Then, the current flowing through the key is written as follows

$$I = I_1 - I_2. \tag{C1.10}$$

The currents depend on time as

$$I_1(t) = A\cos\omega t + B\sin\omega t, \tag{C1.11}$$

$$I_2(t) = D\cos\omega t + F\sin\omega t, \tag{C1.12}$$

The constants A, B, D, F can be determined from the initial values of the currents and their amplitudes by putting down the following set of equations

$$I_1(0) = A = I_0, \tag{C1.13}$$

$$A^2 + B^2 = J_1^2, \tag{C1.14}$$

$$I_2(0) = D = I_0, \tag{C1.15}$$

$$D^2 + F^2 = J_2^2. \tag{C1.16}$$

Solving (C1.13)-(C1.16) it is found that

$$B = 2I_0, \tag{C1.17}$$

$$F = -I_0, \tag{C1.18}$$

The sign in F is chosen negative, since at the time moment of the key shortening the current in the coil $2L$ decreases.

Thus, the dependence of the currents on time takes the following form

$$I_1(t) = I_0(\cos\omega t + 2\sin\omega t), \tag{C1.19}$$

$$I_2(t) = I_0(\cos\omega t - \sin\omega t). \tag{C1.20}$$

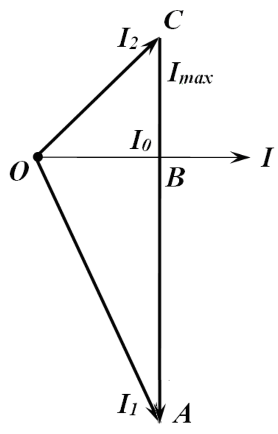
In accordance with (C.10), the current in the key is dependent on time according to

$$I(t) = I_1(t) - I_2(t) = 3I_0\sin\omega t. \tag{C1.21}$$

Hence, the amplitude of the current in the key is obtained as

$$I_{\max} = 3I_0 = \omega q_0 = \frac{q_0}{\sqrt{2LC}}. \tag{C1.22}$$

Method 2. Vector diagram



Instead of determining the coefficients A, B, D, F the vector diagram shown in the figure on the right can be used. The segment AC represents the current sought and its projection on the current axis is zero at the time of the key shortening. The current I_1 in the coil of inductance L grows at the same time moment because the capacitor $2C$ continues to discharge, thus, this current is depicted in the figure by the segment OA. The current I_2 in the coil of inductance $2L$ decreases at the time of the key shortening since it continues to charge the capacitor $2C$, that is why this current is depicted in the figure by the segment OC.

It is known for above that $OB = I_0, OA = \sqrt{5}I_0, OC = \sqrt{2}I_0$. Hence, it is found from the Pythagorean theorem that

$$AB = \sqrt{OA^2 - OB^2} = 2I_0, \quad \text{(C2.1)}$$

$$BC = \sqrt{OC^2 - OB^2} = I_0, \quad \text{(C2.2)}$$

Thus, the current sought is found as

$$I_{\max} = AC = AB + BC = 3I_0 = \omega q_0 = \frac{q_0}{\sqrt{2LC}}. \quad \text{(C2.3)}$$

Method 3. Heuristic approach

It is clear that the current through the key performs harmonic oscillations with the frequency

$$\omega = \frac{1}{\sqrt{2LC}}. \quad \text{(C3.1)}$$

and it is equal to zero at the time of the key shortening, i.e.

$$I(t) = I_{\max} \sin \omega t. \quad \text{(C3.2)}$$

Since the current is equal to zero at the time of the key shortening, then the current amplitude is equal to the current derivative at this time moment divided by the oscillation frequency. Let us find that current derivative. Let the capacitor of capacitance $2C$ have the charge q_1. Then the charge on the capacitor of capacitance C is found from the charge conservation law as

$$q_2 = q_0 - q_1. \quad \text{(C3.3)}$$

After shortening the key the rate of current change in the coil of inductance L is obtained as

$$\dot{I}_1 = \frac{q_1}{2LC}, \quad \text{(C3.4)}$$

whereas in the coil of inductance $2L$ it is equal to

$$\dot{I}_2 = -\frac{q_0 - q_1}{2LC}. \quad \text{(C3.5)}$$

Since the voltage polarity on the capacitors are opposite, then the current derivative with respect to time finally takes the form

$$\dot{I} = \dot{I}_1 - \dot{I}_2 = \frac{q_0}{2LC} = \omega^2 q_0. \quad \text{(C3.6)}$$

Note that this derivative is independent of the time of the key shortening!

Hence, the maximum current is found as

$$I_{\max} = \frac{\dot{I}}{\omega} = \omega q_0 = \frac{q_0}{\sqrt{2LC}}, \quad \text{(C3.7)}$$

and it is independent of the time of the key shortening!

Problem 2. Van der Waals equation of state (11 points)

In a well-known model of an ideal gas, whose equation of state obeys the Clapeyron-Mendeleev law, the following important physical effects are neglected. First, molecules of a real gas have a finite size and, secondly, they interact with one another. In all parts of this problem *one mole of water* is considered.

Part A. Non-ideal gas equation of state (2 points)

Taking into account the finite size of the molecules, the gaseous equation of state takes the form

$$P(V-b)=RT, \tag{1}$$

where P, V, T stands for the gas pressure, its volume per mole and temperature, respectively, R denotes the universal gas constant, and b is a specific constant extracting some volume.

A1	Estimate b and express it in terms of the diameter of the molecules d. **(0.3 points)**

With account of intermolecular attraction forces, van der Waals proposed the following equation of state that neatly describes both the gaseous and liquid states of matter

$$\left(P+\frac{a}{V^2}\right)(V-b)=RT. \tag{2}$$

where a is another specific constant.

At temperatures T below a certain critical value T_c the isotherm of equation (2) is well represented by a non-monotonic curve 1 shown in Figure 1 which is then called van der Waals isotherm. In the same figure curve 2 shows the isotherm of an ideal gas at the same temperature. A real isotherm differs from the van der Waals isotherm by a straight segment $\boldsymbol{AB}$ drawn at some constant pressure P_{LG}. This straight segment is located between the volumes V_L and V_G, and corresponds to the equilibrium of the liquid phase (indicated by L) and the gaseous phase (referred to by G). From the second law of thermodynamics J. Maxwell showed that the pressure P_{LG} must be chosen such that the areas *I* and *II* shown in Figure 1 must be equal.

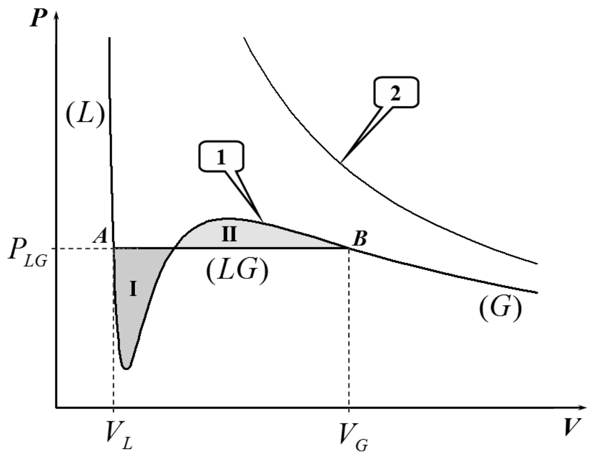


Figure 1. Van der Waals isotherm of gas/liquid (curve 1) and the isotherm of an ideal gas (curve 2).

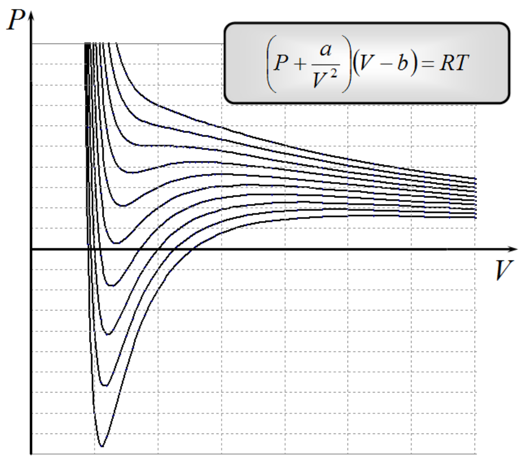


Figure 2. Several isotherms for van der Waals equation of state.

With increasing temperature the straight segment AB on the isotherm shrinks to a single point when the temperature and the pressure reaches some values T_c and $P_{LG}=P_c$, respectively. The parameters P_c and T_c are called critical and can be measured experimentally with high degree of accuracy.

A2	Express the van der Waals constants a and b in terms of T_c and P_c. **(1.3 points)**
A3	For water $T_c = 647$ K and $P_c = 2.2\cdot 10^7$ Pa. Calculate a_w and b_w for water. **(0.2 points)**
A4	Estimate the diameter of water molecules d_w. **(0.2 points)**

Part B. Properties of gas and liquid (6 points)

This part of the problem deals with the properties of water in the gaseous and liquid states at temperature $T = 100$ °C. The saturated vapor pressure at this temperature is known to be $p_{LG} = p_0 = 1.0 \cdot 10^5$ Pa, and the molar mass of water is $\mu = 1.8 \cdot 10^{-2}\ \frac{\text{kg}}{\text{mole}}$.

Gaseous state

It is reasonable to assume that the inequality $V_G \gg b$ is valid for the description of water properties in a gaseous state.

B1	Derive the formula for the volume V_G and express it in terms of R, T, p_0, and a. **(0.8 points)**

Almost the same volume V_{G0} can be approximately evaluated using the ideal gas law.

B2	Evaluate in percentage the relative decrease in the gas volume due to intermolecular forces, $\frac{\Delta V_G}{V_{G0}} = \frac{V_{G0} - V_G}{V_{G0}}$. **(0.3 points)**

If the system volume is reduced below V_G, the gas starts to condense. However, thoroughly purified gas can remain in a mechanically metastable state (called supercooled vapor) until its volume reaches a certain value $V_{G\min}$.

The condition of mechanical stability of supercooled gas at constant temperature is written as: $\frac{dP}{dV} < 0$.

B3	Find and evaluate how many times the volume of water vapor can be reduced and still remains in a metastable state. In other words, what is $V_G / V_{G\min}$? **(0.7 points)**

Liquid state

For the van der Waals' description of water in a liquid state it is reasonable to assume that the following inequality holds $P \ll a/V^2$.

B4	Express the volume of liquid water V_L in terms of a, b, R, and T. **(1.0 points)**

Assuming that $bRT \ll a$, find the following characteristics of water. *Do not be surprised if some of the data evaluated do not coincide with the well-known tabulated values!*

B5	Express the liquid water density ρ_L in some of the terms of μ, a, b, R and evaluate it. **(0.5 points)**
B6	Express the volume thermal expansion coefficient $\alpha = \frac{1}{V_L}\frac{\Delta V_L}{\Delta T}$ in terms of a, b, R, and evaluate it. **(0.6 points)**
B7	Express the specific heat of water vaporization L in terms of μ, a, b, R and evaluate it. **(1.1 points)**
B8	Considering the monomolecular layer of water, estimate the surface tension σ of water. **(1.2 points)**

Part C. Liquid-gas system (3 points)

From Maxwell's rule (equalities of areas, by applying trivial integration) and the van der Waals' equation of state together with the approximations made in Part B, it can be shown that the saturated vapor pressure p_{LG} depends on temperature T as follows

$$\ln p_{LG} = A + \frac{B}{T}, \tag{3}$$

where A and B are some constants, that can be expressed in terms of a and b as $A = \ln\left(\frac{a}{b^2}\right) - 1\,;\ B = -\frac{a}{bR}$

W. Thomson showed that the pressure of saturated vapor depends on the curvature of the liquid surface. Consider a liquid that does not wet the material of a capillary (contact angle 180°). When the capillary is immersed into the liquid, the liquid in the capillary drops to a certain level because of the surface tension (see Figure 3).

C1	Find a small change in pressure Δp_T of the saturated vapor over the curved surface of liquid and express it in terms of the vapor density ρ_s, the liquid density ρ_L, the surface tension σ and the radius of surface curvature r. **(1.3 points)**

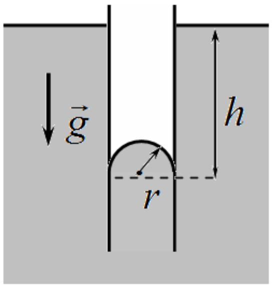


Figure 3. Capillary immersed in a liquid that does not wet its material

Metastable states, considered in part B3, are widely used in real experimental setups, such as the cloud chamber designed for registration of elementary particles. They also occur in natural phenomena such as the formation of morning dew. Supercooled vapor is subject to condensation by forming liquid droplets. Very small droplets evaporate quickly but large enough ones can still grow.

C2	Suppose that at the evening temperature of $t_e = 20$ °C the water vapor in the air was saturated, but in the morning the ambient temperature has fallen by a small amount of $\Delta t = 5.0$ °C. Assuming that the vapor pressure has remained unchanged, estimate the minimum radius of droplets that can grow. Use the tabulated value of water surface tension $\sigma = 7.3 \cdot 10^{-2}\ N/m$. **(1.7 points)**

Problem 2.Van der Waals equation of state

Solution

Part A. Non-ideal gas equation of state

A1. If $V = b$is substituted into the equation of state, then the gas pressure turns infinite. It is obvious that this is the moment when all the molecules are tightly packed. Therefore, the parameter b is approximately equal to the volume of all molecules, i.e.

$$b = N_A d^3 \quad (A1.1)$$

A2. In the most general case thevan der Waals equation of state can be rewritten as

$$P_c V^3 - (RT_c + bP_c)V^2 + aV - ab = 0 \quad (A2.1).$$

Since at the critical values of the gas parameters the straight line disappears, then, the solution of (A2.1) must have one real triple root, i.e. it can be rewritten as follows

$$P_c (V - V_c)^3 = 0 \quad (A2.2).$$

Comparing the coefficients of expression (A2.1) and (A2.2), the following set of equations is obtained

$$\begin{cases} 3P_c V_c = RT_c + bP_c \\ 3P_c V_c^2 = a \\ P_c V_c^3 = ab \end{cases} \quad (A2.3).$$

Solution to the set (A2.3) is the following formulas for the van der Waals coefficients

$$a = \frac{27R^2T_c^2}{64P_c} \quad (A2.4),$$

$$b = \frac{RT_c}{8P_c} \quad (A2.5).$$

Alternative solution

The critical parameters are achieved in the presence of an inflection point in the isotherm, at which the first and second derivatives are both zero. Therefore, they are defined by thefollowingconditions

$$\left(\frac{dP}{dV}\right)_T = 0 \quad (A2.6),$$

and

$$\left(\frac{d^2P}{dV^2}\right)_T = 0 \quad (A2.7).$$

Thus, the following set of equations is obtained

$$\begin{cases} -\frac{RT_c}{(V_c-b)^2} + \frac{2a}{V_c^3} = 0 \\ \frac{2RT_c}{(V_c-b)^3} - \frac{6a}{V_c^4} = 0 \\ \left(P_c + \frac{a}{V_c^2}\right)(V_c - b) = RT_c \end{cases} \quad (A2.8),$$

which has the same solution (A2.4) and (A2.5).

A3. Numericalcalculationsforwaterproduce the following result

$$a_w = 0.56\,\frac{\mathrm{m^6 \cdot Pa}}{\mathrm{mole^2}} \quad (A3.1).$$

$$b_w = 3.1 \cdot 10^{-5}\,\frac{\mathrm{m^3}}{\mathrm{mole}} \quad (A3.2).$$

A4. From equations (A1.4) and (A3.2) it is found that

$$d_w = \sqrt[3]{\frac{b}{N_A}} = 3.7 \cdot 10^{-10}\mathrm{m} \approx 4 \cdot 10^{-10}\mathrm{m} \quad (A4.1).$$

Part B. Properties of gas and liquid

B1. Usingtheinequality$V_G \gg b$, the van der Waals equation of state can be written as

$$\left(p_0 + \frac{a}{V_G^2}\right)V_G = RT \quad (B1.1),$$

which has the following solutions

$$V_G = \frac{RT}{2p_0}\left(1 \pm \sqrt{1 - \frac{4ap_0}{R^2T^2}}\right) \quad (B1.2).$$

Smaller root in (B1.2) gives the volume in an unstable state on the rising branch of thevan der Waals isotherm. The volume of gas is given by the larger root, since at $a = 0$an expression for the volume of an ideal gasshould be obtained, i.e.

$$V_G = \frac{RT}{2p_0}\left(1 + \sqrt{1 - \frac{4ap_0}{R^2T^2}}\right) \quad \text{(B1.3).}$$

For given values of the parameters the value$\frac{ap_0}{(RT)^2} = 5.8 \cdot 10^{-3}$. It can therefore be assumed that$\frac{ap_0}{(RT)^2} \ll 1$, then (B1.3)takes the form

$$V_G \approx \frac{RT}{p_0}\left(1 - \frac{ap_0}{R^2T^2}\right) = \frac{RT}{p_0} - \frac{a}{RT} \quad \text{(B1.4).}$$

B2. For an ideal gas

$$V_{G0} = \frac{RT}{p_0} \quad \text{(B2.1),}$$

hence,

$$\left(\frac{\Delta V_G}{V_{G0}}\right) = \frac{V_{G0} - V_G}{V_{G0}} = \frac{1}{2}\left(1 - \sqrt{1 - \frac{4ap_0}{R^2T^2}}\right) \approx \frac{ap_0}{R^2T^2} = 0.58\%. \quad \text{(B2.2)}$$

B3. Mechanical stability of a thermodynamic system is inpower provided that

$$\left(\frac{dP}{dV}\right)_T < 0. \quad \text{(B3.1)}$$

The minimum volume, in which the mattercan still exist in the gaseous state, corresponds to a point in which

$$V_{Gmin} \to \left(\frac{dP}{dV}\right)_T = 0 \quad \text{(B3.2).}$$

Using the van der Waals equation of state (B3.2) is written as

$$\left(\frac{dP}{dV}\right)_T = -\frac{RT}{(V-b)^2} + \frac{2a}{V^3} = 0 \quad \text{(B3.3).}$$

From (B3.2) and (B3.3), and with the help of$V_{Gmin} \gg b$, it is found that

$$V_{Gmin} = \frac{2a}{RT} \quad \text{(B3.4).}$$

Thus,

$$\frac{V_G}{V_{Gmin}} = \frac{R^2T^2}{2ap_0} = 86 \quad \text{(B3.5).}$$

B4. Usingtheinequality$P \ll a/V^2$, the van der Waals equation of state is written as

$$\frac{a}{V_L^2}(V_L - b) = RT, \quad \text{(B4.1)}$$

whose solution is

$$V_L = \frac{a}{2RT}\left(1 \pm \sqrt{1 - \frac{4bRT}{a}}\right) \quad \text{(B4.2).}$$

In this case, the smaller root shouldbe taken, since at$T \to 0$the liquid volume$V_L = b$ must be obtained according to (B4.1), i.e.

$$V_L = \frac{a}{2RT}\left(1 - \sqrt{1 - \frac{4bRT}{a}}\right) \approx b\left(1 + \frac{bRT}{a}\right). \quad \text{(B4.3).}$$

B5. Since (B4.3) givesthevolumeoftheonemoleofwaterits mass density is easily found as

$$\rho_L = \frac{\mu}{V_L} = \frac{\mu}{b\left(1 + \frac{bRT}{a}\right)} \approx \frac{\mu}{b} = 5.8 \cdot 10^2 \, \frac{\text{kg}}{\text{m}^3} \quad \text{(B5.1).}$$

B6. Inaccordancewith (B4.3) the volume thermal expansion coefficient is derived as

$$\alpha = \frac{1}{V_L}\frac{\Delta V_L}{\Delta T} = \frac{bR}{a + bRT} \approx \frac{bR}{a} = 4.6 \cdot 10^{-4} \text{K}^{-1} \quad \text{(B6.1).}$$

B7. The heat, required to convert the liquid to gas, is used to overcome the intermolecular forces that create negative pressure a/V^2, therefore,

$$E = L\mu \approx \int_{V_L}^{V_G} \frac{a}{V^2} dV = a\left(\frac{1}{V_L} - \frac{1}{V_G}\right) \quad \text{(B7.1),}$$

and using$V_G \gg V_L$, (B7.1) yields

$$L = \frac{a}{\mu V_L} = \frac{a}{\mu b\left(1 + \frac{bRT}{a}\right)} \approx \frac{a}{\mu b} = 1.0 \cdot 10^6 \, \frac{\text{J}}{\text{kg}} \quad \text{(B7.2).}$$

B8. Consider some water of volume$V$. To make a monolayer of thickness d out of it, the following work must be done

$$A = 2\sigma S \qquad \text{(B8.1).}$$

Fabrication of the monomolecular layer may be interpreted as the evaporation of an equivalent volume of water which requires the following amount of heat

$$Q = Lm \qquad \text{(B8.2),}$$

where the mass is given by

$$m = \rho S d \qquad \text{(B8.3).}$$

Using (A4.1a), (B5.1)and(B7.2), one finally gets

$$\sigma = \frac{a}{2b^2} d_w = 0.12 \cdot 10^{-2}\,\frac{\mathrm{N}}{\mathrm{m}} \qquad \text{(B8.4).}$$

Part C. Liquid-gas systems

C1. At equilibrium, the pressure in the liquid and gas should be equalat all depths. The pressure$p$in the fluid at the depth $h$is related to the pressure of saturated vapor above the flat surface by

$$p = p_0 + \rho_L g h \qquad \text{(C1.1).}$$

The surface tension creates additional pressure defined by the Laplace formula as

$$\Delta p_L = \frac{2\sigma}{r} \qquad \text{(C1.2).}$$

The same pressure$p$inthefluidatthedepth$h$ depends on the vapor pressure p_h over the curved liquid surface and its radiusofcurvature as

$$p = p_h + \frac{2\sigma}{r} \qquad \text{(C1.3).}$$

Furthermore, the vapor pressure at different heights are related by

$$p_h = p_0 + \rho_S g h \qquad \text{(C1.4).}$$

Solving (C1.1)-(C1.4), it is found that

$$h = \frac{2\sigma}{(\rho_L - \rho_S) g r} \qquad \text{(C1.5).}$$

Hence,the pressure difference sought is obtained as

$$\Delta p_T = p_h - p_0 = \rho_S g h = \frac{2\sigma}{r}\frac{\rho_S}{\rho_L - \rho_S} \approx \frac{2\sigma}{r}\frac{\rho_S}{\rho_L}. \qquad \text{(C1.6).}$$

Note that the vapor pressure over the convex surface of the liquid is larger than the pressure above the flat surface.

C2. Let $P_e$be vapor pressure at a temperature T_e, and $P_e - \Delta P_e$be vapor pressure at a temperature $T_e - \Delta T_e$. In accordance with equation (3) from problem statement, whentheambient temperature falls by an amount of $\Delta T_e$the saturated vapor pressure changes by an amount

$$\Delta P_e = P_e \frac{a}{bRT_e^2} \Delta T_e \qquad \text{(C2.1).}$$

In accordance with the Thomson formula obtained in part **C1**, the pressure of saturated vapor above the droplet increases by the amountof Δp_T. While a droplet is small in size, the vapor above its surface remains unsaturated. Whena droplet hasgrownuptoacertainminimumsize, thevaporaboveitssurface turns saturated.

Since the pressure remains unchanged, the following condition must hold

$$P_e - \Delta P_e + \Delta p_T = P_e \qquad \text{(C2.2).}$$

Assuming the vapor is almost ideal gas, its density can be found as

$$\rho_S = \frac{\mu P_e}{RT_e} \ll \rho_L \qquad \text{(C2.3).}$$

From equations (C2.1)-(C2.3), (B5.1) and (C1.6) one finds

$$\frac{2\sigma}{r}\frac{\mu P_e}{RT_e\left(\frac{\mu}{b}\right)} = P_e \frac{a\Delta T_e}{bRT_e^2} \qquad \text{(C2.4).}$$

Thus, it is finally obtained that

$$r = \frac{2\sigma b^2 T_e}{a \Delta T_e} = 1.5 \cdot 10^{-8}\mathrm{m} \qquad \text{(C2.5).}$$

Problem3. Simplest model of gas discharge (10 points)

An electric current flowing through a gas is called a gas discharge. There are many types of gas discharges including glow discharge in lighting lamps, arc discharge inwelding and the well known spark discharge that occurs between the clouds and the earth in the form of lightning.

Part A. Non-self-sustained gas discharge (4.8points)

In this part of the problem the so-called non-self-sustained gas discharge is studied. To maintain it a permanent operationan external ionizer is needed, which creates Z_{ext} pairs of singly ionized ions and free electrons per unit volume and per unit time uniformly in the volume.

When an external ionizer is switched on, the number of electrons and ions starts to grow. Unlimited increase in the number densities of electrons and ions in the gas is prevented by the recombination process in which a free electron recombineswith an ion to form a neutral atom. The number of recombining eventsZ_{rec} that occurs in the gas per unit volume and per unit time is given by

$$Z_{\text{rec}} = r n_e n_i,$$

where r is a constant called the recombination coefficient, and n_e,$n_i$denotethe electron and ion number densities, respectively.

Suppose that at time $t = 0$the external ionizer is switched on and the initial number densities of electrons and ions in the gas are both equal to zero. Then, the electron number density $n_e(t)$ depends on time$t$as follows:

$$n_e(t) = n_0 + a \tanh bt,$$

where n_0, a and $b$are some constants, and $\tanh x$stands for the hyperbolic tangent.

A1	Find$n_0, a, b$and express them in terms ofZ_{ext} and$r$.**(1.8points)**

Assume that there are two external ionizers available. When the first one is switched on, the electron number density in the gas reaches its equilibrium value of be $n_{e1} = 12 \cdot 10^{10}\,\text{cm}^{-3}$. When the second external ionizer is switched on, the electron number density reaches its equilibrium value of $n_{e2} = 16 \cdot 10^{10}\,\text{cm}^{-3}$.

A2	Find the electron number density$n_e$at equilibriumwhen both external ionizers are switched on simultaneously.**(0.6 points)**

Attention!*In what follows it is assumed that the external ionizer is switched on for quite long period of time such that all processes have become stationary and do not depend on time. Completely neglect the electric field due the charge carriers.*

Assume that the gas fills in the tube between the two parallel conductive plates of area S separated by the distance $L \ll \sqrt{S}$ from each other. The voltage U is applied across the plates to create an electric field between them. Assume that the number densities of both kinds of charge carriers remain almost constant along the tube.

Assumethat both the electrons (denoted by thesubscript$e$) and the ions (denoted by the subscript$i$) acquire the same ordered speed$v$due to the electric field strength$E$ found as

$$v = \beta E,$$

where $b$isa constant called charge mobility.

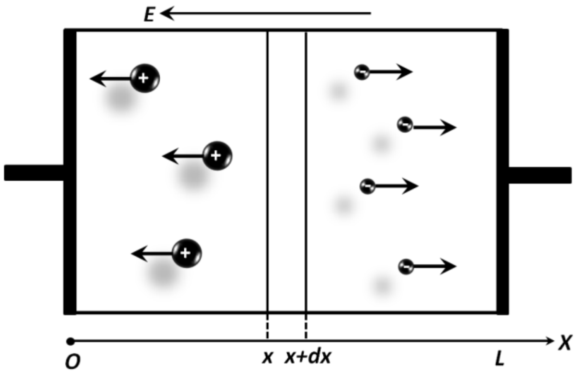
A3	Express the electric current$I$in the tube in terms of $U, \beta, L, S, Z_{\text{ext}}, r$ and $e$whichis the elementary charge.**(1.7 points)**
A4	Find the resistivityρ_{gas} of the gas at sufficiently small values of the voltage applied and express it in terms of$\beta, L, Z_{\text{ext}}, r$ and e.**(0.7 points)**

PartB. Self-sustained gas discharge (5.2 points)

In this part of the problemthe ignition of the self-sustained gas discharge is consideredto show how the electric current in the tube becomes self-maintaining.

Attention!*In the sequel assume that the external ionizer continues to operatewith the same* Z_{ext} *rate, neglect the electric field due to the charge carriers such that the electric field is uniform along the tube, and the recombination can be completely ignored.*

For the self-sustained gas discharge there are two important processes not considered above. The first process isasecondary electron emission, and the second one isa formation of electron avalanche. The secondary electron emission occurs when ions hit on the negative electrode, called a cathode, and the electrons are knocked out of it to move towards the positive electrode, called an anode. The ratio of the number of the knocked electrons$\dot{N}_e$per unit timeto the number of ions$\dot{N}_i$ hitting the cathode per unit time is called the coefficient of the secondary electron emission,$\gamma = \dot{N}_e/\dot{N}_i$.The formation of the electron avalanche isexplained as follows. The electric field accelerates free electrons which acquire enough kinetic energy to ionize the atoms in the gas by hitting them. As a result the number of free electrons moving towards the anodesignificantly increases. This process is described by the Townsend coefficient α, which characterizes an increase in the number of electronsdN_edue to moving N_e electrons that have passed the distancedl, i.e.



$$\frac{dN_e}{dl} = \alpha N_e.$$

The total current$I$ in any cross section of the gas tube consists of the ion $I_i(x)$and the electron$I_e(x)$ currents which, in the steady state, depend on the coordinate x, shown in the figureabove. The electron current $I_e(x)$ varies along the x-axis according to the formula

$$I_e(x) = C_1 e^{A_1 x} + A_2,$$

where A_1, A_2, C_1 are some constants.

B1	Find$A_1, A_2$and express them in terms of$Z_{\text{ext}}, \alpha, e, L, S$.**(2 points)**

The ion current $I_i(x)$ varies along the x-axis according to the formula

$$I_i(x) = C_2 + B_1 e^{B_2 x},$$

where B_1, B_2, C_2 are some constants.

B2	Find$B_1, B_2$and express them in terms of$Z_{\text{ext}}, \alpha, e, L, S, C_1$.**(0.6 points)**
B3	Write down the condition for $I_i(x)$at$x = L$.**(0.3 points)**
B4	Write down the condition for$I_i(x)$and$I_e(x)$at $x = 0$.**(0.6 points)**
B5	Find the total current$I$and express it in terms of$Z_{\text{ext}}, \alpha, \gamma, e, L, S$.Assume that it remains finite **(1.2 points)**

Let the Townsend coefficient αbe constant. When the length of the tube turns out greater than some critical value, i.e. $L > L_{\text{cr}}$, the external ionizer can be turned off and the discharge becomes self-sustained.

B6	Find$L_{cr}$and express it in terms of$Z_{\text{ext}}, \alpha, \gamma, e, L, S$.**(0.5 points)**

Problem 3. Simplest model of gas discharge

Solution

Part A. Non-self-sustained gas discharge

A1. Let us derive an equation describing the change of the electron number density with time. It is determined by the two processes; the generation of ion pairs by external ionizer and the recombination of electrons with ions. At ionization process electrons and ions are generated in pairs, and at recombination processthey disappear in pairs as well.Thus, their concentrations are alwaysequal at any given time, i.e.

$$n(t) = n_e(t) = n_i(t) \qquad \text{(A1.1).}$$

Then the equation describing the numberdensityevolution of electrons and ions in time can be written as

$$\frac{dn(t)}{dt} = Z_{ext} - rn(t)^2 \qquad \text{(A1.2).}$$

It is easy to show that at$t \to 0$ the function $\tanh bt \to 0$, therefore, by virtue of the initial condition $n(0) = 0$,one finds

$$n_0 = 0 \qquad \text{(A1.3).}$$

Substituting $n_e(t) = a \tanh bt$ in (A1.2) and separating it in the independent functions (hyperbolic, or 1 ande^x), one gets

$$a = \sqrt{\frac{Z_{ext}}{r}} \qquad \text{(A1.4),}$$

$$b = \sqrt{rZ_{ext}} \qquad \text{(A1.5).}$$

A2. According to equation (A1.4) the number density of electronsat steady-state is expressed in terms of the external ionizer activity as

$$n_{e1} = \sqrt{\frac{Z_{ext\,1}}{r}} \qquad \text{(A2.1),}$$

$$n_{e2} = \sqrt{\frac{Z_{ext\,2}}{r}} \qquad \text{(A2.2),}$$

$$n_e = \sqrt{\frac{Z_{ext\,1}+Z_{ext\,2}}{r}} \qquad \text{(A2.3).}$$

Thus,the following analogue of the Pythagorean theorem is obtained as

$$n_e = \sqrt{n_{e1}^2 + n_{e2}^2} = 20.0 \cdot 10^{10}\,\mathrm{cm}^{-3}. \qquad \text{(A2.4)}$$

A3. In the steady state, the balance equations of electrons and ions in the tube volume take the form

$$Z_{ext}SL = rn_e n_i SL + \frac{I_e}{e} \qquad \text{(A3.1),}$$

$$Z_{ext}SL = rn_e n_i SL + \frac{I_i}{e} \qquad \text{(A3.2).}$$

It follows from equations (A3.1) and (A3.2) that the ion and electron currents are equal, i.e.

$$I_e = I_i \qquad \text{(A3.3).}$$

At the same time the total current in each tube section is the sum of the electron and ion currents

$$I = I_e + I_i \qquad \text{(A3.4).}$$

By definition ofthe current density the following relations hold

$$I_e = \frac{I}{2} = en_e vS = e\beta n_e ES \qquad \text{(A3.5),}$$

$$I_i = \frac{I}{2} = en_i vS = e\beta n_i ES \qquad \text{(A3.6).}$$

Substituting (A3.5) and (A3.6) into (A3.1) and (A3.2), the following quadratic equation for the current is derived

$$Z_{ext}SL = rSL\left(\frac{I}{2e\beta ES}\right)^2 + \frac{I}{2e} \qquad \text{(A3.7).}$$

The electric field strength in the gas is equal to

$$E = \frac{U}{L} \qquad \text{(A3.8).}$$

and solution to the quadratic equation (A3.7) takes the form

$$I = \frac{e\beta^2 U^2 S}{rL^3}\left(-1 \pm \sqrt{1 + \frac{4rZ_{ext}L^4}{\beta^2 U^2}}\right) \qquad \text{(A3.9).}$$

It is obvious that only positive root does make sense, i.e.

$$I = \frac{e\beta^2 U^2 S}{rL^3}\left(\sqrt{1+\frac{4rZ_{ext}L^4}{\beta^2 U^2}}-1\right) \quad \text{(A3.10).}$$

A4. At low voltages (A3.10) simplifies and gives the following expression

$$I = 2Ue\beta\sqrt{\frac{Z_{ext}}{r}}\frac{S}{L}. \quad \text{(A4.1)}$$

which is actually the Ohm law.

Using the well-known relation

$$R = \frac{U}{I} \quad \text{(A4.2)}$$

together with

$$R = \rho\frac{L}{S} \quad \text{(A4.3),}$$

one gets

$$\rho = \frac{1}{2e\beta}\sqrt{\frac{r}{Z_{ext}}} \quad \text{(A4.4).}$$

Part B. Self-sustained gas discharge

B1. Consider a gas layer located between $x$and$x + dx$.The rate of change in the electron number inside the layer due to the electric current is givenfor a small time interval dt by

$$dN_e^I = \frac{I_e(x+dx)-I_e(x)}{e}dt = \frac{1}{e}\frac{dI_e(x)}{dx}dxdt. \quad \text{(B1.1).}$$

This change is due to the effect of the external ionization and the electron avalanche formation.

The external ionizer creates the following number of electrons in the volumeSdx

$$dN_e^{ext} = Z_{ext}Sdxdt \quad \text{(B1.2).}$$

whereas the electron avalanche produces the number of electrons found as

$$dN_e^a = \alpha N_e dl = n_e Sdxvdt = \alpha\frac{I_e(x)}{e}dxdt \quad \text{(B1.3).}$$

The balance equationfor the number of electrons is written as

$$dN_e^I = dN_e^{ext} + dN_e^a \quad \text{(B1.4),}$$

whichresults in the following differential equation for the electron current

$$\frac{dI_e(x)}{dx} = eZ_{ext}S + \alpha I_e(x) \quad \text{(B1.5).}$$

On substituting$I_e(x) = C_1e^{A_1x} + A_2$,one derives

$$A_1 = \alpha \quad \text{(B1.6),}$$

$$A_2 = -\frac{eZ_{ext}S}{\alpha} \quad \text{(B1.7).}$$

B2. Given the fact that the ions flow in the direction opposite to the electron motion,the balance equationfor the number of ionsis written as

$$dN_i^I = dN_i^{ext} + dN_i^a \quad \text{(B2.1),}$$

where

$$dN_i^I = \frac{I_i(x)-I_i(x+dx)}{e}dt = -\frac{1}{e}\frac{dI_i(x)}{dx}dxdt \quad \text{(B2.2).}$$

$$dN_i^{ext} = Z_{ext}Sdxdt \quad \text{(B2.3).}$$

$$dN_i^a = \alpha\frac{I_e(x)}{e}dxdt \quad \text{(B2.4).}$$

Hence, the following differential equation for the ion current is obtained

$$-\frac{dI_i(x)}{dx} = eZ_{ext}S + \alpha I_e(x). \quad \text{(B2.5)}$$

Onsubstituting the previouslyfound electron current together with the ion current,$I_i(x) = C_2 + B_1e^{B_2x}$,yields

$$B_1 = -C_1 \quad \text{(B2.6),}$$

$$B_2 = \alpha \quad \text{(B2.7).}$$

B3. Sincetheionsstartstomovefrom the anode located at$x = L$, the following condition holds

$$I_i(L) = 0 \quad \text{(B3.1).}$$

B4. By definition of secondary electron emission coefficient the following condition should be imposed

$$I_e(0) = \gamma I_i(0) \quad \text{(B4.1).}$$

B5. Total current in each tube section is the sum of the electron and ion currents:

$$I = I_e + I_i = C_2 - \frac{eZ_{ext}S}{\alpha} \quad \text{(B5.1).}$$

Aftersubstituting the boundary conditions (B3.1) and (B4.1):

$$C_2 - C_1 e^{\alpha L} = 0 \quad \text{(B5.2)}$$

and

$$C_1 - \frac{eZ_{ext}S}{\alpha} = \gamma(C_2 - C_1) \quad \text{(B5.3).}$$

Solving (B5.2) and (B5.3) one can obtain:

$$C_2 = \frac{eZ_{ext}S}{\alpha}\left(\frac{1}{e^{-\alpha L}(1+\gamma)-\gamma}\right) \quad \text{(B5.4).}$$

So the total current:

$$I = \frac{eZ_{ext}S}{\alpha}\left(\frac{1}{e^{-\alpha L}(1+\gamma)-\gamma} - 1\right) \quad \text{(B5.5).}$$

B6. When the discharge gap length is increased, the denominator in formula (B5.1) decreases. At that moment, when it turns zero, the electric current in the gas becomes self-sustaining and external ionizer can be turned off. Thus,

$$L_{cr} = \frac{1}{\alpha}\ln\left(1 + \frac{1}{\gamma}\right) \quad \text{(B6.1).}$$

Particles from the Sun[1] (Total Marks: 10)

Photons from the surface of the Sun and neutrinos from its core can tell us about solar temperatures and also confirm that the Sun shines because of nuclear reactions.

Throughout this problem, take the mass of the Sun to be $M_\odot = 2.00 \times 10^{30}$ kg, its radius, $R_\odot = 7.00 \times 10^8$ m, its luminosity (radiation energy emitted per unit time), $L_\odot = 3.85 \times 10^{26}$ W, and the Earth-Sun distance, $d_\odot = 1.50 \times 10^{11}$ m.

Note:

(i) $\int xe^{ax}dx = \left(\frac{x}{a} - \frac{1}{a^2}\right)e^{ax} + \text{ constant}$

(ii) $\int x^2e^{ax}dx = \left(\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3}\right)e^{ax} + \text{ constant}$

(iii) $\int x^3e^{ax}dx = \left(\frac{x^3}{a} - \frac{3x^2}{a^2} + \frac{6x}{a^3} - \frac{6}{a^4}\right)e^{ax} + \text{ constant}$

A Radiation from the sun :

A1	Assume that the Sun radiates like a perfect blackbody. Use this fact to calculate the temperature, T_s, of the solar surface.	**0.3**

The spectrum of solar radiation can be approximated well by the Wien distribution law. Accordingly, the solar energy incident on any surface on the Earth per unit time per unit frequency interval, $u(\nu)$, is given by

$$u(\nu) = A\frac{R_\odot^2}{d_\odot^2}\frac{2\pi h}{c^2}\nu^3 \exp(-h\nu/k_B T_s),$$

where ν is the frequency and A is the area of the surface normal to the direction of the incident radiation.

Now, consider a solar cell which consists of a thin disc of semiconducting material of area, A, placed perpendicular to the direction of the Sun's rays.

A2	Using the Wien approximation, express the total radiated solar power, P_{in}, incident on the surface of the solar cell, in terms of A, $R_\odot$, $d_\odot$, T_s and the fundamental constants c, h, k_B.	**0.3**
A3	Express the number of photons, $n_\gamma(\nu)$, per unit time per unit frequency interval incident on the surface of the solar cell in terms of A, $R_\odot$, $d_\odot$, T_s, ν and the fundamental constants c, h, k_B.	**0.2**

The semiconducting material of the solar cell has a "band gap" of energy, E_g. We assume the following model. Every photon of energy $E \geq E_g$ excites an electron across the band gap. This electron contributes an energy, E_g, as the useful output energy, and any extra energy is dissipated as heat (not converted to useful energy).

A4	Define $x_g = h\nu_g/k_B T_s$ where $E_g = h\nu_g$. Express the useful output power of the cell, P_{out}, in terms of x_g, A, $R_\odot$, $d_\odot$, T_s and the fundamental constants c, h, k_B.	**1.0**
A5	Express the efficiency, η, of this solar cell in terms of x_g.	**0.2**
A6	Make a qualitative sketch of η versus x_g. The values at $x_g = 0$ and $x_g \to \infty$ should be clearly shown. What is the slope of $\eta(x_g)$ at $x_g = 0$ and $x_g \to \infty$?	**1.0**
A7	Let x_0 be the value of x_g for which η is maximum. Obtain the cubic equation that gives x_0. Estimate the value of x_0 within an accuracy of ± 0.25. Hence calculate $\eta(x_0)$.	**1.0**
A8	The band gap of pure silicon is $E_g = 1.11$ eV. Calculate the efficiency, η_{Si}, of a silicon solar cell using this value.	**0.2**

[1] *Amol Dighe (TIFR), Anwesh Mazumdar (HBCSE-TIFR) and Vijay A. Singh (ex-National Coordinator, Science Olympiads) were the principal authors of this problem. The contributions of the Academic Committee, Academic Development Group and the International Board are gratefully acknowledged.*

In the late nineteenth century, Kelvin and Helmholtz (KH) proposed a hypothesis to explain how the Sun shines. They postulated that starting as a very large cloud of matter of mass, $M_{\odot}$, and negligible density, the Sun has been shrinking continuously. The shining of the Sun would then be due to the release of gravitational potential energy through this slow contraction.

A9	Let us assume that the density of matter is uniform inside the Sun. Find the total gravitational potential energy, Ω, of the Sun at present, in terms of G, $M_{\odot}$ and $R_{\odot}$.	**0.3**
A10	Estimate the maximum possible time, τ_{KH} (in years), for which the Sun could have been shining, according to the KH hypothesis. Assume that the luminosity of the Sun has been constant throughout this period.	**0.5**

The τ_{KH} calculated above does not match the age of the solar system estimated from studies of meteorites. This shows that the energy source of the Sun cannot be purely gravitational.

B Neutrinos from the Sun :

In 1938, Hans Bethe proposed that nuclear fusion of hydrogen into helium in the core of the Sun is the source of its energy. The net nuclear reaction is:

$$4\,{}^{1}\mathrm{H} \longrightarrow {}^{4}\mathrm{He} + 2\mathrm{e}^{+} + 2\nu_{\mathrm{e}}$$

The "electron neutrinos", ν_{e}, produced in this reaction may be taken to be massless. They escape the Sun and their detection on the Earth confirms the occurrence of nuclear reactions inside the Sun. Energy carried away by the neutrinos can be neglected in this problem.

B1	Calculate the flux density, Φ_{ν}, of the number of neutrinos arriving at the Earth, in units of $\mathrm{m}^{-2}\mathrm{s}^{-1}$. The energy released in the above reaction is $\Delta E = 4.0 \times 10^{-12}\mathrm{J}$. Assume that the energy radiated by the Sun is entirely due to this reaction.	**0.6**

Travelling from the core of the Sun to the Earth, some of the electron neutrinos, ν_e, are converted to other types of neutrinos, ν_{x}. The efficiency of the detector for detecting ν_{x} is 1/6 of its efficiency for detecting ν_{e}. If there is no neutrino conversion, we expect to detect an average of N_1 neutrinos in a year. However, due to the conversion, an average of N_2 neutrinos (ν_{e} and ν_{x} combined) are actually detected per year.

B2	In terms of N_1 and N_2, calculate what fraction, f, of ν_{e} is converted to ν_{x}.	**0.4**

In order to detect neutrinos, large detectors filled with water are constructed. Although the interactions of neutrinos with matter are very rare, occasionally they knock out electrons from water molecules in the detector. These energetic electrons move through water at high speeds, emitting electromagnetic radiation in the process. As long as the speed of such an electron is greater than the speed of light in water (refractive index, n), this radiation, called Cherenkov radiation, is emitted in the shape of a cone.

B3	Assume that an electron knocked out by a neutrino loses energy at a constant rate of α per unit time, while it travels through water. If this electron emits Cherenkov radiation for a time, Δt, determine the energy imparted to this electron (E_{imparted}) by the neutrino, in terms of α, Δt, n, m_{e} and c. (Assume the electron to be at rest before its interaction with the neutrino.)	**2.0**

The fusion of H into He inside the Sun takes place in several steps. Nucleus of ${}^{7}\mathrm{Be}$ (rest mass, m_{Be}) is produced in one of these intermediate steps. Subsequently, it can absorb an electron, producing a ${}^{7}\mathrm{Li}$ nucleus (rest mass, $m_{\mathrm{Li}} < m_{\mathrm{Be}}$) and emitting a ν_{e}. The corresponding nuclear reaction is:

$${}^{7}\mathrm{Be} + \mathrm{e}^{-} \longrightarrow {}^{7}\mathrm{Li} + \nu_{\mathrm{e}}$$

When a Be nucleus ($m_{\mathrm{Be}} = 11.65 \times 10^{-27}$ kg) is at rest and absorbs an electron also at rest, the emitted neutrino has energy $E_{\nu} = 1.44 \times 10^{-13}$ J. However, the Be nuclei are in random thermal motion due to the temperature T_{c} at the core of the Sun, and act as moving neutrino sources. As a result, the energy of emitted neutrinos fluctuates with a root mean square (rms) value ΔE_{rms}.

B4	If ΔE_{rms}=5.54×10^{-17}J, calculate the rms speed of the Be nuclei, V_{Be}, and hence estimate T_{c}. (Hint: ΔE_{rms} depends on the rms value of the component of velocity along the line of sight).	**2.0**

Particles from the Sun[1]

Photons from the surface of the Sun and neutrinos from its core can tell us about solar temperatures and also confirm that the Sun shines because of nuclear reactions.

Throughout this problem, take the mass of the Sun to be $M_\odot = 2.00 \times 10^{30}$ kg, its radius, $R_\odot = 7.00 \times 10^{8}$ m, its luminosity (radiation energy emitted per unit time), $L_\odot = 3.85 \times 10^{26}$ W, and the Earth-Sun distance, $d_\odot = 1.50 \times 10^{11}$ m.

Note:

(i) $\displaystyle\int x e^{ax} dx = \left(\frac{x}{a} - \frac{1}{a^2}\right) e^{ax} + \text{constant}$

(ii) $\displaystyle\int x^2 e^{ax} dx = \left(\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3}\right) e^{ax} + \text{constant}$

(iii) $\displaystyle\int x^3 e^{ax} dx = \left(\frac{x^3}{a} - \frac{3x^2}{a^2} + \frac{6x}{a^3} - \frac{6}{a^4}\right) e^{ax} + \text{constant}$

A. Radiation from the Sun :

(A1) Assume that the Sun radiates like a perfect blackbody. Use this fact to calculate the temperature, T_s , of the solar surface. [0.3]

Solution:

Stefan's law: $L_\odot = (4\pi R_\odot^2)(\sigma T_s^4)$

$$T_s = \left(\frac{L_\odot}{4\pi R_\odot^2 \sigma}\right)^{1/4} = 5.76 \times 10^3 \text{ K}$$

The spectrum of solar radiation can be approximated well by the Wien distribution law. Accordingly, the solar energy incident on any surface on the Earth per unit time per unit frequency interval, $u(\nu)$, is given by

$$u(\nu) = A \frac{R_\odot^2}{d_\odot^2} \frac{2\pi h}{c^2} \nu^3 \exp(-h\nu/k_B T_s),$$

where A is the area of the surface normal to the direction of the incident radiation.

Now, consider a solar cell which consists of a thin disc of semiconducting material of area, A, placed perpendicular to the direction of the Sun's rays.

(A2) Using the Wien approximation, express the total power, P_{in}, incident on the surface of the solar cell, in terms of A, $R_\odot$, $d_\odot$, T_s and the fundamental constants c, h, k_B. [0.3]

[1] *Amol Dighe (TIFR), Anwesh Mazumdar (HBCSE-TIFR) and Vijay A. Singh (ex-National Coordinator, Science Olympiads) were the principal authors of this problem. The contributions of the Academic Committee, Academic Development Group and the International Board are gratefully acknowledged.*

Solution:

$$P_{\text{in}} = \int_0^\infty u(\nu)d\nu = \int_0^\infty A\,\frac{R_\odot^2}{d_\odot^2}\,\frac{2\pi h}{c^2}\,\nu^3 \exp(-h\nu/k_{\text{B}}T_{\text{s}})d\nu$$

Let $x = \dfrac{h\nu}{k_{\text{B}}T_{\text{s}}}$. Then, $\nu = \dfrac{k_{\text{B}}T_{\text{s}}}{h}x \qquad d\nu = \dfrac{k_{\text{B}}T_{\text{s}}}{h}dx.$

$$P_{\text{in}} = \frac{2\pi h A R_\odot^2}{c^2 d_\odot^2}\,\frac{(k_{\text{B}}T_{\text{s}})^4}{h^4}\int_0^\infty x^3 e^{-x}dx = \frac{2\pi k_{\text{B}}^4}{c^2h^3}\,T_{\text{s}}{}^4\,A\,\frac{R_\odot^2}{d_\odot^2}\cdot 6 = \frac{12\pi k_{\text{B}}^4}{c^2h^3}\,T_{\text{s}}{}^4\,A\,\frac{R_\odot^2}{d_\odot^2}$$

(A3) Express the number of photons, $n_\gamma(\nu)$, per unit time per unit frequency interval incident on the surface of the solar cell in terms of A, $R_\odot$, $d_\odot$, T_{s} ν and the fundamental constants c, h, k_{B}. [0.2]

Solution:

$$\begin{aligned} n_\gamma(\nu) &= \frac{u(\nu)}{h\nu} \\ &= A\,\frac{R_\odot^2}{d_\odot^2}\,\frac{2\pi}{c^2}\,\nu^2 \exp(-h\nu/k_{\text{B}}T_{\text{s}}) \end{aligned}$$

The semiconducting material of the solar cell has a "band gap" of energy, E_{g}. We assume the following model. Every photon of energy $E \geq E_{\text{g}}$ excites an electron across the band gap. This electron contributes an energy, E_{g} , as the useful output energy, and any extra energy is dissipated as heat (not converted to useful energy).

(A4) Define $x_{\text{g}} = h\nu_{\text{g}}/k_{\text{B}}T_{\text{s}}$ where $E_{\text{g}} = h\nu_{\text{g}}$. Express the useful output power of the cell, P_{out}, in terms of x_{g}, A, $R_\odot$, $d_\odot$, T_{s} and the fundamental constants c, h, k_{B}. [1.0]

Solution:

The useful power output is the useful energy quantum per photon, $E_{\text{g}} \equiv h\nu_{\text{g}}$, multiplied by the number of photons with energy, $E \geq E_{\text{g}}$.

$$\begin{aligned} P_{\text{out}} &= h\nu_{\text{g}} \int_{\nu_{\text{g}}}^\infty n_\gamma(\nu)d\nu \\ &= h\nu_{\text{g}}\,A\,\frac{R_\odot^2}{d_\odot^2}\,\frac{2\pi}{c^2}\int_{\nu_{\text{g}}}^\infty \nu^2 \exp(-h\nu/k_{\text{B}}T_{\text{s}})d\nu \\ &= k_{\text{B}}T_{\text{s}}x_{\text{g}}\,A\,\frac{R_\odot^2}{d_\odot^2}\,\frac{2\pi}{c^2}\left(\frac{k_{\text{B}}T_{\text{s}}}{h}\right)^3\int_{x_{\text{g}}}^\infty x^2e^{-x}dx \\ &= \frac{2\pi k_{\text{B}}^4}{c^2h^3}\,T_{\text{s}}{}^4\,A\,\frac{R_\odot^2}{d_\odot^2}\,x_{\text{g}}(x_{\text{g}}^2+2x_{\text{g}}+2)e^{-x_{\text{g}}} \end{aligned}$$

(A5) Express the efficiency, η, of this solar cell in terms of x_{g}. [0.2]

Solution:

Efficiency $\eta = \dfrac{P_{\text{out}}}{P_{\text{in}}} = \dfrac{x_{\text{g}}}{6}(x_{\text{g}}^2 + 2x_{\text{g}} + 2)e^{-x_{\text{g}}}$

(A6) Make a qualitative sketch of η versus x_{g}. The values at $x_{\text{g}} = 0$ and $x_{\text{g}} \to \infty$ should be clearly shown. What is the slope of $\eta(x_{\text{g}})$ at $x_{\text{g}} = 0$ and $x_{\text{g}} \to \infty$? [1.0]

Solution:

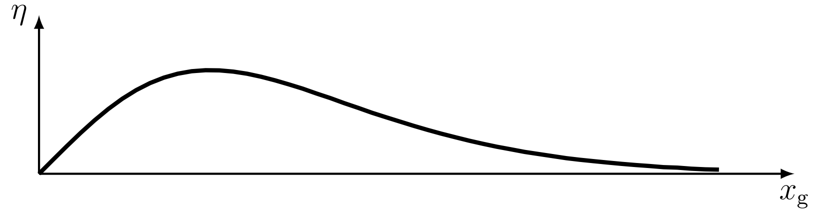
$$\eta = \frac{1}{6}(x_{\text{g}}^3 + 2x_{\text{g}}^2 + 2x_{\text{g}})e^{-x_{\text{g}}}$$

Put limiting values, $\eta(0) = 0 \qquad \eta(\infty) = 0.$

Since the polynomial has all positive coefficients, it increases monotonically; the exponential function decreases monotonically. Therefore, η has only one maximum.

$$\frac{\mathrm{d}\eta}{\mathrm{d}x_{\text{g}}} = \frac{1}{6}(-x_{\text{g}}^3 + x_{\text{g}}^2 + 2x_{\text{g}} + 2)e^{-x_{\text{g}}}$$

$$\left.\frac{\mathrm{d}\eta}{\mathrm{d}x_{\text{g}}}\right|_{x_{\text{g}}=0} = \frac{1}{3} \qquad \left.\frac{\mathrm{d}\eta}{\mathrm{d}x_{\text{g}}}\right|_{x_{\text{g}}\to\infty} = 0$$



(A7) Let x_0 be the value of x_{g} for which η is maximum. Obtain the cubic equation that gives x_0. Estimate the value of x_0 within an accuracy of ± 0.25. Hence calculate $\eta(x_0)$. [1.0]

Solution:

The maximum will be for $\dfrac{\mathrm{d}\eta}{\mathrm{d}x_{\text{g}}} = \dfrac{1}{6}(-x_{\text{g}}^3 + x_{\text{g}}^2 + 2x_{\text{g}} + 2)e^{-x_{\text{g}}} = 0$

$$\Rightarrow p(x_{\text{g}}) \equiv x_{\text{g}}^3 - x_{\text{g}}^2 - 2x_{\text{g}} - 2 = 0$$

<u>A Numerical Solution by the Bisection Method:</u>

Now,

$$\begin{aligned}
p(0) &= -2 \\
p(1) &= -4 \\
p(2) &= -2 \\
p(3) &= 10 && \Rightarrow \quad 2 < x_0 < 3 \\
p(2.5) &= 2.375 && \Rightarrow \quad 2 < x_0 < 2.5 \\
p(2.25) &= -0.171 && \Rightarrow \quad 2.25 < x_0 < 2.5
\end{aligned}$$

The approximate value of x_{g} where η is maximum is $x_0 = 2.27$.

Alternative methods leading to the same result are acceptable.

$\eta(2.27) = 0.457$

(A8) The band gap of pure silicon is $E_{\mathrm{g}} = 1.11\,\mathrm{eV}$. Calculate the efficiency, η_{Si}, of a silicon solar cell using this value. [0.2]

Solution:

$$x_{\mathrm{g}} = \frac{1.11 \times 1.60 \times 10^{-19}}{1.38 \times 10^{-23} \times 5763} = 2.23$$

$$\eta_{\mathrm{Si}} = \frac{x_{\mathrm{g}}}{6}(x_{\mathrm{g}}^2 + 2x_{\mathrm{g}} + 2)e^{-x_{\mathrm{g}}} = 0.457$$

In the late nineteenth century, Kelvin and Helmholtz (KH) proposed a hypothesis to explain how the Sun shines. They postulated that starting as a very large cloud of matter of mass, $M_\odot$, and negligible density, the Sun has been shrinking continuously. The shining of the Sun would then be due to the release of gravitational energy through this slow contraction.

(A9) Let us assume that the density of matter is uniform inside the Sun. Find the total gravitational potential energy, Ω, of the Sun at present, in terms of G, $M_\odot$and $R_\odot$. [0.3]

Solution:

The total gravitational potential energy of the Sun: $\Omega = -\int_0^{M_\odot} \frac{Gm\,\mathrm{d}m}{r}$

For constant density, $\rho = \frac{3M_\odot}{4\pi R_\odot^3}$ $\qquad m = \frac{4}{3}\pi r^3 \rho$ $\qquad \mathrm{d}m = 4\pi r^2 \rho \mathrm{d}r$

$$\Omega = -\int_0^{R_\odot} G\left(\frac{4}{3}\pi r^3 \rho\right)(4\pi r^2 \rho)\frac{\mathrm{d}r}{r} = -\frac{16\pi^2 G\rho^2}{3}\frac{R_\odot^5}{5} = -\frac{3}{5}\frac{GM_\odot^2}{R_\odot}$$

(A10) Estimate the maximum possible time τ_{KH} (in years), for which the Sun could have been shining, according to the KH hypothesis. Assume that the luminosity of the Sun has been constant throughout this period. [0.5]

Solution:

$$\tau_{\mathrm{KH}} = \frac{-\Omega}{L_\odot}$$

$$\tau_{\mathrm{KH}} = \frac{3GM_\odot^2}{5R_\odot L_\odot} = 1.88 \times 10^7 \mathrm{years}$$

The τ_{KH} calculated above does not match the age of the solar system estimated from studies of meteorites. This shows that the energy source of the Sun cannot be purely gravitational.

B. Neutrinos from the Sun:

In 1938, Hans Bethe proposed that nuclear fusion of hydrogen into helium in the core of the Sun is the source of its energy. The net nuclear reaction is:

$$4\,{}^1\mathrm{H} \longrightarrow {}^4\mathrm{He} + 2\mathrm{e}^+ + 2\nu_\mathrm{e}$$

The "electron neutrinos", ν_e, produced in this reaction may be taken to be massless. They escape the Sun and their detection on Earth confirms the occurrence of nuclear reactions inside the Sun. Energy carried away by the neutrinos can be neglected in this problem.

(B1) Calculate the flux density, Φ_ν, of the number of neutrinos arriving at the Earth, in units of $\mathrm{m}^{-2}\,\mathrm{s}^{-1}$. The energy released in the above reaction is $\Delta E = 4.0 \times 10^{-12}$ J. Assume that the energy radiated by the Sun is almost entirely due to this reaction. [0.6]

Solution:

$$4.0 \times 10^{-12}\ \mathrm{J} \leftrightarrow 2\nu$$

$$\Rightarrow \Phi_\nu = \frac{L_\odot}{4\pi d_\odot^2\,\delta E} \times 2 = \frac{3.85 \times 10^{26}}{4\pi \times (1.50 \times 10^{11})^2 \times 4.0 \times 10^{-12}} \times 2 = 6.8 \times 10^{14}\,\mathrm{m}^{-2}\,\mathrm{s}^{-1}.$$

Travelling from the core of the Sun to the Earth, some of the electron neutrinos, ν_e, are converted to other types of neutrinos, ν_x. The efficiency of the detector for detecting ν_x is 1/6th of its efficiency for detecting ν_e. If there is no neutrino conversion, we expect to detect an average of N_1 neutrinos in a year. However, due to the conversion, an average of N_2 neutrinos (ν_e and ν_x combined) are actually detected per year.

(B2) In terms of N_1 and N_2, calculate what fraction, f, of ν_e is converted to ν_x. [0.4]

Solution:

$$\begin{aligned} N_1 &= \epsilon N_0 \\ N_e &= \epsilon N_0(1-f) \\ N_x &= \epsilon N_0 f/6 \\ N_2 &= N_e + N_x \end{aligned}$$

OR

$$(1-f)N_1 + \frac{f}{6}N_1 = N_2$$

$$\Rightarrow f = \frac{6}{5}\left(1 - \frac{N_2}{N_1}\right)$$

In order to detect neutrinos, large detectors filled with water are constructed. Although the interactions of neutrinos with matter are very rare, occasionally they knock out electrons from water molecules in the detector. These energetic electrons move through water at high speeds, emitting electromagnetic radiation in the process. As long as the speed of such an electron is greater than the speed of light in water (refractive index, n), this radiation, called Cherenkov radiation, is emitted in the shape of a cone.

(B3) Assume that an electron knocked out by a neutrino loses energy at a constant rate of α per unit time, while it travels through water. If this electron emits Cherenkov radiation for a time Δt, determine the energy imparted to this electron (E_{imparted}) by the neutrino, in terms of $\alpha, \Delta t, n, m_{\text{e}}, c$. (Assume the electron to be at rest before its interaction with the neutrino.) [2.0]

Solution:

When the electron stops emitting Cherenkov radiation, its speed has reduced to $v_{\text{stop}} = c/n$.

Its total energy at this time is

$$E_{\text{stop}} = \frac{m_{\text{e}}c^2}{\sqrt{1 - v_{\text{stop}}^2/c^2}} = \frac{nm_{\text{e}}c^2}{\sqrt{n^2 - 1}}$$

The energy of the electron when it was knocked out is

$$E_{\text{start}} = \alpha\Delta t + \frac{nm_{\text{e}}c^2}{\sqrt{n^2 - 1}}$$

Before interacting, the energy of the electron was equal to $m_{\text{e}}c^2$.

Thus, the energy imparted by the neutrino is

$$E_{\text{imparted}} = E_{\text{start}} - m_{\text{e}}c^2 = \alpha\Delta t + \left(\frac{n}{\sqrt{n^2 - 1}} - 1\right) m_{\text{e}}c^2$$

The fusion of H into He inside the Sun takes place in several steps. Nucleus of ^{7}Be (rest mass, m_{Be}) is produced in one of these intermediate steps. Subsequently, it can absorb an electron, producing a ^{7}Li nucleus (rest mass $m_{\text{Li}} < m_{\text{Be}}$) and emitting a ν_{e}. The corresponding nuclear reaction is:

$$^7\text{Be} + \text{e}^- \longrightarrow {}^7\text{Li} + \nu_{\text{e}} \ .$$

When a Be nucleus ($m_{\text{Be}} = 11.65 \times 10^{-27}$ kg) is at rest and absorbs an electron also at rest, the emitted neutrino has energy $E_\nu = 1.44 \times 10^{-13}$ J. However, the Be nuclei are in random thermal motion due to the temperature T_{c} at the core of the Sun, and act as moving neutrino sources. As a result, the energy of emitted neutrinos fluctuates with a root mean square value ΔE_{rms}.

(B4) If $\Delta E_{\text{rms}} = 5.54 \times 10^{-17}$ J, calculate the rms speed of the Be nuclei, V_{Be} and hence estimate T_{c}. (Hint: ΔE_{rms} depends on the rms value of the component of velocity along the line of sight.)

[2.0]

Solution:

Moving ^{7}Be nuclei give rise to Doppler effect for neutrinos. Since the fractional change in energy ($\Delta E_{\rm rms}/E_\nu \sim 10^{-4}$) is small, the Doppler shift may be considered in the non-relativistic limit (a relativistic treatment gives almost same answer). Taking the line of sight along the z-direction,

$$\begin{aligned}\frac{\Delta E_{\rm rms}}{E_\nu} &= \frac{v_{z,rms}}{c}\\ &= 3.85\times 10^{-4}\end{aligned}$$

$$= \frac{1}{\sqrt{3}}\frac{V_{\rm Be}}{c}$$

$\Rightarrow V_{\rm Be} = \sqrt{3}\times 3.85\times 10^{-4}\times 3.00\times 10^{8}\,\mathrm{m\,s^{-1}} = 2.01\times 10^{5}\,\mathrm{m\,s^{-1}}$.

The average temperature is obtained by equating the average kinetic energy to the thermal energy.

$$\frac{1}{2}m_{\rm Be}V_{\rm Be}^2 = \frac{3}{2}k_{\rm B}T_{\rm c}$$

$$\Rightarrow \qquad T_{\rm c} = 1.13\times 10^{7}\,\mathrm{K}$$

The Extremum Principle[1] (Total Marks: 10)

A The Extremum Principle in Mechanics

Consider a horizontal frictionless $x-y$ plane shown in Fig. 1. It is divided into two regions, I and II, by a line AB satisfying the equation $x = x_1$. The potential energy of a point particle of mass m in region I is $V = 0$ while it is $V = V_0$ in region II. The particle is sent from the origin O with speed v_1 along a line making an angle θ_1 with the x-axis. It reaches point P in region II traveling with speed v_2 along a line that makes an angle θ_2 with the x-axis. Ignore gravity and relativistic effects in this entire task T-2 (all parts).

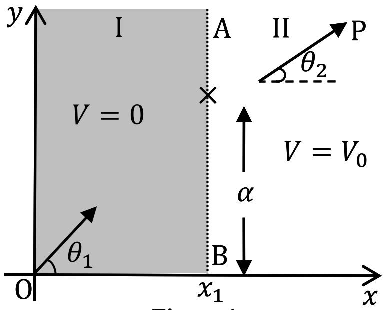


Figure 1

A1	Obtain an expression for v_2 in terms of m, v_1 and V_0.	**0.2**
A2	Express v_2 in terms of v_1, θ_1 and θ_2.	**0.3**

We define a quantity called action $A = m\int v(s)ds$, where ds is the infinitesimal length along the trajectory of a particle of mass m moving with speed $v(s)$. The integral is taken over the path. As an example, for a particle moving with constant speed v on a circular path of radius R, the action A for one revolution will be $2\pi mRv$. For a particle with constant energy E, it can be shown that of all the possible trajectories between two fixed points, the actual trajectory is the one on which A defined above is an extremum (minimum or maximum). Historically this is known as the Principle of Least Action (PLA).

A3	PLA implies that the trajectory of a particle moving between two fixed points in a region of constant potential will be a straight line. Let the two fixed points O and P in Fig. 1 have coordinates (0,0) and (x_0, y_0) respectively and the boundary point where the particle transits from region I to region II have coordinates (x_1, α). Note that x_1 is fixed and the action depends on the coordinate α only. State the expression for the action $A(\alpha)$. Use PLA to obtain the relationship between v_1/v_2 and these coordinates.	**1.0**

B The Extremum Principle in Optics

A light ray travels from medium I to medium II with refractive indices n_1 and n_2 respectively. The two media are separated by a line parallel to the x-axis. The light ray makes an angle i_1 with the y-axis in medium I and i_2 in medium II (see Fig. 2). To obtain the trajectory of the ray, we make use of another extremum (minimum or maximum) principle known as Fermat's principle of least time.

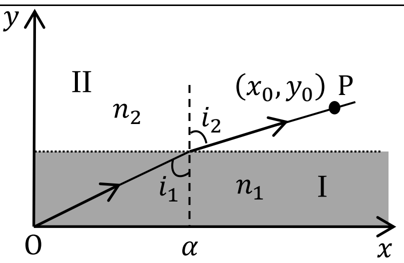


Figure 2

B1	The principle states that between two fixed points, a light ray moves along a path such that time taken between the two points is an extremum. Derive the relation between $\sin i_1$ and $\sin i_2$ on the basis of Fermat's principle.	**0.5**

Shown in Fig. 3 is a schematic sketch of the path of a laser beam incident horizontally on a solution of sugar in which the concentration of sugar decreases with height. As a consequence, the refractive index of the solution also decreases with height.

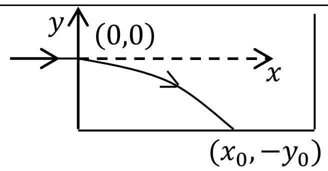


Figure 3: Tank of Sugar Solution

B2	Assume that the refractive index $n(y)$ depends only on y. Use the equation obtained in B1 to obtain the expression for the slope dy/dx of the beam's path in terms of refractive index n_0 at $y = 0$ and $n(y)$.	**1.5**
B3	The laser beam is directed horizontally from the origin (0,0) into the sugar solution at a height y_0 from the bottom of the tank as shown in figure 3. Take $n(y) = n_0 - ky$ where n_0 and k are positive constants. Obtain an expression for x in terms of y and related quantities for the actual trajectory of the laser beam.	**1.2**

[1] *Manoj Harbola (IIT-Kanpur) and Vijay A. Singh (ex-National Coordinator, Science Olympiads) were the principal authors of this problem. The contributions of the Academic Committee, Academic Development Group and the International Board are gratefully acknowledged.*

	You may use: $\int \sec\theta d\theta = \ln(\sec\theta + \tan\theta) +$ constant, where $\sec\theta = 1/\cos\theta$ or $\int \frac{dx}{\sqrt{x^2-1}} = \ln\left(x + \sqrt{x^2-1}\right) +$ constant	
B4	Obtain the value of x_0, the point where the beam meets the bottom of the tank. Take $y_0 = 10.0$ cm, $n_0 = 1.50$, $k = 0.050$ cm^{-1} (1 cm = 10^{-2} m).	**0.8**

C The Extremum Principle and the Wave Nature of Matter

We now explore the connection between the PLA and the wave nature of a moving particle. For this we assume that a particle moving from O to P can take all possible trajectories and we will seek a trajectory that depends on the constructive interference of de Broglie waves.

C1	As the particle moves along its trajectory by an infinitesimal distance Δs, relate the change $\Delta\varphi$ in the phase of its de Broglie wave to the change ΔA in the action and the Planck constant.	**0.6**
C2	Recall the problem from part A where the particle traverses from O to P (see Fig. 4). Let an opaque partition be placed at the boundary AB between the two regions. There is a small opening CD of width d in AB such that $d \ll (x_0 - x_1)$ and $d \ll x_1$. Consider two extreme paths OCP and ODP such that OCP lies on the classical trajectory discussed in part A. Obtain the phase difference $\Delta\varphi_{CD}$ between the two paths to first order.	**1.2**

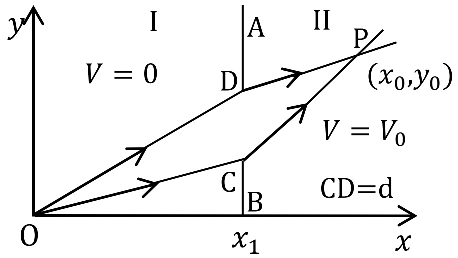


Figure 4

D Matter Wave Interference

Consider an electron gun at O which directs a collimated beam of electrons to a narrow slit at F in the opaque partition A_1B_1 at $x = x_1$ such that OFP is a straight line. P is a point on the screen at $x = x_0$ (see Fig. 5). The speed in I is $v_1 = 2.0000 \times 10^7$ m s^{-1} and $\theta = 10.0000^\circ$. The potential in II is such that speed $v_2 = 1.9900 \times 10^7$ m s^{-1}. The distance $x_0 - x_1$ is 250.00 mm (1mm $= 10^{-3}$m). Ignore electron-electron interaction.

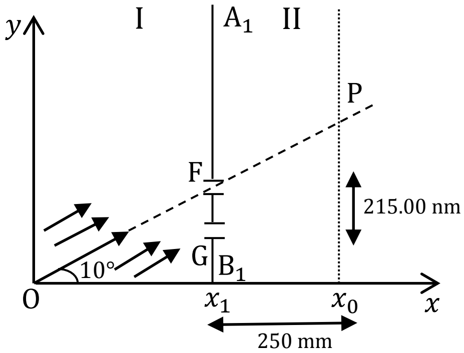


Figure 5

D1	If the electrons at O have been accelerated from rest, calculate the accelerating potential U_1.	**0.3**
D2	Another identical slit G is made in the partition A_1B_1 at a distance of 215.00 nm (1nm $= 10^{-9}$m) below slit F (Fig. 5). If the phase difference between de Broglie waves arriving at P through the slits F and G is $2\pi\beta$, calculate β.	**0.8**
D3	What is the smallest distance Δy from P at which null (zero) electron detection maybe expected on the screen? [Note: you may find the approximation $\sin(\theta + \Delta\theta) \approx \sin\theta + \Delta\theta\cos\theta$ useful]	**1.2**
D4	The beam has a square cross section of 500nm × 500nm and the setup is 2 m long. What should be the minimum flux density I_{min} (number of electrons per unit normal area per unit time) if, on an average, there is at least one electron in the setup at a given time?	**0.4**

The Extremum Principle[1]

A. The Extremum Principle in Mechanics

Consider a horizontal frictionless x-y plane shown in Fig. 1. It is divided into two regions, I and II, by a line AB satisfying the equation $x = x_1$. The potential energy of a point particle of mass m in region I is $V = 0$ while it is $V = V_0$ in region II. The particle is sent from the origin O with speed v_1 along a line making an angle θ_1 with the x-axis. It reaches point P in region II traveling with speed v_2 along a line that makes an angle θ_2 with the $x-$axis. Ignore gravity and relativistic effects in this entire task T-2 (all parts).

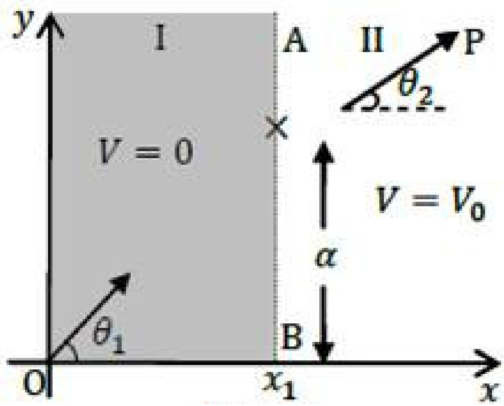


Figure 1

(A1) Obtain an expression for v_2 in terms of m, v_1 and V_0 . [0.2]

Solution:

From the principle of Conservation of Mechanical Energy

$$\frac{1}{2}mv_1^2 = \frac{1}{2}mv_2^2 + V_0$$

$$v_2 = (v_1^2 - \frac{2V_0}{m})^{1/2}$$

(A2) Express v_2 in terms of v_1, θ_1 and θ_2. [0.3]

Solution:

At the boundary there is an impulsive force ($\propto dV/dx$) in the $-x$ direction. Hence only the velocity component in the $x-$direction v_{1x} suffers change . The component in the $y-$direction remains unchanged. Therefore

$v_{1y} = v_{2y}$

$v_1 \sin\theta_1 = v_2 \sin\theta_2$

We define a quantity called action $A = m\int v(s)\,ds$, where ds is the infinitesimal length along the trajectory of a particle of mass m moving with speed $v(s)$. The integral is taken over the path. As an example. for a particle moving with constant speed v on a circular path of radius R, the action A for one revolution will be $2\pi mRv$. For a particle with constant energy E, it can be shown that of all the possible trajectories between two fixed points, the actual trajectory is the one on which A defined above is an extremum (minimum or maximum). Historically this is known as the Principle of Least Action (PLA).

[1] *Manoj Harbola (IIT-Kanpur) and Vijay A. Singh (ex-National Coordinator, Science Olympiads) were the principal authors of this problem. The contributions of the Academic Committee, Academic Development Group and the International Board are gratefully acknowledged.*

(A3) PLA implies that the trajectory of a particle moving between two fixed points in a region of constant potential will be a straight line. Let the two fixed points O and P in Fig. 1 have coordinates (0,0) and (x_0,y_0) respectively and the boundary point where the particle transits from region I to region II have coordinates (x_1,α). Note x_1 is fixed and the action depends on the coordinate α only. State the expression for the action $A(\alpha)$. Use PLA to obtain the the relationship between v_1/v_2 and these coordinates. [1.0]

Solution:

By definition $A(\alpha)$ from O to P is

$$A(\alpha) = mv_1\sqrt{x_1^2 + \alpha^2} + mv_2\sqrt{(x_0 - x_1)^2 + (y_0 - \alpha)^2}$$

Differentiating w.r.t. α and setting the derivative of $A(\alpha)$ to zero

$$\frac{v_1\alpha}{(x_1^2 + \alpha^2)^{1/2}} - \frac{v_2(y_0 - \alpha)}{[(x_0 - x_1)^2 + (y_0 - \alpha)^2]^{1/2}} = 0$$

$$\therefore \frac{v_1}{v_2} = \frac{(y_0 - \alpha)\,(x_1^2 + \alpha^2)^{1/2}}{\alpha\,[(x_0 - x_1)^2 + (y_0 - \alpha)^2]^{1/2}}$$

Note this is the same as A2, namely $v_1 \sin\theta_1 = v_2 \sin\theta_2$.

B. The Extremum Principle in Optics

A light ray travels from medium I to medium II with refractive indices n_1 and n_2 respectively. The two media are separated by a line parallel to the x-axis. The light ray makes an angle i_1 with the y-axis in medium I and i_2 in medium II (see Fig. 2). To obtain the trajectory of the ray, we make use of another extremum (minimum or maximum) principle known as Fermat's principle of least time.

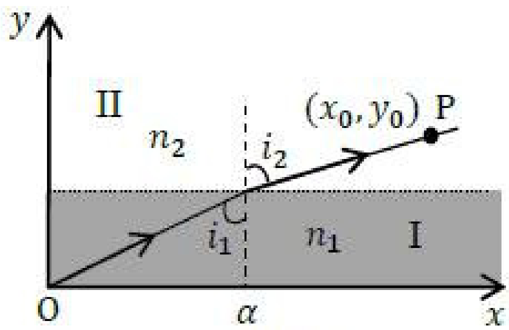


Figure 2

(B1) The principle states that between two fixed points, a light ray moves along a path such that the time taken between the two points is an extremum. Derive the relation between $\sin i_1$ and $\sin i_2$ on the basis of Fermat's principle. [0.5]

Solution:

The speed of light in medium I is c/n_1 and in medium II is c/n_2, where c is the speed of light in vacuum. Let the two media be separated by the fixed line $y = y_1$. Then time $T(\alpha)$ for light to travel from origin (0,0) and (x_0,y_0) is

$$T(\alpha) = n_1(\sqrt{y_1^2 + \alpha^2})/c + n_2(\sqrt{(x_0 - \alpha)^2 + (y_0 - y_1)^2})/c$$

Differentiating w.r.t. α and setting the derivative of $T(\alpha)$ to zero

$$\frac{n_1\alpha}{(y_1^2+\alpha^2)^{1/2}} - \frac{n_2(y_0-\alpha)}{[(x_0-\alpha)^2+(y_0-y_1)^2]^{1/2}} = 0$$

$$\therefore \quad n_1 \sin i_1 = n_2 \sin i_2$$

[Note: Derivation is similar to A3. This is Snell's law.]

Shown in Fig. 3 is a schematic sketch of the path of a laser beam incident horizontally on a solution of sugar in which the concentration of sugar decreases with height. As a consequence, the refractive index of the solution also decreases with height.

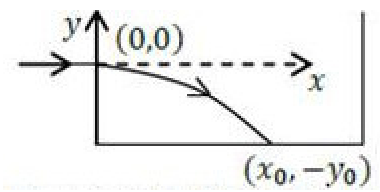


Figure 3

(B2) Assume that the refractive index n(y) depends only on y. Use the equation obtained in B1 to obtain the expresssion for the slope dy/dx of the beam's path in terms of n_0 at $y = 0$ and $n(y)$. [1.5]

Solution:

From Snell's law $n_0 \sin i_0 = n(y) \sin i$

Then, $\quad \dfrac{dy}{dx} = -\cot i$

$$n_0 \sin i_0 = \frac{n(y)}{\sqrt{1+(\frac{dy}{dx})^2}}$$

$$\frac{dy}{dx} = -\sqrt{\left(\frac{n(y)}{n_0 \sin i_0}\right)^2 - 1}$$

(B3) The laser beam is directed horizontally from the origin (0,0) into the sugar solution at a height y_0 from the bottom of the tank as shown. Take $n(y) = n_0 - ky$ where n_0 and k are positive constants. Obtain an expression for x in terms of y and related quantities. You may use: $\int \sec\theta d\theta = \ln(\sec\theta + \tan\theta)$ + constant $\sec\theta = 1/\cos\theta$ or $\int \frac{dx}{\sqrt{x^2-1}} = \ln(x+\sqrt{x^2-1})$ + constant. [1.2]

Solution:

$$\int \frac{dy}{\sqrt{(\frac{n_0-ky}{n_0 \sin\ i_0})^2 - 1}} = -\int dx$$

Note $i_0 = 90^o$ so $\sin i_0 = 1$.

Method I We employ the substitution

$$\xi = \frac{n_0 - ky}{n_0}$$

$$\int \frac{d\xi(-\frac{n_0}{k})}{\sqrt{\xi^2 - 1}} = -\int dx$$

Let $\xi = \sec\theta$. Then

$$\frac{n_0}{k}\ln(\sec\theta + \tan\theta) = x + c$$

Or METHOD II

We employ the substition

$$\xi = \frac{n_0 - ky}{n_0}$$

$$\int \frac{d\xi(-\frac{n_0}{k})}{\sqrt{\xi^2 - 1}} = -\int dx$$

$$-\frac{n_0}{k}\ln\left(\frac{n_0 - ky}{n_0} + \sqrt{(\frac{n_0 - ky}{n_0})^2 - 1}\right) = -x + c$$

Now continuing

Considering the substitutions and boundary condition, $x = 0$ for $y = 0$ we obtain that the constant $c = 0$.

Hence we obtain the following trajectory:

$$x = \frac{n_0}{k}\ln\left(\frac{n_0 - ky}{n_0} + \sqrt{(\frac{n_0 - ky}{n_0})^2 - 1}\right)$$

(B4) Obtain the value of x_0, the point where the beam meets the bottom of the tank. Take y_0 = 10.0 cm, $n_0 = 1.50$, $k = 0.050$ cm^{-1} (1 cm = 10^{-2} m). [0.8]

Solution:

Given $y_0 = 10.0$ cm. $\quad n_0 = 1.50 \quad k = 0.050$ cm^{-1}

From (B3)

$$x_0 = \frac{n_0}{k}\ln\left[\left(\frac{n_0 - ky}{n_0}\right) + \left(\left(\frac{n_0 - ky}{n_0}\right)^2 - 1\right)^{1/2}\right]$$

Here $y = -y_0$

$$x_0 = \frac{n_0}{k} \ln \left[\frac{(n_0 + ky_0)}{n_0} + \left(\frac{(n_0 + ky_0)^2}{n_0^2} - 1 \right)^{1/2} \right]$$
$$= 30 \ln \left[\frac{2}{1.5} + \left(\left(\frac{2}{1.5} \right)^2 - 1 \right)^{1/2} \right]$$
$$= 30 \ln \left[\frac{4}{3} + \left(\frac{7}{9} \right)^{1/2} \right]$$
$$= 30 \ln \left[\frac{4}{3} + 0.88 \right]$$
$$= 24.0 \text{ cm}$$

C. The Extremum Principle and the Wave Nature of Matter

We now explore between the PLA and the wave nature of a moving particle. For this we assume that a particle moving from O to P can take all possible trajectories and we will seek a trajectory that depends on the constructive interference of de Broglie waves.

(C1) As the particle moves along its trajectory by an infinitesimal distance Δs, relate the change $\Delta\phi$ in the phase of its de Broglie wave to the change ΔA in the action and the Planck constant. [0.6]

Solution:

From the de Broglie hypothesis

$$\lambda \to \lambda_{dB} = h/mv$$

where λ is the de Broglie wavelength and the other symbols have their usual meaning

$$\Delta\phi = \frac{2\pi}{\lambda}\Delta s$$
$$= \frac{2\pi}{h} mv\Delta s$$
$$= \frac{2\pi\Delta A}{h}$$

(C2) Recall the problem from part A where the particle traverses from O to P (see Fig. 4). Let an opaque partition be placed at the boundary AB between the two regions. There is a small opening CD of width d in AB such that $d \ll (x_0 - x_1)$ and $d \ll x_1$. Consider two extreme paths OCP and ODP such that OCP lies on the classical trajectory discussed in part A. Obtain the phase difference $\Delta\phi_{CD}$ between the two paths to first order. [1.2]

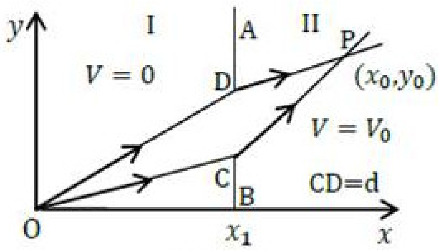
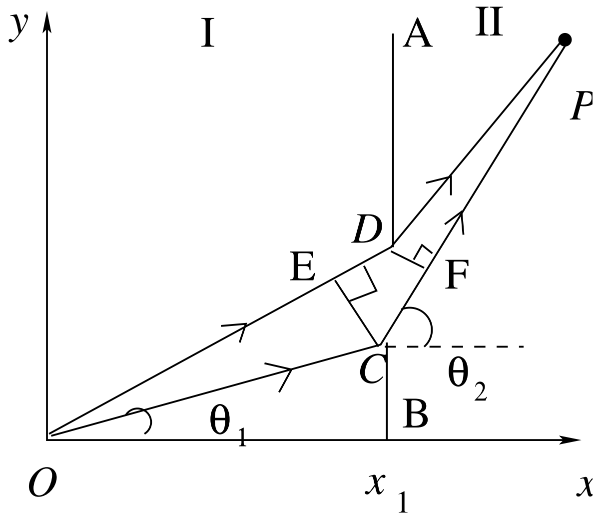


Figure 4

Solution:



Consider the extreme trajectories OCP and ODP of (C1)
The geometrical path difference is ED in region I and CF in region II.
This implies (note: $d \ll (x_0 - x_1)$ and $d \ll x_1$)

$$\Delta\phi_{CD} = \frac{2\pi d \sin\theta_1}{\lambda_1} - \frac{2\pi d \sin\theta_2}{\lambda_2}$$

$$\begin{aligned}\Delta\phi_{CD} &= \frac{2\pi m v_1 d \sin\theta_1}{h} - \frac{2\pi m v_2 d \sin\theta_2}{h} \\ &= 2\pi \frac{md}{h}(v_1 \sin\theta_1 - v_2 \sin\theta_2) \\ &= 0 \qquad \text{(from A2 or B1)}\end{aligned}$$

Thus near the clasical path there is invariably constructive interference.

D. Matter Wave Interference

Consider an electron gun at O which directs a collimated beam of electrons to a narrow slit at F in the opaque partition A_1B_1 at $x = x_1$ such that OFP is a straight line. P is a point on the screen at $x = x_0$ (see Fig. 5). The speed in I is $v_1 = 2.0000 \times 10^7$ m s^{-1} and $\theta = 10.0000°$. The potential in region II is such that the speed $v_2 = 1.9900 \times 10^7$ m s^{-1}. The distance $x_0 - x_1$ is 250.00 mm (1 mm $= 10^{-3}$ m). Ignore electron-electron interaction.

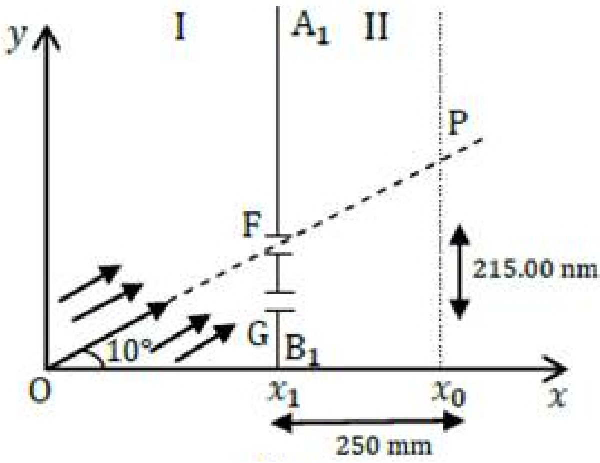


Figure 5

(D1) If the electrons at O have been accelerated from rest, calculate the accelerating potential U_1. [0.3]

Solution:

$$qU_1 = \frac{1}{2}\, mv^2$$

$$= \frac{9.11 \times 10^{-31} \times 4 \times 10^{14}}{2} J$$

$$= 2 \times 9.11 \times 10^{-17} J$$

$$= \frac{2 \times 9.11 \times 10^{-17}}{1.6 \times 10^{-19}}\, eV$$

$$= 1.139 \times 10^3\ eV \quad (\simeq 1100\ eV)$$

$$U_1 = 1.139 \times 10^3\ V$$

(D2) Another identical slit G is made in the partition A_1B_1 at a distance of 215.00 nm (1 nm $= 10^{-9}$ m) below slit F (Fig. 5). If the phase difference between de Broglie waves ariving at P from F and G is $2\,\pi\,\beta$, calculate β. [0.8]

Solution: Phase difference at P is

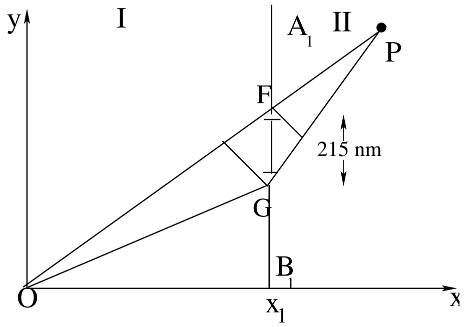
$$\Delta\phi_P = \frac{2\pi d \sin\theta}{\lambda_1} - \frac{2\pi d \sin\theta}{\lambda_2}$$

$$= 2\pi(v_1 - v_2)\frac{md}{h} \sin 10° = 2\pi\beta$$

$$\beta = 5.13$$

(D3) What is is the smallest distance Δy from P at which null (zero) electron detection maybe expected on the screen? [Note: you may find the approximation $\sin(\theta + \Delta\theta) \approx \sin\theta + \Delta\theta\cos\theta$ useful] [1.2]

Solution:



From previous part for null (zero) electron detection $\Delta\phi = 5.5 \times 2\pi$

$$\therefore mv_1\frac{d\sin\theta}{h} - \frac{mv_2 d\sin(\theta+\Delta\theta)}{h} = 5.5$$

$$\begin{aligned} sin(\theta+\Delta\theta) &= \frac{\dfrac{mv_1 d\sin\theta}{h} - 5.5}{\dfrac{mv_2 d}{h}} \\ &= \frac{v_1}{v_2}\sin\theta - \frac{h}{m}\frac{5.5}{v_2 d} \\ &= \frac{2}{1.99}\sin 10^\circ - \frac{5.5}{1374.78 \times 1.99 \times 10^7 \times \times 2.15 \times 10^{-7}} \\ &= 0.174521 - 0.000935 \end{aligned}$$

This yields $\Delta\theta = -0.0036^\circ$

The closest distance to P is

$$\begin{aligned} \Delta y &= (x_0 - x_1)(\tan(\theta+\Delta\theta) - \tan\theta) \\ &= 250(\tan 9.9964 - \tan 10) \\ &= -0.0162mm \\ &= -16.2\mu\,\mathrm{m} \end{aligned}$$

The negative sign means that the closest minimum is below P.

Approximate Calculation for θ and Δy
Using the approximation $\sin(\theta+\Delta\theta) \approx \sin\theta + \Delta\theta\cos\theta$
The phase difference of $5.5 \times 2\pi$ gives

$$mv_1\frac{d\sin 10^\circ}{h} - mv_2\frac{d(\sin 10^\circ + \Delta\theta\cos 10^\circ)}{h} = 5.5$$

From solution of the previous part

$$mv_1\frac{d\sin 10^\circ}{h} - mv_2\frac{dsin10^\circ}{h} = 5.13$$

Therefore

$$mv_2 \frac{d\Delta\theta \cos 10^\circ}{h} = 0.3700$$

This yields $\Delta\theta \approx 0.0036^\circ$
$\Delta y = -0.0162\,\text{mm} = -16.2\mu\text{m}$ as before

(D4) The electron beam has a square cross section of 500 nm $\times$ 500 nm and the setup is 2 m long. What should be the minimum beam flux density I_{min} (number of electrons per unit normal area per unit time) if, on an average, there is at least one electron in the setup at a given time? [0.4]

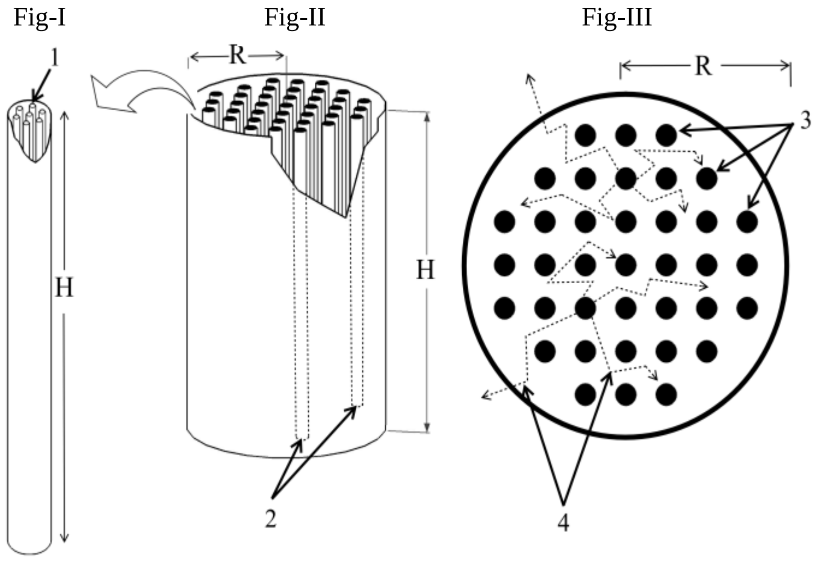
Solution: The product of the speed of the electrons and number of electron per unit volume on an average yields the intensity.
Thus $N = 1 =$ Intensity $\times$ Area $\times$ Length/ Electron Speed
$= I_{min} \times 0.25 \times 10^{-12} \times 2/2 \times 10^7$
This gives $I_{min} = 4\times\ 10^{19}\ \ \text{m}^{-2}\ \text{s}^{-1}$

R. Bach, D. Pope, Sy-H Liou and H. Batelaan, New J. of Physics Vol. 15, 033018 (2013).

The Design of a Nuclear Reactor[1] **(Total Marks: 10)**

Uranium occurs in nature as UO_2 with only 0.720% of the uranium atoms being ^{235}U. Neutron induced fission occurs readily in ^{235}U with the emission of 2-3 fission neutrons having high kinetic energy. This fission probability will increase if the neutrons inducing fission have low kinetic energy. So by reducing the kinetic energy of the fission neutrons, one can induce a chain of fissions in other ^{235}U nuclei. This forms the basis of the power generating nuclear reactor (NR).

A typical NR consists of a cylindrical tank of height H and radius R filled with a material called moderator. Cylindrical tubes, called fuel channels, each containing a cluster of cylindrical fuel pins of natural UO_2 in solid form of height H, are kept axially in a square array. Fission neutrons, coming outward from a fuel channel, collide with the moderator, losing energy and reach the surrounding fuel channels with low enough energy to cause fission (Figs I-III). Heat generated from fission in the pin is transmitted to a coolant fluid flowing along its length. In the current problem we shall study some of the physics behind the (A) Fuel Pin, (B) Moderator and (C) NR of cylindrical geometry.



Schematic sketch of the Nuclear Reactor (NR)
Fig-I: Enlarged view of a fuel channel (1-*Fuel Pins*)
Fig-II: A view of the NR (2-*Fuel Channels*)
Fig-III: Top view of NR (3-*Square Arrangement of Fuel Channels* and 4-*Typical Neutron Paths*).
Only components relevant to the problem are shown (e.g. control rods and coolant are not shown).

A Fuel Pin

Data for UO_2	1. Molecular weight $M_w = 0.270$ kg mol^{-1}	2. Density $\rho = 1.060\times10^4$ kg m^{-3}
	3. Melting point $T_m = 3.138\times10^3$ K	4. Thermal conductivity $\lambda = 3.280$ W m^{-1} K^{-1}

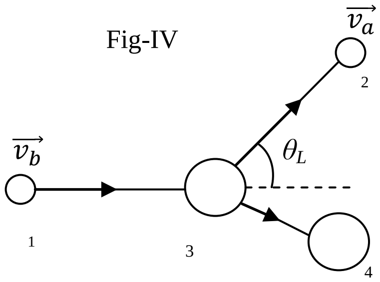
A1	Consider the following fission reaction of a stationary ^{235}U after it absorbs a neutron of negligible kinetic energy. $$^{235}U + {}^{1}n \rightarrow {}^{94}Zr + {}^{140}Ce + 2\,{}^{1}n + \Delta E$$ Estimate ΔE (in MeV) the total fission energy released. The nuclear masses are: $m(^{235}U) = 235.044$ u; $m(^{94}Zr) = 93.9063$ u; $m(^{140}Ce) = 139.905$ u; $m(^{1}n) = 1.00867$ u and 1 u = 931.502 MeV c^{-2}. Ignore charge imbalance.	**0.8**
A2	Estimate N the number of ^{235}U atoms per unit volume in natural UO_2.	**0.5**
A3	Assume that the neutron flux density, $\varphi = 2.000\times10^{18}$ m^{-2} s^{-1} on the fuel is uniform. The fission cross-section (effective area of the target nucleus) of a ^{235}U nucleus is $\sigma_f = 5.400\times10^{-26}$ m^2. If 80.00% of the fission energy is available as heat, estimate Q (in W m^{-3}), the rate of heat production in the pin per unit volume. 1MeV = 1.602×10^{-13} J	**1.2**
A4	The steady-state temperature difference between the center (T_c) and the surface (T_s) of the pin can be expressed as $T_c - T_s = k\,F(Q,a,\lambda)$, where $k = 1/4$ is a dimensionless constant and a is the radius of the pin. Obtain $F(Q,a,\lambda)$ by dimensional analysis. Note that λ is the thermal conductivity of UO_2.	**0.5**

[1] *Joseph Amal Nathan (BARC) and Vijay A. Singh (ex-National Coordinator, Science Olympiads) were the principal authors of this problem. The contributions of the Academic Committee, Academic Development Group and the International Board are gratefully acknowledged.*

A5	The desired temperature of the coolant is 5.770×10^{2} K. Estimate the upper limit a_u on the radius a of the pin.	**1.0**

B The Moderator

Consider the two dimensional elastic collision between a neutron of mass 1 u and a moderator atom of mass A u. Before collision all the moderator atoms are considered at rest in the laboratory frame (LF). Let $\overrightarrow{v_b}$ and $\overrightarrow{v_a}$ be the velocities of the neutron before and after collision respectively in the LF. Let $\overrightarrow{v_m}$ be the velocity of the center of mass (CM) frame relative to LF and θ be the neutron scattering angle in the CM frame. All the particles involved in collisions are moving at nonrelativistic speeds.

B1	The collision in LF is shown schematically, where θ_L is the scattering angle (Fig-IV). Sketch the collision schematically in CM frame. Label the particle velocities for 1, 2 and 3 in terms of $\overrightarrow{v_b}$, $\overrightarrow{v_a}$ and $\overrightarrow{v_m}$. Indicate the scattering angle θ. Fig-IV  *Collision in the Laboratory Frame* 1-*Neutron* before collision 2-*Neutron* after collision 3-*Moderator Atom* before collision 4-*Moderator Atom* after collision	**1.0**
B2	Obtain v and V, the speeds of the neutron and moderator atom in the CM frame after collision, in terms of A and v_b.	**1.0**
B3	Derive an expression for $G(\alpha, \theta) = E_a/E_b$, where E_b and E_a are the kinetic energies of the neutron, in the LF, before and after the collision respectively and $\alpha \equiv [(A-1)\ /\ (A+1)]^2$.	**1.0**
B4	Assume that the above expression holds for D_2O molecule. Calculate the maximum possible fractional energy loss $f_l \equiv \frac{E_b - E_a}{E_b}$ of the neutron for the D_2O (20 u) moderator.	**0.5**

C The Nuclear Reactor

To operate the NR at any constant neutron flux ψ (steady state), the leakage of neutrons has to be compensated by an excess production of neutrons in the reactor. For a reactor in cylindrical geometry the leakage rate is $k_1\,[(2.405/R)^2 + (\pi/H)^2]\,\psi$ and the excess production rate is $k_2\,\psi$. The constants k_1 and k_2 depend on the material properties of the NR.

C1	Consider a NR with $k_1 = 1.021\times10^{-2}$ m and $k_2 = 8.787\times10^{-3}$ m^{-1}. Noting that for a fixed volume the leakage rate is to be minimized for efficient fuel utilization, obtain the dimensions of the NR in the steady state.	**1.5**
C2	The fuel channels are in a square arrangement (Fig-III) with the nearest neighbour distance 0.286 m. The effective radius of a fuel channel (if it were solid) is 3.617×10^{-2} m. Estimate the number of fuel channels F_n in the reactor and the mass M of UO_2 required to operate the NR in steady state.	**1.0**

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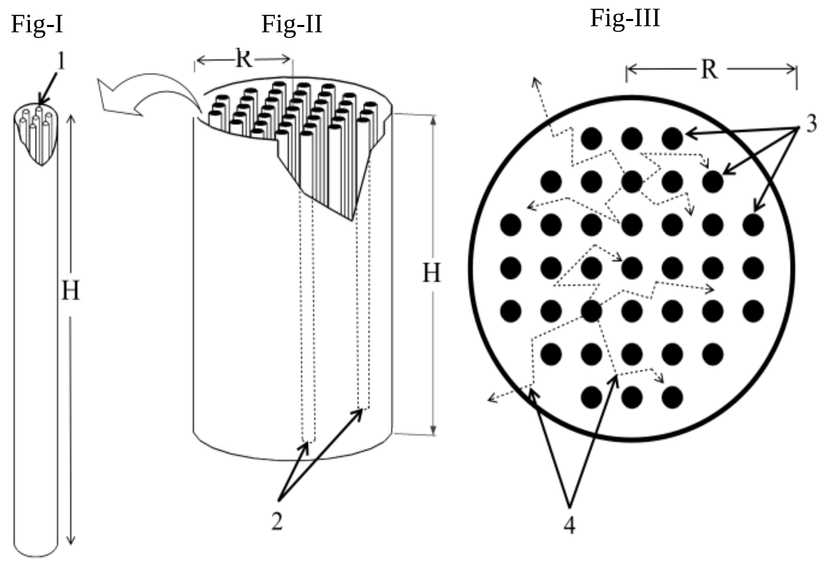

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A1 Consider the following fission reaction of a stationary ^{235}U after it absorbs a neutron of negligible kinetic energy.

$$^{235}\mathrm{U} + ^{1}\mathrm{n} \longrightarrow ^{94}\mathrm{Zr} + ^{140}\mathrm{Ce} + 2\,^{1}\mathrm{n} + \Delta E$$

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Estimate ΔE (in MeV) the total fission energy released. The nuclear masses are: $m(^{235}\text{U}) = 235.044$ u; $m(^{94}\text{Zr}) = 93.9063$ u; $m(^{140}\text{Ce}) = 139.905$ u; $m(^{1}\text{n}) = 1.00867$ u and 1 u = 931.502 MeV c^{-2}. Ignore charge imbalance. [0.8]

Solution: $\Delta E = 208.684$ MeV

Detailed solution: The energy released during the transformation is

$$\Delta E = [m(^{235}\text{U}) + m(^{1}\text{n}) - m(^{94}\text{Zr}) - m(^{140}\text{Ce}) - 2m(^{1}\text{n})]\text{c}^2$$

Since the data is supplied in terms of unified atomic masses (u), we have

$$\Delta E = [m(^{235}\text{U}) - m(^{94}\text{Zr}) - m(^{140}\text{Ce}) - m(^{1}\text{n})]\text{c}^2$$

$= 208.684$ MeV [***Acceptable Range*** **(208.000 to 209.000)**]
from the given data.

A2 Estimate N the number of ^{235}U atoms per unit volume in natural UO_2. [0.5]

Solution: $N = 1.702 \times 10^{26}$ m^{-3}

Detailed solution: The number of UO_2 molecules per m^3 of the fuel N_1 is given in the terms of its density ρ, the Avogadro number N_A and the average molecular weight M_w as

$$N_1 = \frac{\rho N_A}{M_w} = \frac{10600 \times 6.022 \times 10^{23}}{0.270} = 2.364 \times 10^{28} \text{ m}^{-3}$$

Each molecule of UO_2 contains one uranium atom. Since only 0.72% of these are ^{235}U,

$$N = 0.0072 \times N_1 = 1.702 \times 10^{26} \text{ m}^{-3}$$ [***Acceptable Range*** **(1.650 to 1.750)**]

A3 Assume that the neutron flux $\phi = 2.000 \times 10^{18}$ m^{-2} s^{-1} on the fuel is uniform. The fission cross-section (effective area of the target nucleus) of a ^{235}U nucleus is $\sigma_f = 5.400 \times 10^{-26}$ m^2. If 80.00% of the fission energy is available as heat, estimate Q (in W m^{-3}) the rate of heat production in the pin per unit volume. 1MeV = 1.602 $\times 10^{-13}$ J. [1.2]

Solution: $Q = 4.917 \times 10^8$ W/m^3

Detailed solution: It is given that 80% of the fission energy is available as heat thus the heat energy available per fission E_f is from a-(i)

$$E_f = 0.8 \times 208.7 \text{ MeV} = 166.96 \text{ MeV} = 2.675 \times 10^{-11} \text{ J}$$

The total cross-section per unit volume is $N \times \sigma_f$. Thus the heat produced per unit

volume per unit time Q is
$Q = N \times \sigma_f \times \phi \times E_f$
$= (1.702 \times 10^{26}) \times (5.4 \times 10^{-26}) \times (2 \times 10^{18}) \times (2.675 \times 10^{-11})$ W/m^3
$= 4.917 \times 10^8$ W/m^3 [***Acceptable Range* (4.800 to 5.000)**]

A4 The steady-state temperature difference between the center (T_c) and the surface (T_s) of the pin can be expressed as $T_c - T_s = kF(Q, a, \lambda)$ where $k = 1/4$ is a dimensionless constant and a is the radius of the pin. Obtain $F(Q, a, \lambda)$ by dimensional analysis. [0.5]

Solution: $T_c - T_s = \dfrac{Qa^2}{4\lambda}$.

Detailed solution: The dimensions of $T_c - T_s$ is temperature. We write this as $T_c - T_s = [K]$. Once can similarly write down the dimensions of Q, a and λ. Equating the temperature to powers of Q, a and λ, one could state the following dimensional equation:
$K = Q^\alpha a^\beta \lambda^\gamma$
$= [ML^{-1}T^{-3}]^{\ \alpha}\ [L]^{\ \beta}\ [ML^1T^{-3}K^{-1}]^{\ \gamma}$
This yields the following algebraic equations
γ = -1 equating powers of temperature
$\alpha + \gamma = 0$ equating powers of mass or time. From the previous equation we get $\alpha = 1$
Next $-\alpha + \beta + \gamma = 0$ equating powers of length. This yields $\beta = 2$.
Thus we obtain $T_c - T_s = \dfrac{Qa^2}{4\lambda}$ where we insert the dimensionless factor $1/4$ as suggested in the problem. ***No penalty if the factor 1/4 is not written.***

Note: Same credit for alternate ways of obtaining α, β, γ.

A5 The desired temperature of the coolant is 5.770 $\times 10^2$ K. Estimate the upper limit a_u on the radius a of the pin. [1.0]

Solution: $a_u = 8.267 \times 10^{-3}$ m.

Detailed solution: The melting point of UO_2 is 3138 K and the maximum temperature of the coolant is 577 K. This sets a limit on the maximum permissible temperature ($T_c - T_s$) to be less than (3138 - 577 = 2561 K) to avoid "meltdown". Thus one may take a maximum of ($T_c - T_s$) = 2561 K.
Noting that $\lambda = 3.28$ W/m - K, we have

$$a_u^2 = \frac{2561 \times 4 \times 3.28}{4.917 \times 10^8}$$

Where we have used the value of Q from A2. This yields $a_u \backsimeq 8.267 \times 10^{-3}$ m. So $a_u = 8.267 \times 10^{-3}$ m constitutes an upper limit on the radius of the fuel pin.

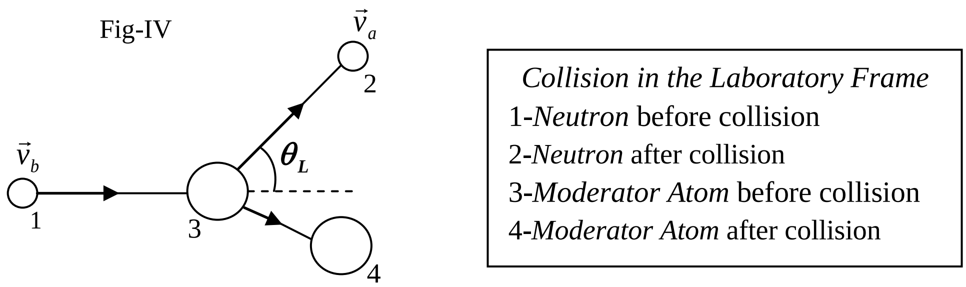
Note: The Tarapur 3 & 4 NR in Western India has a fuel pin radius of 6.090×10^{-3} m.

B. The Moderator

Consider the two dimensional elastic collision between a neutron of mass 1 u and a moderator atom of mass A u. Before collision all the moderator atoms are considered at rest in the laboratory frame (LF). Let $\vec{v_b}$ and $\vec{v_a}$ be the velocities of the neutron before and after collision respectively in the LF. Let $\vec{v_m}$ be the velocity of the center of mass (CM) frame relative to LF and θ be the neutron scattering angle in the CM frame. All the particles involved in collisions are moving at non-relativistic speeds

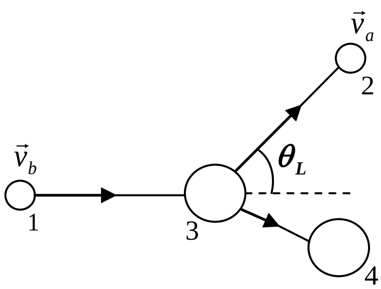
B1 The collision in LF is shown schematically with θ_L as the scattering angle (Fig-IV). Sketch the collision schematically in CM frame. Label the particle velocities for 1, 2 and 3 in terms of $\vec{v_b}$, $\vec{v_a}$ and $\vec{v_m}$. Indicate the scattering angle θ. [1.0]

Fig-IV

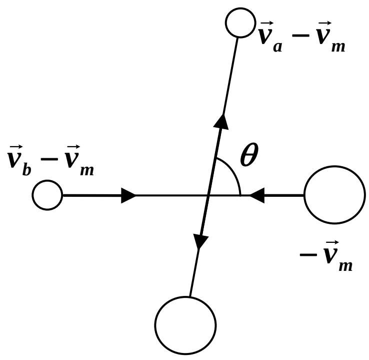


Solution:

Laboratory Frame



Center of Mass Frame



B2 Obtain v and V, the speeds of the neutron and the moderator atom in the CM frame after the collision, in terms of A and v_b. [1.0]

Solution: ***Detailed solution***: Before the collision in the CM frame $(v_b - v_m)$ and v_m will be magnitude of the velocities of the neutron and moderator atom respectively. From momentum conservation in the CM frame, $v_b - v_m = Av_m$ gives $v_m = \frac{v_b}{A+1}$.

After the collision, let v and V be magnitude of the velocities of neutron and moderator atom respectively in the CM frame. From conservation laws,

$$v = AV \qquad \text{and} \qquad \frac{1}{2}(v_b - v_m)^2 + \frac{1}{2}Av_m^2 = \frac{1}{2}v^2 + \frac{1}{2}AV^2.(\rightarrow [0.2 + 0.2])$$

Solving gives $v = \frac{Av_b}{A+1}$ and $V = \frac{v_b}{A+1}$. **(OR)** From definition of center of mass frame $v_m = \frac{v_b}{A+1}$. Before the collision in the CM frame $v_b - v_m = \frac{Av_b}{A+1}$ and v_m will be magnitude of the velocities of the neutron and moderator atom respectively. In elastic collision the particles are scattered in the opposite direction in the CM frame and so the speeds remain same $v = \frac{Av_b}{A+1}$ and $V = \frac{v_b}{A+1}$ ($\rightarrow [0.2 + 0.1]$).

Note: Alternative solutions are worked out in the end and will get appropriate weightage.

B3 Derive an expression for $G(\alpha, \theta) = E_a/E_b$, where E_b and E_a are the kinetic energies of the neutron, in the LF, before and after the collision respectively, and $\alpha \equiv [(A-1)/(A+1)]^2$, [1.0]

Solution:

$$G(\alpha, \theta) = \frac{E_a}{E_b} = \frac{A^2 + 2A\cos\theta + 1}{(A+1)^2} = \frac{1}{2}\left[(1+\alpha) + (1-\alpha)\cos\theta\right].$$

<u>Detailed solution</u>: Since $\overrightarrow{v_a} = \overrightarrow{v} + \overrightarrow{v_m}$, $v_a^2 = v^2 + v_m^2 + 2vv_m\cos\theta$ ($\rightarrow [0.3]$). Substituting the values of v and v_m, $v_a^2 = \frac{A^2v_b^2}{(A+1)^2} + \frac{v_b^2}{(A+1)^2} + \frac{2Av_b^2}{(A+1)^2}\cos\theta$ ($\rightarrow [0.2]$), so

$$\frac{v_a^2}{v_b^2} = \frac{E_a}{E_b} = \frac{A^2 + 2A\cos\theta + 1}{(A+1)^2}.$$

$$G(\alpha, \theta) = \frac{A^2+1}{(A+1)^2} + \frac{2A}{(A+1)^2}\cos\theta = \frac{1}{2}\left[(1+\alpha) + (1-\alpha)\cos\theta\right].$$

Alternate form

$$= 1 - \frac{(1-\alpha)(1-\cos\theta)}{2}.$$

Note: Alternative solutions are worked out in the end and will get appropriate weightage.

B4 Assume that the above expression holds for D_2O molecule. Calculate the maximum possible fractional energy loss $f_l \equiv \frac{E_b - E_a}{E_b}$ of the neutron for the D_2O (20 u) moderator. [0.5]

Solution: $f_l = 0.181$

<u>Detailed solution</u>: The maximum energy loss will be when the collision is head on ie., E_a will be minimum for the scattering angle $\theta = \pi$.

So $E_a = E_{min} = \alpha E_b$.

For D_2O, $\alpha = 0.819$ and maximum fractional loss $\left(\frac{E_b - E_{min}}{E_b}\right) = 1 - \alpha = 0.181$. [***Acceptable Range*** **(0.170 to 0.190)**]

C. The Nuclear Reactor

To operate the NR at any constant neutron flux Ψ (steady state), the leakage of neutrons has to be compensated by an excess production of neutrons in the reactor. For a reactor in cylindrical geometry the leakage rate is $k_1\left[\left(\frac{2.405}{R}\right)^2+\left(\frac{\pi}{H}\right)^2\right]\Psi$ and the excess production rate is $k_2\Psi$. The constants k_1 and k_2 depend on the material properties of the NR.

C1 Consider a NR with $k_1 = 1.021\times10^{-2}$ m and $k_2 = 8.787\times10^{-3}$ m^{-1}. Noting that for a fixed volume the leakage rate is to be minimized for efficient fuel utilisation obtain the dimensions of the NR in the steady state. [1.5]

Solution: $R = 3.175$ m, $H = 5.866$ m.

Detailed solution: For constant volume $V = \pi R^2 H$,

$$\frac{d}{dH}\left[\left(\frac{2.405}{R}\right)^2+\left(\frac{\pi}{H}\right)^2\right]=0,$$

$$\frac{d}{dH}\left[\frac{2.405^2\pi H}{V}+\frac{\pi^2}{H^2}\right]=\frac{2.405^2\pi}{V}-2\frac{\pi^2}{H^3}=0,$$

gives $\left(\frac{2.405}{R}\right)^2 = 2\left(\frac{\pi}{H}\right)^2$.

For steady state,

$$1.021\times10^{-2}\left[\left(\frac{2.405}{R}\right)^2+\left(\frac{\pi}{H}\right)^2\right]\Psi = 8.787\times10^{-3}\ \Psi.$$

Hence $H = 5.866$ m [***Acceptable Range* (5.870 to 5.890)**]
$R = 3.175$ m [***Acceptable Range* (3.170 to 3.180)**].

Alternative Non-Calculus Method to Optimize

Minimisation of the expression $\left(\frac{2.405}{R}\right)^2+\left(\frac{\pi}{H}\right)^2$**, for a fixed volume** $V = \pi R^2 H$:

Substituting for R^2 in terms of V, H we get $\dfrac{2.405^2\pi H}{V}+\dfrac{\pi^2}{H^2}$,

which can be written as, $\dfrac{2.405^2\pi H}{2V}+\dfrac{2.405^2\pi H}{2V}+\dfrac{\pi^2}{H^2}$.

Since all the terms are positive applying AMGM inequality for three positive terms we get

$$\frac{\frac{2.405^2\pi H}{2V}+\frac{2.405^2\pi H}{2V}+\frac{\pi^2}{H^2}}{3}\geq\sqrt[3]{\frac{2.405^2\pi H}{2V}\times\frac{2.405^2\pi H}{2V}\times\frac{\pi^2}{H^2}}=\sqrt[3]{\frac{2.405^4\pi^4}{4V^2}}.$$

The RHS is a constant. The LHS is always greater or equal to this constant implies that this is the minimum value the LHS can achieve. The minimum is achieved when all the three positive terms are equal, which gives the condition $\frac{2.405^2 \pi H}{2V} = \frac{\pi^2}{H^2} \Rightarrow \left(\frac{2.405}{R}\right)^2 = 2\left(\frac{\pi}{H}\right)^2$.

For steady state,

$$1.021 \times 10^{-2} \left[\left(\frac{2.405}{R}\right)^2 + \left(\frac{\pi}{H}\right)^2\right] \Psi = 8.787 \times 10^{-3} \ \Psi.$$

Hence $H = 5.866$ m [***Acceptable Range* (5.870 to 5.890)**]
$R = 3.175$ m [***Acceptable Range* (3.170 to 3.180)**].

Note: Putting the condition in the RHS gives the minimum as $\frac{\pi^2}{H^2}$. From the condition we get $\frac{\pi^3}{H^3} = \frac{2.405^2 \pi^2}{2V} \Rightarrow \frac{\pi^2}{H^2} = \sqrt[3]{\frac{2.405^4 \pi^4}{4V^2}}$.

Note: The radius and height of the Tarapur 3 & 4 NR in Western India is 3.192 m and 5.940 m respectively.

C2 The fuel channels are in a square arrangement (Fig-III) with nearest neighbour distance 0.286 m. The effective radius of a fuel channel (if it were solid) is 3.617×10^{-2} m. Estimate the number of fuel channels F_n in the reactor and the mass M of UO_2 required to operate the NR in steady state. [1.0]

Solution: $F_n = 387$ and $M = 9.892 \times 10^4$kg.

<u>Detailed solution</u>: Since the fuel channels are in square pitch of 0.286 m, the effective area per channel is 0.286^2 m^2 $= 8.180 \times 10^{-2}$ m^2.

The cross-sectional area of the core is $\pi R^2 = 3.142 \times (3.175)^2 = 31.67$ m^2, so the maximum number of fuel channels that can be accommodated in the cylinder is the integer part of $\frac{31.67}{0.0818} = 387$.

Mass of the fuel=387×Volume of the rod×density

$$= 387 \times (\pi \times 0.03617^2 \times 5.866) \times 10600 = 9.892 \times 10^4 \text{kg}.$$

$F_n = 387$ [***Acceptable Range* (380 to 394)**]
$M = 9.892 \times 10^4$kg [***Acceptable Range* (9.000 to 10.00)**]

Note 1: (Not part of grading) The total volume of the fuel is $387 \times (\pi \times 0.03617^2 \times 5.866) = 9.332$ m^3. If the reactor works at 12.5 % efficieny then using the result of a-(iii) we have that the power output of the reactor is $9.332 \times 4.917 \times 10^8 \times 0.125 =$

573 MW.

Note 2: The Tarapur 3 & 4 NR in Western India has 392 channels and the mass of the fuel in it is 10.15 $\times 10^4$ kg. It produces 540 MW of power.

Alternative Solutions to sub-parts B2 and B3: Let σ be the scattering angle of the Moderator atom in the LF, taken clockwise with respect to the initial direction of the neutron before collision. Let U be the speed of the Moderator atom, in the LF, after collision. From momentum and kinetic conservation in LF we have

$$v_b = v_a \cos\theta_L + AU\cos\sigma, \tag{1}$$
$$0 = v_a \sin\theta_L - AU\sin\sigma, \tag{2}$$
$$\frac{1}{2}v_b^2 = \frac{1}{2}AU^2 + \frac{1}{2}v_a^2. \tag{3}$$

Squaring and adding eq(1) and (2) to eliminate σ and from eq(3) we get

$$A^2U^2 = v_a^2 + v_b^2 - 2v_a v_b \cos\theta_L,$$
$$A^2U^2 = Av_b^2 - Av_a^2, \tag{4}$$

which gives

$$2v_a v_b \cos\theta_L = (A+1)v_a^2 - (A-1)v_b^2. \tag{5}$$

(ii) Let v be the speed of the neutron after collision in the COMF. From definition of center of mass frame $v_m = \dfrac{v_b}{A+1}$.

$v_a \sin\theta_L$ and $v_a\cos\theta_L$ are the perpendicular and parallel components of v_a, in the LF, resolved along the initial direction of the neutron before collision. Transforming these to the COMF gives $v_a \sin\theta_L$ and $v_a\cos\theta_L - v_m$ as the perpendicular and parallel components of v. Substituting for v_m and for $2v_a v_b\cos\theta_L$ from eq(5) in $v = \sqrt{v_a^2\sin^2\theta_L + v_a^2\cos^2\theta_L + v_m^2 - 2v_a v_m\cos\theta_L}$ and simplifying gives $v = \dfrac{Av_b}{A+1}$. Squaring the components of v to eliminate θ_L gives $v_a^2 = v^2 + v_m^2 + 2vv_m\cos\theta$. Substituting for v and v_m and simplifying gives,

$$\frac{v_a^2}{v_b^2} = \frac{E_a}{E_b} = \frac{A^2 + 2A\cos\theta + 1}{(A+1)^2}.$$

$$G(\alpha,\theta) = \frac{E_a}{E_b} = \frac{A^2+1}{(A+1)^2} + \frac{2A}{(A+1)^2}\cos\theta = \frac{1}{2}\left[(1+\alpha) + (1-\alpha)\cos\theta\right].$$

(OR)

(iii) From definition of center of mass frame $v_m = \dfrac{v_b}{A+1}$. After the collision, let v and V be magnitude of the velocities of neutron and moderator atom respectively in the COMF. From conservation laws in the COMF,

$$v = AV \qquad \text{and} \qquad \frac{1}{2}(v_b - v_m)^2 + \frac{1}{2}Av_m^2 = \frac{1}{2}v^2 + \frac{1}{2}AV^2.$$

Solving gives $v = \frac{Av_b}{A+1}$ and $V = \frac{v_b}{A+1}$. We also have $v\cos\theta = v_a\cos\theta_L - v_m$, substituting for v_m and for $v_a\cos\theta_L$ from eq(5) and simplifying gives

$$\frac{v_a^2}{v_b^2} = \frac{E_a}{E_b} = \frac{A^2 + 2A\cos\theta + 1}{(A+1)^2}.$$

$$G(\alpha,\theta)=\frac{E_a}{E_b}=\frac{A^2+1}{(A+1)^2}+\frac{2A}{(A+1)^2}\cos\theta=\frac{1}{2}\left[(1+\alpha)+(1-\alpha)\cos\theta\right].$$

(OR)

(iv) From definition of center of mass frame $v_m=\dfrac{v_b}{A+1}$. After the collision, let v and V be magnitude of the velocities of neutron and moderator atom respectively in the CM frame. From conservation laws in the CM frame,

$$v=AV \qquad \text{and} \qquad \frac{1}{2}(v_b-v_m)^2+\frac{1}{2}Av_m^2=\frac{1}{2}v^2+\frac{1}{2}AV^2.$$

Solving gives $v=\frac{Av_b}{A+1}$ and $V=\frac{v_b}{A+1}$. $U\sin\sigma$ and $U\cos\sigma$ are the perpendicular and parallel components of U, in the LF, resolved along the initial direction of the neutron before collision. Transforming these to the COMF gives $U\sin\sigma$ and $-U\cos\sigma+v_m$ as the perpendicular and parallel components of V. So we get $U^2=V^2\sin^2\theta+V^2\cos^2\theta+v_m^2-2Vv_m\cos\theta$. Since $V=v_m$ we get $U^2=2v_m^2(1-\cos\theta)$. Substituting for U from eq(4) and simplifying gives

$$\frac{v_a^2}{v_b^2}=\frac{E_a}{E_b}=\frac{A^2+2A\cos\theta+1}{(A+1)^2}.$$

$$G(\alpha,\theta)=\frac{E_a}{E_b}=\frac{A^2+1}{(A+1)^2}+\frac{2A}{(A+1)^2}\cos\theta=\frac{1}{2}\left[(1+\alpha)+(1-\alpha)\cos\theta\right].$$

Note: We have $v_a=\dfrac{\sqrt{A^2+2A\cos\theta+1}}{A+1}v_b$. Substituting for v_a, v, v_m in $v\cos\theta=v_a\cos\theta_L-v_m$ gives the relation between θ_L and θ,

$$\cos\theta_L=\frac{A\cos\theta+1}{\sqrt{A^2+2A\cos\theta+1}}.$$

Treating the above equation as quadratic in $\cos\theta$ gives,

$$\cos\theta=\frac{-\sin^2\theta_L\pm\cos\theta_L\sqrt{A^2-\sin^2\theta_L}}{A}.$$

For $\theta_L=0°$ the root with the negative sign gives $\theta=180°$ which is not correct so,

$$\cos\theta=\frac{\cos\theta_L\sqrt{A^2-\sin^2\theta_L}-\sin^2\theta_L}{A}.$$

Substituting the above expression for $\cos\theta$ in the expression for $\dfrac{v_a^2}{v_b^2}$ gives an expression in terms of $\cos\theta_L$

$$\frac{v_a^2}{v_b^2}=\frac{E_a}{E_b}=\frac{A^2+2\cos\theta_L\sqrt{A^2-\sin^2\theta_L}+\cos 2\theta_L}{(A+1)^2}.$$

LIGO-GW150914 (10 points)

In 2015, the gravitational-wave observatory LIGO detected, for the first time, the passing of gravitational waves (GW) through Earth. This event, named GW150914, was triggered by waves produced by two black holes that were orbiting on quasi-circular orbits. This problem will make you estimate some physical parameters of the system, from the properties of the detected signal.

Part A: Newtonian (conservative) orbits (3.0 points)

A.1 Consider a system of two stars with masses M_1, M_2, at locations $\vec{r}_1, \vec{r}_2$, respectively, with respect to the center-of-mass of the system, that is, **1.0pt**

$$M_1\vec{r}_1 + M_2\vec{r}_2 = 0\ . \tag{1}$$

The stars are isolated from the rest of the Universe and moving at non-relativistic velocities. Using Newton's laws, the acceleration vector of mass M_1 can be expressed as

$$\frac{\mathrm{d}^2\vec{r}_1}{\mathrm{d}t^2} = -\alpha\frac{\vec{r}_1}{r_1^n}\ , \tag{2}$$

where $r_1 = |\vec{r}_1|, r_2 = |\vec{r}_2|$. Find $n \in \mathbb{N}$ and $\alpha = \alpha(G, M_1, M_2)$, where G is Newton's constant [$G \simeq 6.67 \times 10^{-11}$ N m^2 kg^{-2}].

A.2 The total energy of the 2-mass system, in circular orbits, can be expressed as: **1.0pt**

$$E = A(\mu, \Omega, L) - G\frac{M\mu}{L}\ , \tag{3}$$

where

$$\mu \equiv \frac{M_1M_2}{M_1+M_2}\ , \qquad M \equiv M_1 + M_2\ , \tag{4}$$

are the *reduced mass* and *total mass* of the system, Ω is the angular velocity of each mass and L is the total separation $L = r_1 + r_2$. Obtain the explicit form of the term $A(\mu, \Omega, L)$.

A.3 Equation 3 can be simplified to $E = \beta G\frac{M\mu}{L}$. Determine the number β. **1.0pt**

Part B: Introducing relativistic dissipation (7.0 points)

The correct theory of gravity, *General Relativity*, was formulated by Einstein in 1915, and predicts that gravity travels with the speed of light. The messengers carrying information about the interaction are called GWs. GWs are emitted whenever masses are accelerated, making the system of masses lose energy.

Consider a system of two point-like particles, isolated from the rest of the Universe. Einstein proved that for small enough velocities the emitted GWs: 1) have a frequency which is twice as large as the orbital frequency; 2) can be characterized by a luminosity, i.e. emitted power $\mathcal{P}$, which is dominated by Einstein's quadrupole formula,

$$\mathcal{P} = \frac{G}{5c^5}\sum_{i=1}^{3}\sum_{j=1}^{3}\left(\frac{\mathrm{d}^3 Q_{ij}}{\mathrm{d}t^3}\right)\left(\frac{\mathrm{d}^3 Q_{ij}}{\mathrm{d}t^3}\right) . \tag{5}$$

Here, c is the velocity of light $c \simeq 3\times 10^8$ m/s. For a system of 2 pointlike particles orbiting on the $x-y$ plane, Q_{ij} is the following table (i,j label the row/column number)

$$Q_{11} = \sum_{A=1}^{2}\frac{M_A}{3}\left(2x_A^2 - y_A^2\right) , \qquad Q_{22} = \sum_{A=1}^{2}\frac{M_A}{3}\left(2y_A^2 - x_A^2\right) , \qquad Q_{33} = -\sum_{A=1}^{2}\frac{M_A}{3}(x_A^2 + y_A^2) , \tag{6}$$

$$Q_{12} = Q_{21} = \sum_{A=1}^{2} M_A\, x_A\, y_A , \tag{7}$$

and $Q_{ij} = 0$ for all other possibilities. Here, (x_A, y_A) is the position of mass A in the center-of-mass frame.

B.1 For the circular orbits described in A.2 the components of Q_{ij} can be expressed as a function of time t as: 1.0pt

$$Q_{ii} = \frac{\mu L^2}{2}\left(a_i + b_i \cos kt\right) , \qquad Q_{ij} \overset{i\neq j}{=} \frac{\mu L^2}{2} c_{ij} \sin kt . \tag{8}$$

Determine k in terms of Ω and the numerical values of the constants a_i, b_i, c_{ij}.

B.2 Compute the power $\mathcal{P}$ emitted in gravitational waves for that system, and obtain: 1.0pt

$$\mathcal{P} = \xi \frac{G}{c^5}\mu^2 L^4 \Omega^6 . \tag{9}$$

What is the number ξ? [If you could not obtain ξ, use $\xi = 6.4$ in the following.]

B.3 In the absence of GW emission the two masses will orbit on a fixed circular orbit indefinitely. However, the emission of GWs causes the system to lose energy and to slowly evolve towards smaller circular orbits. Obtain that the rate of change $\frac{\mathrm{d}\Omega}{\mathrm{d}t}$ of the orbital angular velocity takes the form 1.0pt

$$\left(\frac{\mathrm{d}\Omega}{\mathrm{d}t}\right)^3 = (3\xi)^3 \frac{\Omega^{11}}{c^{15}}\left(GM_{\mathrm{c}}\right)^5 , \tag{10}$$

where M_{c} is called the *chirp mass*. Obtain M_{c} as a function of M and μ. This mass determines the increase in frequency during the orbital decay. [The name "chirp" is inspired by the high pitch sound (increasing frequency) produced by small birds.]

B.4 Using the information provided above, relate the orbital angular velocity Ω with the GW frequency f_{GW}. Knowing that, for any smooth function $F(t)$ and $a \neq 1$, **2.0pt**

$$\frac{\mathrm{d}F(t)}{\mathrm{d}t} = \chi F(t)^a \qquad \Rightarrow \qquad F(t)^{1-a} = \chi(1-a)(t-t_0)\,, \tag{11}$$

where χ is a constant and t_0 is an integration constant, show that (10) implies that the GW frequency is

$$f_{\text{GW}}^{-8/3} = 8\pi^{8/3}\xi\left(\frac{GM_c}{c^3}\right)^{(2/3)+p}(t_0-t)^{2-p} \tag{12}$$

and determine the constant p.

On September 14, 2015 GW150914 was registered by the LIGO detectors, consisting of two L-shaped arms, each 4 km long. These arms changed by a relative length according to Fig. 1. The arms of the detector respond linearly to a passing gravitational wave, and the response pattern mimics the wave. This wave was created by two black holes on quasi-circular orbits; the loss of energy through gravitational radiation caused the orbit to shrink and the black holes to eventually collide. The collision point corresponds, roughly, to the peak of the signal after point D, in Fig. 1.

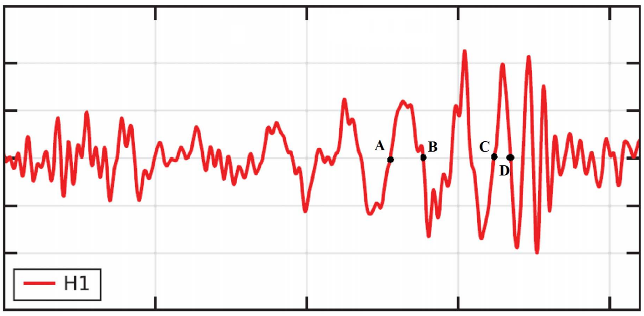


Figure 1. Strain, i.e. relative variation of the size of each arm, at the LIGO detector H1. The horizontal axis is time, and the points A, B, C, D correspond to $t = 0.000, 0.009, 0.034, 0.040$ seconds, respectively.

B.5 From the figure, estimate $f_{\text{GW}}(t)$ at **1.0pt**

$$t_{\overline{\text{AB}}} = \frac{t_{\text{B}}+t_{\text{A}}}{2} \qquad \text{and} \qquad t_{\overline{\text{CD}}} = \frac{t_{\text{D}}+t_{\text{C}}}{2}\,. \tag{13}$$

Assuming that (12) is valid all the way until the collision (which strictly speaking is not true) and that the two objects have equal mass, estimate the chirp mass, M_c, and total mass of the system, in terms of solar masses $M_\odot \simeq 2\times10^{30}$ kg.

B.6 Estimate the minimal orbital separation between the two objects at $t_{\overline{\text{CD}}}$. Hence estimate a maximum size for each object, R_{max}. Obtain $R_\odot/R_{\text{max}}$ to compare this size with the radius of our Sun, $R_\odot \simeq 7\times10^5$ km. Estimate also their orbital linear velocity at the same instant, v_{col}, comparing it with the speed of light, v_{col}/c. **1.0pt**

Where is the neutrino? (10 points)

When two protons collide with a very high energy at the Large Hadron Collider (LHC), several particles may be produced as a result of that collision, such as electrons, muons, neutrinos, quarks, and their respective anti-particles. Most of these particles can be detected by the particle detector surrounding the collision point. For example, quarks undergo a process called *hadronisation* in which they become a shower of subatomic particles, called "jet". In addition, the high magnetic field present in the detectors allows even very energetic charged particles to curve enough for their momentum to be determined. The ATLAS detector uses a superconducting solenoid system that produces a constant and uniform 2.00 Tesla magnetic field in the inner part of the detector, surrounding the collision point. Charged particles with momenta below a certain value will be curved so strongly that they will loop repeatedly in the field and most likely not be measured. Due to its nature, the neutrino is not detected at all, as it escapes through the detector without interacting.

Data: Electron rest mass, $m = 9.11 \times 10^{-31}$ kg; Elementary charge, $e = 1.60 \times 10^{-19}$ C;

Speed of light, $c = 3.00 \times 10^{8}$ m s^{-1}; Vacuum permittivity, $\epsilon_0 = 8.85 \times 10^{-12}$ F m^{-1}

Part A. ATLAS Detector physics (4.0 points)

A.1	Derive an expression for the cyclotron radius, r, of the circular trajectory of an electron acted upon by a magnetic force perpendicular to its velocity, and express that radius as a function of its kinetic energy, K; charge modulus, e; mass, m; and magnetic field, B. Assume that the electron is a non-relativistic classical particle.	0.5pt

Electrons produced inside the ATLAS detector must be treated relativistically. However, the formula for the cyclotron radius also holds for relativistic motion when the relativistic momentum is considered.

A.2	Calculate the minimum value of the momentum of an electron that allows it to escape the inner part of the detector in the radial direction. The inner part of the detector has a cylindrical shape with a radius of 1.1 meters, and the electron is produced in the collision point exactly in the center of the cylinder. Express your answer in MeV/c.	0.5pt

When accelerated perpendicularly to the velocity, relativistic particles of charge e and rest mass m emitt electromagnetic radiation, called synchrotron radiation. The emitted power is given by

$$P = \frac{e^2 a^2 \gamma^4}{6\pi\epsilon_0 c^3}$$

where a is the acceleration and $\gamma = [1-(v/c)^2]^{-1/2}$.

A.3	A particle is called ultrarelativistic when its speed is very close to the speed of light. For an ultrarelativistic particle the emitted power can be expressed as: $$P = \xi \frac{e^4}{\epsilon_0 m^k c^n} E^2 B^2 \,,$$ where ξ is a real number, n, k are integers, E is the energy of the charged particle and B is the magnetic field. Find ξ, n and k.	1.0pt

A.4	In the ultrarelativistic limit, the energy of the electron as a function of time is: $$E(t) = \frac{E_0}{1+\alpha E_0 t},$$ where E_0 is the initial energy of the electron. Find α as a function of e, c, B, ϵ_0 and m.	1.0pt

A.5	Consider an electron produced at the collision point along the radial direction with an energy of 100 GeV. Estimate the amount of energy that is lost due to synchrotron radiation until the electron escapes the inner part of the detector? Express your answer in MeV.	0.5pt

A.6	Find an expression for the cyclotron frequency of the electron as a function of time in the ultrarelativistic limit.	0.5pt

Part B. Finding the neutrino (6.0 points)

The collision between the two protons shown in Figure 1 leads to the production of a top quark (t) and an anti-top quark ($\bar{t}$), the heaviest elementary particles ever detected. The top quark decays into a W^+ boson and a bottom quark (b), while the anti-top quark decays into a W^- boson and an anti-bottom quark ($\bar{b}$). In the case depicted in Figure 1, the W^+ boson decays into a anti-muon (μ^+) and a neutrino (ν), and the W^- boson decays into a quark and an anti-quark. The task of this problem is to reconstruct the full momentum of the neutrino using the momenta of some detected particles. **For simplicity, all particles and jets in this problem will be considered massless, except for the top quark and W$^\pm$ bosons.**

The momenta of the top quark decay products can be determined from the experiment (see Table), except for the neutrino momentum component along the (z-axis). The total linear momentum of the final state particles caught by the detector is only zero on the transverse plane (xy plane), and not along the collision line (z-axis). As such, one can find the transverse momentum of the neutrino from the missing momentum in the transverse plane.

On June 4, 2015, the ATLAS experiment at the LHC recorded a proton-proton collision at 00:21:24 GMT+1 like the one represented in Figure 1.

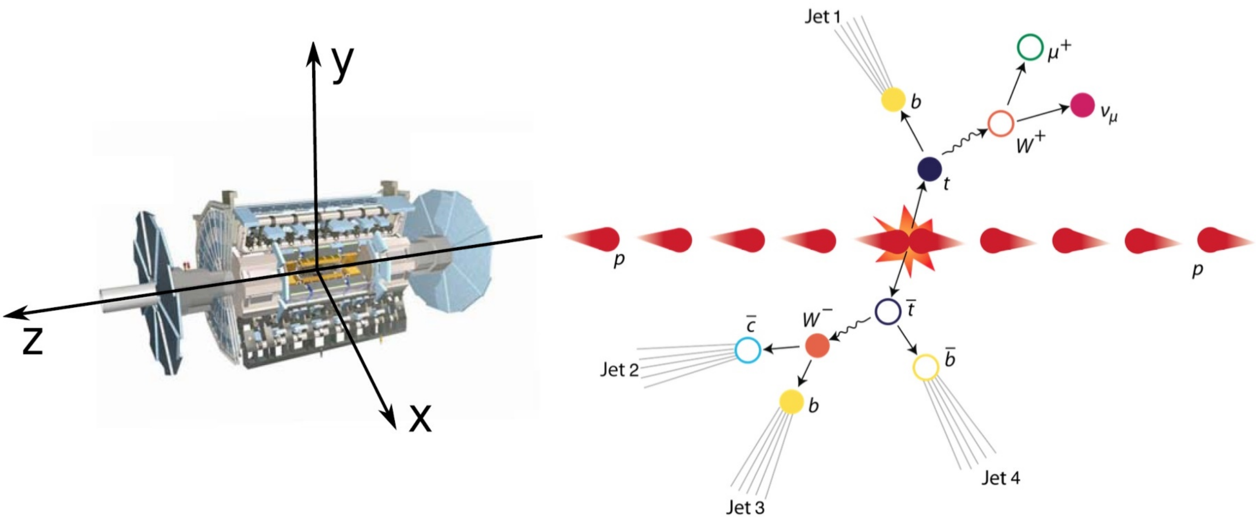


Figure 1. Schematic representation of the ATLAS detector coordinate system (left) and proton-proton collision (right).

The linear momenta of the three final-state particles coming from the top quark decay, including the neutrino, is presented below for each component.

Particle	p_x (GeV/c)	p_y (GeV/c)	p_z (GeV/c)
anti-muon (μ^+)	-24.7	-24.9	-12.4
jet 1 (j_1)	-14.2	$+50.1$	$+94.1$
neutrino (ν)	-104.1	$+5.3$	---

B.1 Find an equation which relates the square of the W^+ boson mass, m_{W}^2, with the neutrino and anti-muon momentum components presented in the table above. Express your answer in terms of the neutrino and anti-muon transverse momentum, (1.5pt)

$$\vec{p}_{\mathrm{T}}{}^{(\nu)} = p_x{}^{(\nu)}\,\hat{\imath} + p_y{}^{(\nu)}\,\hat{\jmath} \text{ and } \vec{p}_{\mathrm{T}}{}^{(\mu)} = p_x{}^{(\mu)}\,\hat{\imath} + p_y{}^{(\mu)}\,\hat{\jmath},$$

and their z-axis momentum components, $p_z{}^{(\mu)}$ and $p_z{}^{(\nu)}$.

B.2 Assuming a W^+ boson mass of $m_{\mathrm{W}} = 80.4$ GeV/c^2 calculate the two possible solutions for the neutrino momentum along the z-axis, $p_z{}^{(\nu)}$. Express your answer in GeV/c. (1.5pt)

B.3 Calculate the top quark mass for each one of the two previous solutions. Express your answer in GeV/c^2. (1.0pt)
[If you did not obtain the two solutions in B.2, use

$$p_z{}^{(\nu)} = 70 \text{ GeV}/c \quad \text{and} \quad p_z{}^{(\nu)} = -180 \text{ GeV}/c.]$$

The normalised number of collision events for the measurement of the top quark mass (as determined from the experiment), has two components: the so-called "signal" (corresponding to the decay of top quarks) and "background" (corresponding to events from other processes that do not include top quarks). Experimental data include both processes, see Fig. 2.

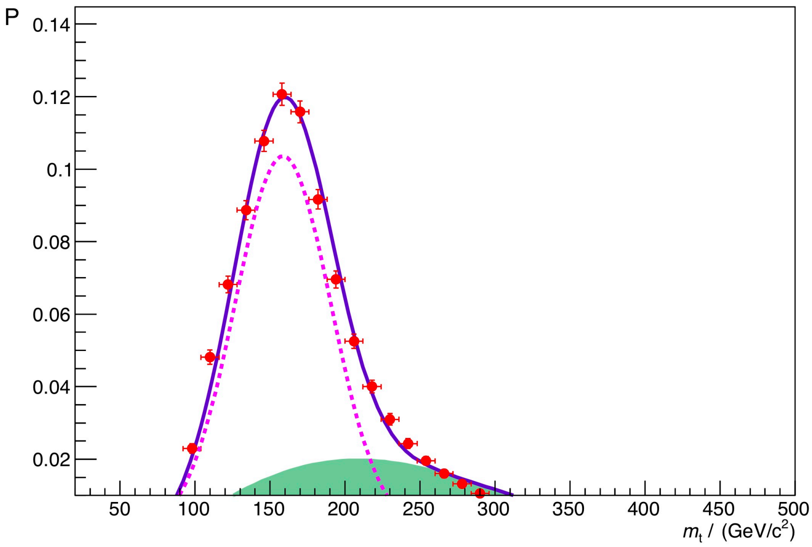


Figure 2. Top quark mass distribution as determined from the experiment, i.e. the normalised number of events plotted against the top quark mass. The dots correspond to the data. The dashed line corresponds to the "signal" and the shade to the "background".

B.4	According to the top quark mass distribution, which one of the two previous solutions is more likely to be the right one? Estimate the probability for the most likely solution.	1.0pt

B.5	Calculate the distance traveled by the top quark before decaying, using the most likely solution. Assume the top quark has a mean lifetime at rest of 5×10^{-25} s.	1.0pt

Physics of Live Systems (10 points)

Data: Normal atmospheric pressure, $P_0 = 1.013 \times 10^5$ Pa $= 760$ mmHg

Part A. The physics of blood flow (4.5 points)

In this part you will analyse two simplified models of blood flow in vessels.

Blood vessels are approximately cylindrical in shape, and it is known that for a steady, non turbulent flow of an incompressible fluid in a rigid cylinder, the difference in pressure of the fluid at the two ends of the cylinder is given by

$$\Delta P = \frac{8\ell\eta}{\pi r^4} Q \,, \tag{1}$$

where ℓ and r are the length and radius of the cylinder, η is the fluid viscosity and Q is the volumetric flow rate, i.e. the fluid volume that passes the cylinder cross section per unit time. This expression is often able to provide the correct order of magnitude for the pressure difference in a vessel, even without taking into account the pulsatile flow, the vessel's compressibility and irregular shape, and the fact that blood is not a simple fluid but a mixture of cells and plasma. Moreover, this expression has the same form as Ohm's law, with the volumetric flow rate being interpreted as a current, the difference in pressure as a voltage, and the factor $R = \frac{8\ell\eta}{\pi r^4}$ as a resistance.

Consider for example the symmetrical network of arterioles (small arteries) depicted in Figure 1 that delivers blood to the capillary bed of a tissue. In this network, at each bifurcation a vessel is divided in two identical vessels. However, the vessels of higher levels are thinner and shorter: consider that the radii and lengths of vessels in two consecutive levels, i and $i+1$, are related by $r_{i+1} = r_i/2^{1/3}$ and by $\ell_{i+1} = \ell_i/2^{1/3}$.

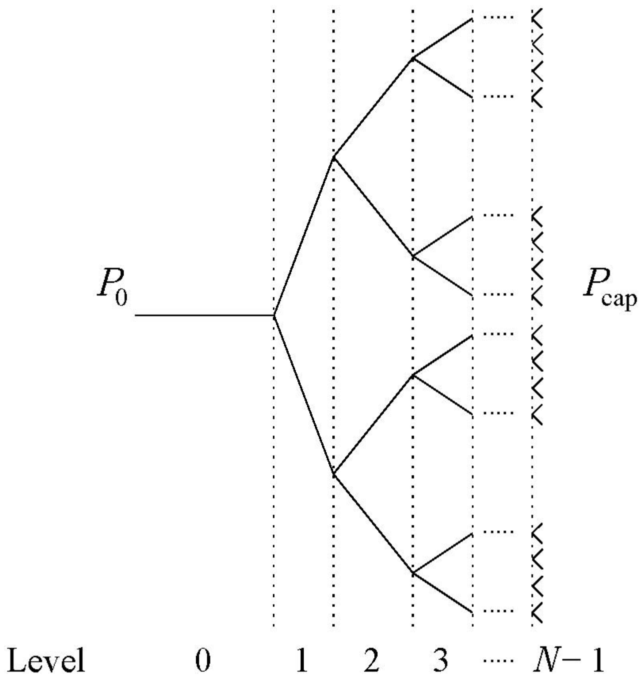


Figure 1. Network of arterioles.

A.1	Obtain an expression for the volumetric flow rate, Q_i, in a vessel at any level i, as a function of the total number of levels N, of the viscosity η, of the radius r_0 and length ℓ_0 of the first vessel, and of the difference $\Delta P = P_0 - P_{\text{cap}}$ between the pressure at the arteriole at level 0, P_0, and the pressure at the capillary bed, P_{cap}.	1.3pt

A.2	Calculate the numerical value of the volumetric flow rate Q_0 of the arteriole at level 0, if its radius is 6.0×10^{-5} m and its length is 2.0×10^{-3} m. Consider that the pressure at the arteriole inlet is 55 mmHg and the vessel network has $N = 6$ levels linking this arteriole to the capillary bed at the pressure 30 mmHg. Consider that the blood viscosity is $\eta = 3.5 \times 10^{-3}$ kg m^{-1} s^{-1}. Express your result in ml/h.	0.5pt

A blood vessel as an LCR circuit

The approximation of rigid cylindrical vessels falls short for several reasons. It is particularly important to include the time dependent flow and to take into account the change in vessel diameter that occurs when the pressure varies during a blood pumping cycle done by the heart. Moreover, it is observed that in the larger vessels the blood pressure varies significantly during a cycle, while in the smaller vessels the amplitude of the oscillations in pressure is much smaller, and the flow is almost time independent.

When the pressure increases in a single elastic vessel, there will be an increase in its diameter, thus permitting to store more fluid in the vessel, and to deliver it when the pressure drops. For this reason, the elastic behaviour of the vessel can be simulated by adding a capacitor to our initial description. Moreover, when taking into account the time dependent blood flow rate, one has to consider the inertia of the fluid, proportional to its density $\rho = 1.05 \times 10^3$ kg m^{-3}. This inertia can be described by an inductance in our model. In Figure 2 we represent the equivalent circuit for a single vessel in this model. The equivalent capacitance and inductance are given by

$$C = \frac{3\ell\pi r^3}{2Eh} \quad \text{and} \quad L = \frac{9\ell\rho}{4\pi r^2}\,, \tag{2}$$

respectively, where h is the width of the vessel wall and E is the artery Young's modulus, a coefficient that describes the alteration in size of the vessel tissue when a force is applied. The Young's modulus has units of pressure and is on the order of $E = 0.06$ MPa for arterioles.

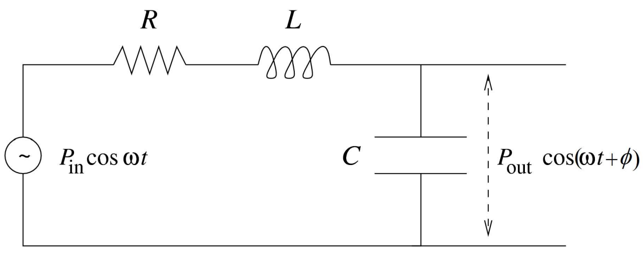


Figure 2. Equivalent electric circuit for a single vessel.

A.3	Obtain, in the stationary regime, the pressure amplitude at the vessel outlet, P_{out}, as a function of the pressure amplitude at the inlet, P_{in}, the equivalent resistance, R, inductance, L and capacitance, C, for a flow with angular frequency ω. Establish the condition between η, ρ, E, h, r and ℓ so that, for low frequencies, the pressure oscillation amplitude at the outlet is smaller than that of P_{in}.	2.0pt

A.4	For the vessel network in **A.2** estimate the maximum arteriole wall thickness h so that the condition established in **A.3** is satisfied (consider that h is level independent).	0.7pt

Part B. Tumour growth (5.5 points)

Tumour growth is a very complex process where biological mechanisms such as cell proliferation and natural selection are intertwined with physics. In this problem we will consider a simplified model of tumour growth that addresses the increase in pressure commonly observed in solid tumors.

Consider a group of normal cells forming a tissue surrounded by an inextensible basement membrane, which forces the tissue to maintain always the same form: a sphere of radius R (Figure 3).

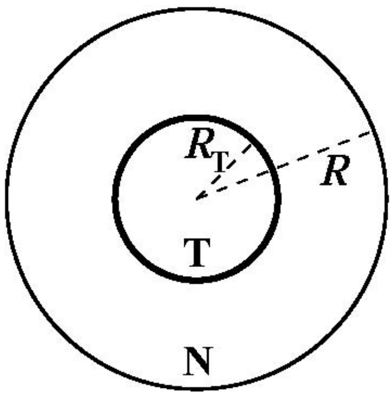


Figure 3. Simplified tumour.

Initially the tissue does not have residual stresses, i.e. the pressure at every point is equal to the atmospheric pressure.

At time $t = 0$, a tumour starts growing at the centre of this sphere and, as it grows, the pressure inside the tissue increases. Consider that both tissues (normal, N, and tumour, T) are compressible such that their densities, ρ_{N} and ρ_{T}, increase linearly with pressure:

$$\rho_{\text{N}} = \rho_0 \left(1 + \frac{p}{K_{\text{N}}}\right) , \quad \rho_{\text{T}} = \rho_0 \left(1 + \frac{p}{K_{\text{T}}}\right) , \tag{3}$$

where ρ_0 is the rest tissue density, p is the pressure difference to the atmospheric pressure and $K_{\text{N}}, K_{\text{T}}$ are the compressibility moduli (bulk moduli) of the normal and tumour tissues, respectively. In general, tumours are stiffer and so they have a higher bulk modulus.

B.1	The mass of normal cells is not altered while the tumour is growing. Obtain the ratio between the tumour volume and the total tissue volume, $v = V_\mathrm{T}/V$, as a function of the ratio between the tumour mass (M_T) and the normal tissue mass (M_N), $\mu = M_\mathrm{T}/M_\mathrm{N}$ and the ratio of the bulk moduli, $\kappa = K_\mathrm{N}/K_\mathrm{T}$.	1.0pt

Hyperthermia is sometimes used together with chemotherapy and radiotherapy in the treatment of cancer. In hyperthermia the cancer cells are selectively heated from the normal human body temperature, 37 °C, to temperatures above 43 °C, inducing their death. Researchers are currently developing carbon nanotubes covered with special proteins capable of binding to tumour cells. When the tissue is irradiated with near-infrared radiation, the nanotubes absorb it in a much greater extent than the surrounding tissues and therefore can be selectively heated as well as the tumour cells to which they are attached.

Consider that the tumour, the normal cells and the surrounding tissue have a constant thermal conductivity k, i.e. in the geometry of this problem, the energy that crosses a spherical surface of radius r per unit time and per unit area is equal to k times the derivative of the temperature with respect to r. The nanotubes are uniformly distributed in the tumour volume and are able to deliver a power $\mathcal{P}$ of thermal energy per unit volume. Assume that the temperature is equal to the normal human body temperature very far away from the tumour.

B.2	Obtain, for the stationary state, the temperature at the centre of the tumour as a function of $\mathcal{P}$, k, the human body temperature and the tumour radius, R_T.	1.7pt

B.3	Obtain the minimum power per unit volume, $\mathcal{P}_\mathrm{min}$, needed to heat up all tumour cells in a tumour with 5.0 cm radius to a temperature larger than 43.0 °C. Take the thermal conductivity of the tissue to be equal to $k = 0.60$ W K^{-1}m^{-1}.	0.5pt

Consider that the tumour is irrigated by a vessel network with a branched structure like in question **A.1**. As the tumour grows, when its pressure p becomes larger than the pressure P_cap at the thinnest vessels, the radii of these vessels will decrease by a small amount δr. If this pressure reaches a critical value p_c (which would correspond to a radius decrease of δr_c), the thinnest vessels would collapse, compromising seriously the irrigation to the tumour. The pressure and the radius change can be related by the following phenomenological relation:

$$\frac{p}{P_\mathrm{cap}} - 1 = \left(\frac{p_\mathrm{c}}{P_\mathrm{cap}} - 1\right)\left(2 - \frac{\delta r}{\delta r_\mathrm{c}}\right)\frac{\delta r}{\delta r_\mathrm{c}} \,. \tag{4}$$

Consider that just the smallest vessels (of level $N-1$) have their radius altered when the tumour increases its pressure.

B.4	In the linear regime (i.e. consider that $p - P_\mathrm{cap}$ is very small), express the relative drop in the flow rate, $\frac{\delta Q_{N-1}}{Q_{N-1}}$, in these thinnest vessels, as a function of the tumour volume ratio $v = V_\mathrm{T}/V$ and K_N, N, p_c, δr_c, r_{N-1}, P_cap.	2.3pt

两个力学问题 (10 分)

开始做题之前，请仔细阅读另外一个信封中的理论考试指南。

A 部分. 隐藏的盘子 (3.5 分)

一个半径为 r_1、厚度为 h_1 的实心木质圆柱体，在其内部挖一个孔，以固定一个半径为 r_2、厚度为 h_2 的圆柱形金属盘。金属盘的对称轴为 B，木质圆柱体的对称轴为 S，二轴互相平行，其距离为 d；金属盘与木质圆柱的上下两底面的距离相等。木质圆柱体的密度为 ρ_1，金属盘的密度 $\rho_2 > \rho_1$。木质圆柱体与金属盘的总质量为 M。

在此问中，将木质圆柱如图 1 所示放置在地面上，木质圆柱可以左右自由滚动。图 1 分别为侧视图与俯视图。

本小题的目标是：求出金属盘的尺寸 r_2 和 h_2 以及在木质圆柱中的位置 d。

在下面的各问题中，当要求用已知物理量表示计算结果时，始终假定以下几个参量是已知的：

$$r_1,\ h_1,\ \rho_1,\ \rho_2, M\,. \tag{1}$$

本小题目标是：通过非接触测量，求出金属盘的尺寸 r_2 和 h_2 以及在木质圆柱中的位置 d。

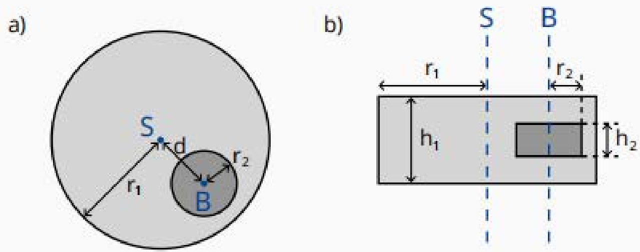


图 1：a) 侧视图；b) 俯视图

b 为整个系统的质心 C 与木质圆柱对称轴 S 之间的距离。为确定 b，我们设计了如下实验：将系统放置在一个水平基座上，呈稳定平衡状态。缓慢倾斜基座使之与地面成 Θ 角 (见图 2)。由于静摩擦力，圆柱与斜面间没有滑动，而只有自由滚动。当圆柱沿斜面滚动一小段距离后，再次进入一个稳定平衡的静止状态，测得再次平衡时圆柱转过的角度为 ϕ。

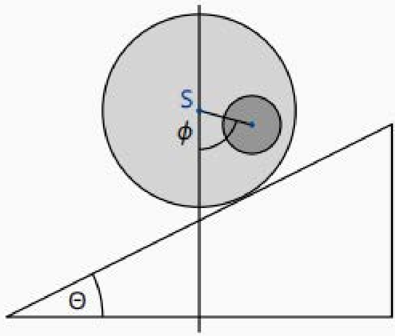


图 2. 倾斜的基座及圆柱

A.1	求出 b 的表达式，用 (1) 中的某参量以及 ϕ 和 Θ 表示。	0.8pt

后面的问题中，我们可以假定 b 为已知。

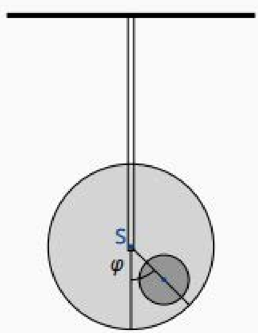


图 3. 悬挂的圆柱。

接下来，我们想测量系统关于 S 轴的转动惯量 I_S。为此，使用硬杆固定 S 轴将圆柱悬挂起来，然后使之相对于平衡位置转过一个小角度并释放，如图 3 所示。可以看到系统转角 φ 作周期运动，周期为 T。

A.2	求出 ϕ 的运动学方程，指出 φ 做何种方式运动？求出系统关于轴 S 的转动惯量 I_S，用 T, b 和 (1) 中的某一已知参量等表示，推导中假定 φ 是很小的小量。	0.5pt

通过问题 **A.1** 和 **A.2** 的测量，我们现在来进一步确定金属盘的几何尺寸及其在木质圆柱中的位置 d。

A.3	求出 d 的表达式，用 b 和 (1) 中的某些已知参量等表示。表达式中还要包含变量 r_2 和 h_2，这两个变量将在问题 **A.5** 中进行计算。	0.4pt

A.4	求出转动惯量 I_S 的表达式，用 b 和 (1) 中的所有已知参量等表示。表达式中还要包含变量 r_2 和 h_2，这将在问题 **A.5** 中进行计算。	0.7pt

A.5	用以上那些结果，求出 r_2 和 h_2 的表达式，用 b, T 和 (1) 中已知的参量表示。为书写简单，你可以在 h_2 的表达式中包含 r_2。	1.1pt

B 部分. 旋转的空间站 (6.5 分)

爱丽丝是一个住在空间站的宇航员。空间站是个半径为 R 的巨大转轮，转轮绕其自身中轴旋转，由此为宇航员提供了一个人造的重力。宇航员生活在转轮的内部边缘处。空间站质量很轻，万有引力可忽略；空间站地面的曲率可忽略。

B.1	空间站以多大角频率 ω_{ss} 旋转，可以使宇航员感受到与在地面相同的重力加速度 g_E?	0.5pt

爱丽丝和她的宇航员朋友鲍勃发生争论。鲍勃不相信他们实际生活在空间站，而是生活在地球。爱丽丝想用物理证明给 Bob 他们确实生活在旋转的空间站。于是，她将质量为 m 的物体挂在弹性系数为 k 的弹簧上，并让其振动。重物只在垂直方向振动，而在水平方向上无运动。

B.2	假设是在地球上实验，重力加速度 g_E 是恒定的，地球上的人可测得上述振动的角频率 ω_E 为多大?	0.2pt

B.3	爱丽丝在空间站做实验，测得的角频率 ω 为多大?	0.6pt

爱丽丝确信她的实验结果能够证明他们生活在一个旋转的空间站。但鲍勃仍保持怀疑，他声称当考虑地球表面以外引力随空间的变化，仍可得到类似现象。接下来我们探究为何鲍勃是对的。

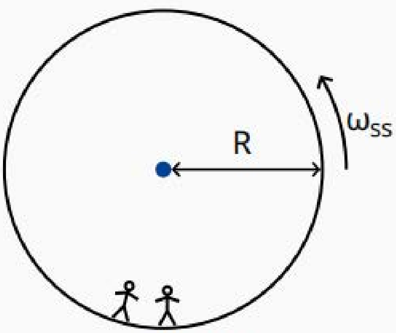


图 4. 空间站

B.4	推导地球表面附近高度 h 为小量处的重力加速度 $g_E(h)$ 的表达式并求出振动角频率 $\tilde{\omega}_E$ (线性近似足矣)。已知地球半径为 R_E。忽略地球自转。	0.8pt

事实上，爱丽丝确实发现对于该空间站，弹簧振子的振动与鲍勃所指出的频率一致。

B.5	空间站的半径 R 为多大时，空间站的振动角频率 ω 和地球上的 $\tilde{\omega}_E$ 相一致？结果用 R_E 表示。	0.3pt

被鲍勃的顽固所激怒，爱丽丝想到了一个实验来证明她的观点。为此她爬到距空间站地面高为 H 的塔顶并下落了一个物体。这个实验可以理解为在一转动参照系中，也可以理解为在一惯性参照系中进行。

在一匀速转动参照系中，宇航员会感受到一个称作科里奥利力的力 $\vec{F}_C$。在角速度为 $\vec{\omega}_{ss}$ 转动参照系中，作用于质量为 m，以速度 $\vec{v}$ 运动的物体上的科里奥利力 $\vec{F}_C$ 表示为：

$$\vec{F}_C = 2m\vec{v} \times \vec{\omega}_{ss} \,. \tag{2}$$

也可以用标量表示为：

$$F_C = 2mv\omega_{ss} \sin\phi \,, \tag{3}$$

式中 ϕ 为速度与旋转轴之间的夹角，科里奥利力的方向与速度 v 和转轴都垂直。力的方向由右手定则确定。

B.6	计算物体落到地面时的水平方向速度 v_x 和水平方向上的位移 d_x（相对于塔基，在垂直于塔的方向上的位移），假定 H 较小，能够保证整个下落过程中加速度不变，并假定 $d_x \ll H$。	1.1pt

为得到一个更好的结果，爱丽丝决定在一个更高的塔实施实验，让她惊奇的是，这一次重物竟然落在了塔基处，即 $d_x = 0$。

B.7	求出发生上述现象（即 $d_x = 0$）时塔的最小高度。	1.3pt

爱丽丝想为说服鲍勃做最后努力。她想用她的弹簧振子演示科里奥利力的效应。于是她改变了原来的实验装置：她将弹簧挂在一个环上，环可以在沿 x 方向的水平杆上无摩擦自由滑动。弹簧本身在 y 方向振动。杆平行于空间站地面，垂直于空间站转轴，xy 平面垂直于空间站转轴，y 方向指向空间站转动中心。

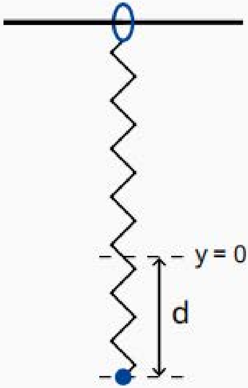


图 5. 装置

B.8 爱丽丝向下方拉动重物离开平衡点（平衡点坐标为 $x=0, y=0$）距离为 d，然后释放（见图 5）。 1.7pt

- 求出 $x(t)$ 和 $y(t)$ 的数学表达式，可假定 $\omega_{ss}d$ 较小，并忽略运动引起的 y 方向上科里奥利力。
- 画出运动轨迹 $(x(t), y(t))$。标示出所有重要特征，如振幅等。

电路中的非线性力学 (10 分)

开始做题之前请仔细阅读另外一个信封中的考试指南

介绍

双稳态非线性半导体元件（例如晶闸管）广泛用于电子开关和电磁振荡发生器。晶闸管应用的主要领域是在电力电子产品中控制交变电流，例如，将兆瓦级别交流电整流为直流电。双稳态元件也可以作为模型系统，以模拟物理学中的自组织现象（B 部分讨论的就是这个问题），还可用于生物学（见 C 部分）和现代非线性科学等领域。

目标

电路中含有非线性 I-V 特性的元件，研究这种电路的不稳定性和复杂动力学。探究此类电路在工程和生物系统建模中的应用。

A 部分 定态与不稳定性（3 分）

图 1 所示的是某非线性元件 X 具有的所谓 S-形的 $I-V$ 特性。在电压范围 $U_{\mathrm{h}} = 4.00\ \mathrm{V}$（保持电压）和 $U_{\mathrm{th}} = 10.0\ \mathrm{V}$（阈值电压）间，此 I-V 特性电压是多值的。为简单起见，图 1 中的图形可被视为分段的线性直线（每个分段都是直线的一段）。特别注意，如果延长上段直线，其延长线将经过原点。这种近似是对真实晶闸管的一个相当好的描述。

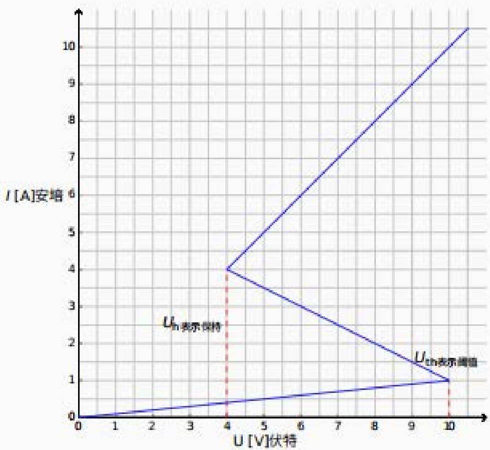


图 1：非线性元件 X 的 $I-V$ 特性图

A.1	使用图 1，分别计算 X 元件 $I-V$ 特性图线上方分段的电阻的值 R_{on}，下方分段的电阻的值 R_{off}。而中间分段的电阻 R_{int} 满足 $$I = I_0 - \frac{U}{R_{\mathrm{int}}}. \qquad (1)$$ 计算参数 I_0 和 R_{int} 的值	0.4pt

如图 2，元件 X 和电阻 R、电感 L 及一个理想电压源 $\mathcal{E}$ 串联在一起，当电路的电流不随时间而变，即 $I(t)=$ 常数 时，我们称此电路处于定态（静态）。

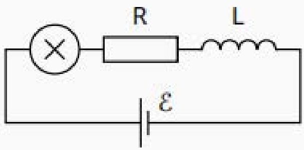


图 2：元件 X、电阻 R、电感 L 和电压源 $\mathcal{E}$ 的电路

A.2	图 2 电路中，当 $\mathcal{E}$ 值不变，$R=3.00\,\Omega$ 时，可能的定态数量是多少？当 $R=1.00\,\Omega$ 时的定态又是几个？	1pt

A.3	在图 2 中，设 $R=3.00\,\Omega, L=1.00\,\mu\mathrm{H}$ 和 $\mathcal{E}=15.0\,\mathrm{V}$，计算定态时非线性元件 X 上的电流 $I_{\mathrm{stationary}}$ 的值和电压 $V_{\mathrm{stationary}}$ 的值。下标 stationary 的意思是定态。	0.6pt

当图 2 处于电流 $I(t)=I_{\mathrm{stationary}}$ 的定态下，如果将电流稍微改变一个微小量 (电流增加或减少) 之后, 电流趋向于返回到定态，则称这种定态是稳定的。如果系统趋向于远离定态, 则它是不稳定的。

A.4	使用 A.3 小题的数值结果，研究 $I(t)=I_{\mathrm{stationary}}$ 的定态的稳定性。此定态是稳定的还是不稳定的？	1pt

B 部分 双稳态非线性元件的物理应用：无线电发射器（5 分）

现在我们研究一个新的电路配置（见图 3）。此时，非线性元件 X 与电容 $C=1.00\,\mu F$ 的电容器并联，然后再与 $R=3.00\,\Omega$ 的电阻和 $\mathcal{E}=15.0\,\mathrm{V}$ 的理想恒压源串联。事实证明这个电路会发生振荡，其非线性元件 X 在一个振荡周期内会从 $I-V$ 特性的一个分段跳变到另一个分段。

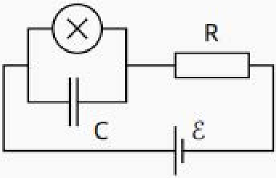


图3：包含元件 X、电容 C、电阻 R 和电压源 $\mathcal{E}$ 的电路。

B.1	在 $I-V$ 图上绘制振荡周期图，包括它的方向（顺时针或逆时针）。并使用方程和作图证明你的答案。	1.8pt

B.2	时间 t_1 和 t_2 为一个振荡周期内 $I-V$ 图线每个分段上经历的时间，求出 t_1 和 t_2 的表达式，并计算出它们的数值。假设 $I-V$ 图中，从一个分段跳跃到另一分段所需的时间忽略不计，求出振荡周期 T 的数值。	1.9pt

B.3	估算一个振荡周期内，非线性元件消耗的平均功率 P。给出其数量级。	0.7pt

图 3 中的电路被用作无线电发射器。出于该用途，元件 X 被连接到长度为 s 的线性天线（长直导线）的一端。天线的另一端是自由的。在天线上形成一个电磁波驻波。沿着天线的电磁波的速度和真空中的是一样的。发射器使用系统的主谐波，其周期为 B.2 小题中的周期 T。

B.4	在天线长度 s 不超过 1km 的情况下，s 的最佳值是多少?	0.6pt

C 部分 双稳态非线性元件的生物学应用：类神经器件（2 分）

在该问题中，我们考虑 双稳态非线性元件在生物过程建模中的应用。人脑中的一个神经元具有以下特性：当受外部信号刺激时，它产生一次振荡，然后返回到它的初始状态。此特性被称为应激性。由于这种特性，脉冲可以在构建神经系统的耦合神经元网络中传播。被设计成模拟应激性和脉冲传播的半导体芯片称为类神经器件(neuristor) （neuristor 由英文中神经元和晶体管两个单词而衍生）

使用我们先前研究的包含非线性元素 X 的电路，我们来模拟类神经器件。为此，将图 3 电路中的电压 $\mathcal{E}$ 减小到值 $\mathcal{E}'=12.0\,\mathrm{V}$，此时振荡停止，并且系统达到它的定态。然后，电压迅速增加回升至 $\mathcal{E}=15.0\,\mathrm{V}$，并在经过一段时间 τ 后（且 $\tau<T$）被再次设置到值 $\mathcal{E}'$（如图 4 所示）。事实证明，存在一个时间临界值 $\tau_{\mathrm{crit.}}$，而且，当 $\tau<\tau_{\mathrm{crit}}$ 和 $\tau>\tau_{\mathrm{crit}}$ 时系统的行为是完全不同的。

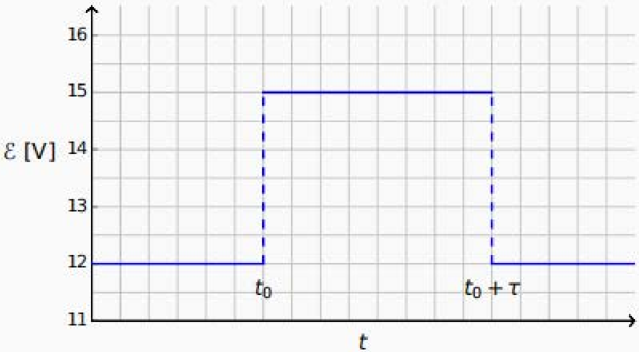


图 4： 电压源的电压随时间变化

C.1	绘制非线性元件 X 上的电流 $I_X(t)$ 随着时间变化的图，分 $\tau < \tau_{\text{crit}}$ 和 $\tau > \tau_{\text{crit}}$ 两种情况分别画出两个图。	1.2pt
C.2	求出电流跳变切换的临界时间 τ_{crit} 的表达式，并计算其数值。	0.6pt
C.3	对于以上电路，当 $\tau = 1.00 \times 10^{-6}$ s 时，能模拟一个类神经器件吗？	0.2pt

大型强子对撞机（10 分）

在开始答题前，请阅读另外一个信封中的理论考试指南。

本题讨论在欧洲核子研究中心（CERN）的粒子加速器：LHC（大型强子对撞机）中的物理问题。CERN 是世界上最大的粒子物理实验室。它的主要研究目标是观测自然的基本规律。通过强磁场下的加速器圆环，两束粒子加速到很高的能量，然后相互碰撞。粒子在加速器圆周上的分布虽然是不均匀的, 但是它们会聚集形成所谓的粒子束。由此碰撞而产生的新的粒子，可以通过一些大型探测器进行观测。表 1 给出了 LHC 的一些参数。

LHC 圆环	
圆环的周长	$26659\ \mathrm{m}$
质子流包含的束数	2808
每一束包含的质子数目	1.15×10^{11}
质子流	
质子的能量	$7.00\ \mathrm{TeV}$
质心能量	$14.0\ \mathrm{TeV}$

表 1: LHC 相关参数的典型值

粒子物理学家对能量、动量和质量采取简便的单位: 能量以电子伏特【eV】为单位。根据定义，$1\,\mathrm{eV}$ 是一个带有基本电荷电量 e 的粒子穿过一伏特的电位差后所获得的能量。（$1\ \mathrm{eV} = 1.602 \cdot 10^{-19}\ \mathrm{kg\ m^2/s^2}$。）

动量以 eV/c 为单位，质量以 eV/c^2 为单位，这里 c 是真空中的光速。因为 1ev 是相当小的能量，粒子物理学家通常使用 MeV（$1\ \mathrm{MeV} = 10^6\ \mathrm{eV}$）、$\mathrm{GeV}$（$1\ \mathrm{GeV} = 10^9\ \mathrm{eV}$）和 TeV（$1\ \mathrm{TeV} = 10^{12}\ \mathrm{eV}$）。

题目的 A 部分研究质子或电子的加速。B 部分关注于如何识别欧洲核子研究中心碰撞产生的各种粒子。

A 部分. LHC 加速器 (6 分)

加速:

假设质子在加速电压 V 下被加速到速度非常接近光速，忽略由于辐射和与其他粒子碰撞引起的任何能量损失。

A.1	求质子的末速度 v 的确切表达式，以加速电压 V 和一些物理常量表示。	0.7pt

设计一个未来在欧洲核子研究中心的实验计划，利用 LHC 产生的质子与能量为 $60.0\ \mathrm{GeV}$ 的电子碰撞。

A.2	对于具有高速能量和低质量的粒子，末速度 v 与光速的相对偏差 $\Delta = (c - v)/c$ 是很小的。求出 Δ 的一阶近似表达式，用加速电压 V 和一些物理常数来表示。对于能量为 $60.0\ \mathrm{GeV}$ 电子，计算 Δ 的值。	0.8pt

我们现在接着研究 LHC 中的质子。假设质子束流的截面是圆形。

A.3	要使质子束在保持在圆形轨道上，求出均匀磁感应强度 B 的表达式。表达式应该只包含质子的能量 E, 周长 L, 一些基本常数和数字。如果能保证对计算结果的影响小于有效数字的最少位数物理量所给出的精度，你可以采取适当的近似。 要获得 $E = 7.00\ \mathrm{TeV}$ 能量的质子，求出均匀磁感应强度 B 的值，计算中忽略质子间的相互碰撞。	1.0pt

辐射功率

加速带电粒子以电磁波的形式向外辐射能量。一个具有恒定角速度做圆周运动的带电粒子，其辐射功率 P_{rad} 仅仅只取决于加速度 a, 所带电荷 q, 光速 c 和自由空间的介电常数 ε_0。

A.4	用量纲分析法，求出辐射功率 P_{rad} 的表达式。	1.0pt

真实的辐射功率公式还包含一个因子 $1/(6\pi)$; 并且由全相对论可以推导出，还需要再乘以一个额外因子 γ^4， 这里 $\gamma = (1-v^2/c^2)^{-\frac{1}{2}}$。

A.5	LHC 的一个质子的能量 $E = 7.00\ \mathrm{TeV}$(在表 1 中已给出)，计算这个质子的总辐射功率 P_{tot} 的值，计算中可以使用适当的近似。	1.0pt

直线加速:

在 CERN，质子由静止状态，经过长度 $d = 30.0\ \mathrm{m}$，电位差 V = 500 MV 的直线加速器加速。假定加速电场是均匀的，这个线性加速器由两个平板组成，图 1 为其示意图。

A.6	求出质子通过这个场的时间 T 的数值。	1.5pt

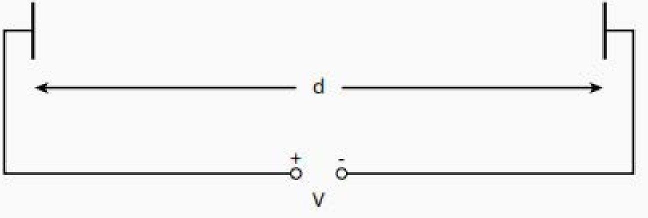


图 1: 加速器模块示意图。

B 部分. 粒子的识别 (4 分)

飞行时间

对碰撞中生成的高能粒子进行识别，对解释碰撞相互作用过程是很重要的。识别粒子的一个简单的方法，是测量一个已知动量的粒子通过长度为 l 的一种叫做飞行时间 (ToF) 探测器所需要的时间 (t)。通过探测器的典型粒子的质量见表 2。

粒子	质量 [MeV/c^2]
氘核	1876
质子	938
带电 K 介子	494
带电 π 介子	140
电子	0.511

表 2: 各种粒子及其质量.

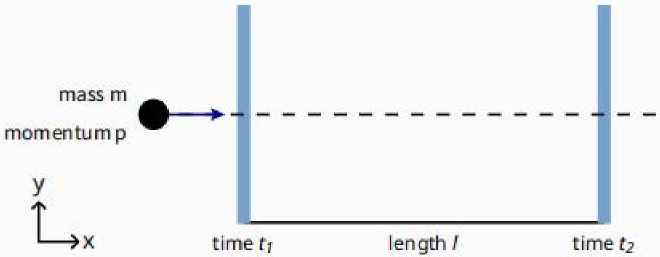
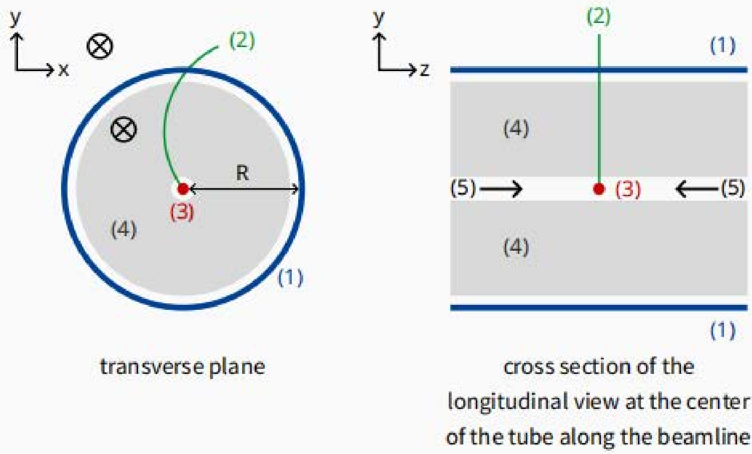


图 2: 飞行时间 (ToF) 探测器结构示意图，图中英文含义：mass 质量，momentum 动量，time 时间，length 长度

B.1	假设带有基本电荷电量 e 的粒子以接近光速 c 的速度，沿直线匀速通过 ToF 探测器，并且飞行的直线轨迹与探测器的两个探测平面垂直 (见图 2)。求粒子的静止质量 m 的函数表达式，用动量 p、飞行长度 l 和飞行时间 t 表示。	0.8pt

B.2	要准确分辨动量均为 1.00 GeV / c 的带电 k 介子和带电 π 介子，计算 ToF 探测器所需的最小长度 l。为了很好区分这两种粒子，要求飞行时间的差异大于 ToF 探测器分辨时间的三倍。对于 ToF 探测器，分辨时间的典型值是 $150\ \mathrm{ps}$ $(1\ \mathrm{ps} = 10^{-12}\ \mathrm{s})$。	0.7pt

在后面的题目中，对产生粒子进行探测的典型 LHC 探测器分为两个：一个是轨迹探测器，另一个是 ToF 探测器。图 3 在与质子束流平行的和垂直的两个平面内展示了装置结构。两个探测器是位于相互作用区的一些管子，粒子束流通过这些管子的中心。轨迹探测器测量带电粒子通过磁场时的运行轨迹，磁场的方向平行于质子束流的方向。通过运行轨道的半径 r 可以确定粒子的横向动量 p_T。因为碰撞时间已知，ToF 探测器只需要用一个管子进行飞行时间测量 (从碰撞到被 ToF 管探测到所经过的时长)。这个 ToF 管位于轨道探测器后面。在本任务中，你可以假设所有由碰撞产生的粒子，其运动方向垂直于质子束流，这意味着新生成的粒子没有沿着质子流方向的纵向动量。



(1) - ToF 管
(2) - 轨迹
(3) - 碰撞点
(4) - 轨迹探测管
(5) - 质子流
⊗- 磁场
图 3：具有一个轨迹探测器和一个 TOF 探测器的粒子探测实验装置结构图。两个探测器是以碰撞区为中心分布的一些管子。左图: 与束流方向垂直的截面上的横向示意图；右图: 与束流方向平行的截面上的纵向示意图。
图中英文含义：transverse plane 横向垂直平面，cross section of the longitudinal view at the center along the beam line of the tube 沿束流方向的过探测管中心的纵向视图

B.3	求粒子质量表达式，用磁感应强度 B, ToF 管半径 R，基本常数，轨迹半径 r 和飞行时间 t（后两个物理量可以测量得到）来表示。	1.7pt

我们探测到四种粒子，现在要对它们进行辨别。轨迹探测器的磁感应强度为 $B = 0.500\ \mathrm{T}$。ToF 管的半径 R 为 $3.70\ \mathrm{m}$。以下是测量结果（$1\ \mathrm{ns} = 10^{-9}\ \mathrm{s}$）：

粒子	轨迹半径 r [m]	飞行时间 t [ns]
A	5.10	20
B	2.94	14
C	6.06	18
D	2.31	25

B.4	通过对粒子质量的计算，分辨出这四种粒子。	0.8pt

Dark Matter

The first formal inference for the existence of dark matter was provided by Fritz Zwicky based on his observation on the dynamics of the Coma galaxy cluster, a cluster of galaxies that consist of about a thousand of galaxies. Zwicky used the Virial theorem to estimate the mass of the galaxy cluster. In a simple sun-planet system, where the planet revolves around the sun in a circular orbit, the Virial theorem states that the kinetic energy of the planet is exactly related to its gravitational potential energy. While in a general case for a system of many particles bounded by some interaction, the Virial theorem, relates the time average total kinetic energy with its time average total potential energy.

In 1933, based on his observation on the velocity of the galaxies near the edge of the Coma galaxy cluster, Zwicky estimated that the cluster has more mass than what was visually observed (i.e. the galaxies). The gravitational attraction from the observable matter (the galaxies) was too small to account for the velocities of the galaxies. Thus there must be some hidden masses that account for such a large velocity. That hidden mass is the dark matter mass. In what follows, assume that the mass of each galaxy is the sum of its visible mass and the mass of the dark matter which moves together with that galaxy, and dark matter interacts with visible matter only gravitationally.

A. Cluster of Galaxies

Consider a cluster of galaxies consist of a large number N of galaxies and dark matter that are distributed homogeneously in a sphere of radius R with the total mass (galaxies and dark matters) of the cluster M. Assume that the average total mass (visible and dark matter) of a galaxy is m.

A.1	Assuming a continuous distribution of matter in the cluster, find the total gravitational potential energy of the cluster, in terms of M and R.	1.0 pt.

Due to cosmological expansion, any distance object will be moving away from an observer on Earth with a speed that depends on the distance from the observer to the object. A certain Lyman (a hydrogen emission spectral line) frequency from a type IA supernova on the i-th galaxy in the galaxy cluster is observed to be f_i , with i = 1,...,N, while the same corresponding Lyman frequency on Earth is f_0.

A.2	Determine the average speed V_{cr} of the whole galaxy cluster moving away from the Earth in terms of f_i (with $i = 1, \ldots, N.$), f_0 and N. Note that a galaxy speed is very small compared to the speed of light c.	0.5 pt.

A.3	By assuming that the galaxies velocities with respect to the center of the cluster are isotropic (the same in all direction), determine the root-mean square speed v_{rms} of the galaxies with respect to the center of the cluster in terms of N, f_i (with $i = 1, \ldots, N.$), and f_0. From this result determine the mean kinetic energy of a galaxy with respect to the center of the cluster in terms of v_{rms} and m.	1.5 pt.

To find the total mass of the cluster, one can use Virial theorem. The theorem stated that for a system of particles bounded by their conservative force,

$$\langle K\rangle_t = -\gamma\langle U\rangle_t,$$

where $\langle K\rangle_t$ is the time average total kinetic energy, $\langle U\rangle_t$ is the time average total potential energy, and γ is a constant. This theorem can be derived by assuming that for a system of particles bounded by its own interaction, the magnitudes of the position and momentum of each particle are finite, and thus the following quantity

$$\Gamma = \sum_i \vec{p_i}\cdot\vec{r_i}$$

is finite.

A.4	Using the fact that the time average over a long period of time of $d\Gamma/dt$ vanishes, $\langle \frac{d\Gamma}{dt}\rangle_t$ = 0, determine γ in the Virial theorem above for the case of gravitational interaction. (Hint: Try to do the problem with the summation in Γ for a small finite number of galaxies).	1.7 pt.
A.5	From the previous results determine the total dark matter mass of the cluster in terms of N, m_g, R and v_{rms}, where m_g is the average total visible mass of a galaxy. Note that the root-mean square speed of the dark matter is the same as that of the galaxies.	0.5 pt.

B. Dark Matter in a Galaxy

Dark matter also exists inside and around a galaxy. Consider a spherical galaxy with a visible edge radius R_g (an approximate outermost distance where a large number of stars still visible, but note that a very small number of stars may still be distributed in the region beyond R_g). Assume that the stars in the galaxy are point particles with an average mass m_s. The stars in the galaxy, distributed homogeneously with a number density n, are assumed to move in circular orbits.

B.1	If the galaxy consists only of stars, find the speed $v(r)$ of a star as a function of its distance to the center of the galaxy and sketch $v(r)$ for $r < R_g$ and $r \geq R_g$.	0.8 pt.

The existence of dark matter can be inferred from the galaxy rotation curve, which is a plot of $v(r)$ obtained from observations. The figure below shows a common pattern of the galaxy rotation curve. You may assume simplifyingly that $v(r)$ is a linear function for $r \leq R_g$ and a constant v_0 for $r > R_g$.

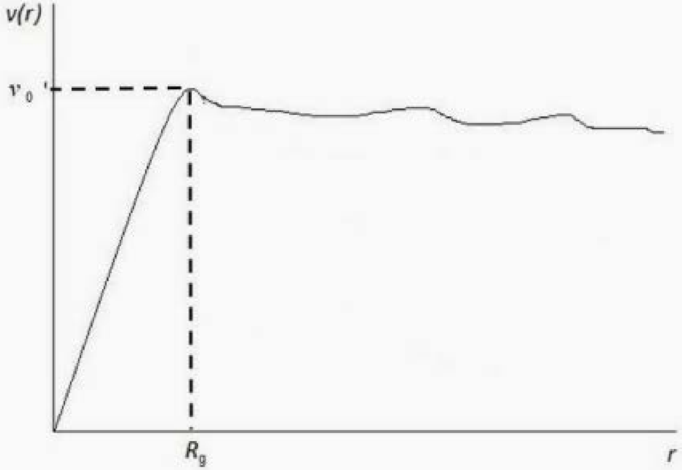


Fig. 1 Plot of galaxy rotation curve in a galaxy.

B.2	Find the total mass m_R of that part of the galaxy which lies within the sphere of radius R_g in terms of v_0 and R_g.	0.5 pt.

The discrepancy between the figure in B.2 and the plot obtained in B.1 indicates the existence of dark matter.

B.3	Determine the dark matter mass density as a function of r, R_g, v_0, n, and m_s for $r < R_g$ and $r \geq R_g$.	1.5 pt.

C. Interstellar Gas and Dark Matter

Now consider a young galaxy whose mass is dominated by interstellar gas and dark matters (neglect the mass of the stars). The interstellar gas is assumed to consist of identical particles of mass m_p. The number density $n(r)$ and temperature $T(r)$ of the gas depend on the distance from the center of the galaxy r. Although many physical processes happen in the gas, we can assume the gas is in a hydrostatic equilibrium due to its pressure and the galaxy gravitational attraction.

C.1	Find the pressure gradient of the gas dP/dr, in terms of $m'(r)$, r and $n(r)$. Here, $m'(r)$ is the total mass of gas and dark matters within a sphere of radius r from the galaxy center.	0.5 pt.

C.2	Assuming the interstellar gas as an ideal gas, find $m'(r)$ in terms of $n(r)$, $T(r)$ and their derivatives with respect to r.	0.5 pt.

Next for simplicity assume that the gas is in isothermal distribution at temperature T_0 and the interstellar gas number density is given by

$$n(r) = \frac{\alpha}{r(\beta + r)^2}$$

,

where α and β are some constants.

C.3	Find the dark matter mass density as a function of r inside the galaxy.	1.0 pt.

Dark Matter

A. Cluster of Galaxies

Question A.1

Answer	Marks
Potential energy for a system of a spherical object with mass $M(r) = \frac{4}{3}\pi r^3 \rho$ and a test particle with mass *dm* at a distance *r* is given by $$dU = -G\frac{M(r)}{r}dm$$	0.2 pts
Thus for a sphere of radius *R* $$U = -\int_0^R G\frac{M(r)}{r}dm = -\int_0^R G\frac{4\pi r^3 \rho}{3r}4\pi r^2 \rho dr = -\frac{16}{3}G\pi^2\rho^2\int_0^R r^4 dr$$ $$= -\frac{16}{15}G\pi^2\rho^2 R^5$$	0.6 pts
Then using the total mass of the system $$M = \frac{4}{3}\pi R^3 \rho$$ we have $$U = -\frac{3}{5}\frac{GM^2}{R}$$	0.2 pts
Total	1.0 pts

Question A.2

Answer	Marks
Using the Doppler Effect, $$f_i = f_0 \frac{1}{1+\beta} \approx f_0(1-\beta),$$ where $\beta = v/c$ and $v << c$. Thus the i-th galaxy moving away (radial) speed is $$V_{ri} = -\frac{f_i - f_0}{f_0} c$$ Alternative without approximation: $$f_i = f_0 \frac{1}{1+\beta}$$ $$V_{ri} = c\left(\frac{f_0}{f_i} - 1\right)$$	0.2 pts
All the galaxies in the galaxy cluster will be moving away together due to the cosmological expansion. Thus the average moving away speed of the N galaxies in the cluster is $$V_{cr} = -\frac{c}{Nf_0}\sum_{i=1}^{N}(f_i - f_0) = -\frac{c}{N}\sum_{i=1}^{N}\left(\frac{f_i}{f_0} - 1\right).$$ Alternative without approximation: $$V_{cr} = \frac{cf_0}{N}\sum_{i=1}^{N}\left(\frac{1}{f_i} - \frac{1}{f_0}\right) = \frac{c}{N}\sum_{i=1}^{N}\left(\frac{f_0}{f_i} - 1\right)$$	0.3 pts
Total	0.5 pts

Question A.3

Answer	Marks
The galaxy moving away speed V_r, in part A.2, is only one component of the three component of the galaxy velocity. Thus the average square speed of each galaxy with respect to the center of the cluster is $$\frac{1}{N}\sum_{i=1}^{N}(\vec{V}_i-\vec{V}_c)^2=\frac{1}{N}\sum_{i=1}^{N}(V_{xi}-V_{xc})^2+(V_{yi}-V_{yc})^2+(V_{zi}-V_{zc})^2$$ Due to isotropic assumption $$\frac{1}{N}\sum_{i=1}^{N}(\vec{V}_i-\vec{V}_c)^2=\frac{3}{N}\sum_{i=1}^{N}(V_{ri}-V_{cr})^2$$	0.5 pts
And thus the root mean square of the galaxy speed with respect to the cluster center is $$v_{rms}=\sqrt{\frac{3}{N}\sum_{i=1}^{N}(V_{ri}-V_{rc})^2}=\sqrt{\frac{3}{N}\sum_{i=1}^{N}(V_{ri}{}^2-2V_{cr}V_{ri}+V_{cr}{}^2)}=\sqrt{\frac{3}{N}\left(\sum_{i=1}^{N}V_{ri}{}^2\right)-3V_{cr}{}^2}$$ $$v_{rms}=c\sqrt{3}\sqrt{\left(\frac{1}{N}\sum_{i=1}^{N}\left(\frac{f_i}{f_0}-1\right)^2\right)-\left(\frac{1}{N}\sum_{i=1}^{N}\left(\frac{f_i}{f_0}-1\right)\right)^2}$$ $$=\frac{c\sqrt{3}}{f_0}\sqrt{\left(\frac{1}{N}\sum_{i=1}^{N}\left(f_i^2-2f_if_0+f_0^2\right)\right)-\left(\left(\frac{1}{N}\sum_{i=1}^{N}f_i\right)^2-2\frac{f_0}{N}\sum_{i=1}^{N}f_i+f_0^2\right)}$$ $$=\frac{c\sqrt{3}}{f_0N}\sqrt{\left(N\sum_{i=1}^{N}f_i^2\right)-\left(\sum_{i=1}^{N}f_i\right)^2}$$ Alternative without approximation:	0.7 pts

$v_{rms} = c\sqrt{3}\sqrt{\left(\frac{1}{N}\sum_{i=1}^{N}\left(\frac{f_0}{f_i}-1\right)^2\right)-\left(\frac{1}{N}\sum_{i=1}^{N}\left(\frac{f_0}{f_i}-1\right)\right)^2}$ $= \frac{c\sqrt{3}}{f_0}\sqrt{\left(\frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{f_i^2}-2\frac{1}{f_i}\frac{1}{f_0}+\frac{1}{f_0^2}\right)\right)-\left(\left(\frac{1}{N}\sum_{i=1}^{N}\frac{1}{f_i}\right)^2-2\frac{1}{N}\frac{1}{f_0}\sum_{i=1}^{N}\frac{1}{f_i}+\frac{1}{f_0^2}\right)}$ $= \frac{cf_0\sqrt{3}}{N}\sqrt{\left(N\sum_{i=1}^{N}\left(\frac{1}{f_i}\right)^2\right)-\left(\sum_{i=1}^{N}\frac{1}{f_i}\right)^2}$	
The mean kinetic energy of the galaxies with respect to the center of the cluster is $K_{ave} = \frac{m}{2}\frac{1}{N}\sum_{i=1}^{N}(\vec{V}_i-\vec{V}_c)^2 = \frac{m}{2}v_{rms}{}^2$	0.3 pts
Total	1.5 pts

Question A.4

Answer	Marks
The time average of $d\Gamma/dt$ vanishes $$\left\langle \frac{d\Gamma}{dt} \right\rangle_t = 0$$ Now $$\frac{d\Gamma}{dt} = \frac{d}{dt}\sum_i \vec{p}_i \cdot \vec{r}_i = \sum_i \frac{d\vec{p}_i}{dt}\cdot \vec{r}_i + \sum_i \vec{p}_i \cdot \frac{d\vec{r}_i}{dt}$$ $$= \sum_i \vec{F}_i \cdot \vec{r}_i + \sum_i m_i \vec{v}_i \cdot \vec{v}_i = \sum_i \vec{F}_i \cdot \vec{r}_i + 2K$$	0.6 pts
Where K is the total kinetic energy of the system. Since the gravitational force on i-th particle comes from its interaction with other particles then $$\sum_i \vec{F}_i \cdot \vec{r}_i = \sum_{i,j\neq i} \vec{F}_{ji} \cdot \vec{r}_i = \sum_{i<j} \vec{F}_{ji} \cdot \vec{r}_i - \sum_{i>j} \vec{F}_{ij} \cdot \vec{r}_i = \sum_{i<j} \vec{F}_{ji} \cdot \vec{r}_i - \sum_{i<j} \vec{F}_{ji} \cdot \vec{r}_j$$ $$= \sum_{i<j} \vec{F}_{ji} \cdot (\vec{r}_i - \vec{r}_j) = -\sum_{i<j} G\frac{m_i m_j}{\lvert \vec{r}_i - \vec{r}_j \rvert^2}\frac{(\vec{r}_i - \vec{r}_j)}{\lvert \vec{r}_i - \vec{r}_j \rvert}\cdot(\vec{r}_i - \vec{r}_j) = -\sum_{i<j} G\frac{m_i m_j}{\lvert \vec{r}_i - \vec{r}_j \rvert} = U_{tot}$$ Alternative proof: $$\sum_i \vec{F}_i . \vec{r}_i = \sum_{i,j\neq i} \vec{F}_{ji} . \vec{r}_i = \vec{F}_{21}.\vec{r}_1 + \vec{F}_{31}.\vec{r}_1 + \vec{F}_{41}.\vec{r}_1 + \cdots + \vec{F}_{N1}.\vec{r}_1 +$$ $$\vec{F}_{12}.\vec{r}_2 + \vec{F}_{32}.\vec{r}_2 + \vec{F}_{42}.\vec{r}_2 + \cdots + \vec{F}_{N2}.\vec{r}_2 +$$ $$\vec{F}_{13}.\vec{r}_3 + \vec{F}_{23}.\vec{r}_3 + \vec{F}_{43}.\vec{r}_3 + \cdots + \vec{F}_{N3}.\vec{r}_3 + \ldots$$ $$\vec{F}_{1N}.\vec{r}_N + \vec{F}_{2N}.N_N + \vec{F}_{3N}.\vec{r}_N + \cdots + \vec{F}_{NN-1}.\vec{r}_{N-1}$$ Collecting terms and noting that $\vec{F}_{ij} = -\vec{F}_{ji}$ we have	0.9 pts

$\vec{F}_{12}.(\vec{r}_2-\vec{r}_1)+\vec{F}_{13}.(\vec{r}_3-\vec{r}_1)+\vec{F}_{14}.(\vec{r}_4-\vec{r}_1)+\cdots+\vec{F}_{23}.(\vec{r}_3-\vec{r}_2)$ $+\vec{F}_{24}.(\vec{r}_4-\vec{r}_2)+\cdots+\vec{F}_{34}.(\vec{r}_4-\vec{r}_3)+\cdots=\sum_{i<j}\vec{F}_{ji}.(\vec{r}_i-\vec{r}_j)$ $=-\sum_{i<j}G\frac{m_i m_j}{\lvert\vec{r}_i-\vec{r}_j\rvert^2}\frac{(\vec{r}_i-\vec{r}_j)}{\lvert\vec{r}_i-\vec{r}_j\rvert}.(\vec{r}_i-\vec{r}_j)=-\sum_{i<j}G\frac{m_i m_j}{\lvert\vec{r}_i-\vec{r}_j\rvert}=U_{tot}$	
Thus we have $$\frac{d\Gamma}{dt}=U+2K$$ And by taking its time average we obtain $\left\langle\frac{d\Gamma}{dt}=U+2K\right\rangle_t=0$ and thus $\langle K\rangle_t=-\frac{1}{2}\langle U\rangle_t$. Therefore $\gamma=\frac{1}{2}$.	0.2 pts
Total	1.7 pts

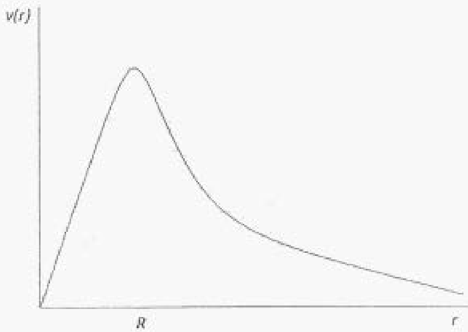
Question A.5

Answer	Marks
Using Virial theorem, and since the dark matter has the same root mean square speed as the galaxy, then we have $$\langle K \rangle_t = -\frac{1}{2}\langle U \rangle_t$$ $$\frac{M}{2}v_{rms}^2 = \frac{1}{2}\frac{3}{5}\frac{GM^2}{R}$$	0.3 pts
From which we have $$M = \frac{5Rv_{rms}^2}{3G}$$	0.1 pts
And the dark matter mass is then $$M_{dm} = \frac{5Rv_{rms}^2}{3G} - Nm_g$$	0.1 pts
Total	0.5 pts

B. Dark Matter in a Galaxy

Question B.1

Answer	Marks
Answer B.1: The gravitational attraction for a particle at a distance r from the center of the sphere comes only from particles inside a spherical volume of radius r. For particle inside the sphere with mass m_s , assuming the particle is orbiting the center of mass in a circular orbit, we have $G\frac{m'(r)m_s}{r^2}=\frac{m_s v_0^2}{r}$	0.3 pts
with $m'(r)$ is the total mass inside a sphere of radius r $m'(r)=\frac{4}{3}\pi r^3 m_s n$ Thus we have $v(r)=\left(\frac{4\pi G n m_s}{3}\right)^{1/2} r$	0.2 pts
While for particle outside the sphere, we have $v(r)=\left(\frac{4\pi G n m_s R^3}{3r}\right)^{1/2}$	0.2 pts

The sketch is given below  Sketch of the rotation velocity vs distance from the center of galaxy	0.1 pts
Total	0.8 pts

Question B.2

Answer	Marks
The total mass can be inferred from $G\frac{m'(R_g)m_s}{R_g^{\;2}}=\frac{m_s v_0^2}{R_g}$ Thus $m_g=m'(R_g)=\frac{v_0^2 R_g}{G}$	0.5 pts
Total	0.5 pts

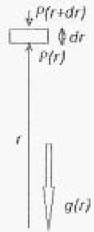
Question B.3

Answer	Marks
Base on the previous answer in B.1, if the mass of the galaxy comes only from the visible stars, then the galaxy rotation curve should fall proportional to $1/\sqrt{r}$ on the outside at a distance $r > R_g$. But in the figure of problem b) the curve remain constant after $r > R_g$, we can infer from $$G\frac{m'(r)m_s}{r^2} = \frac{m_s v_0^2}{r}.$$ to make $v(r)$ constant, then $m'(r)$ should be proportional to r for $r > R_g$, i.e. for $r > R_g$, $m'(r) = Ar$ with A is a constant.	0.3 pts
While for $r < R_g$, to obtain a linear plot proportional to r, then $m'(r)$ should be proportional to r^3, i.e. $m'(r) = Br^3$.	0.3 pts
Thus for $r < R_g$ we have $$m'(r) = \int_0^r \rho_t(r) 4\pi r'^2\, dr' = Br^3$$ $$dm'(r) = \rho_t(r) 4\pi r^2 dr = 3Br^2 dr$$ Thus total mass density $\rho_t(r) = \frac{3B}{4\pi}$	0.2 pts
$m_R = \int_0^{R_g} \frac{3B}{4\pi} 4\pi r'^2\, dr' = BR_g^3$ or $B = \frac{m_R}{R_g^3} = \frac{v_0^2}{GR_g^2}$ Thus the dark matter mass density $\rho(r) = \frac{3v_0^2}{4\pi G R_g^2} - nm_s$	0.2 pts

While for $r > R_g$ we have $m'(r) = \int_0^{R_g} \rho(r')4\pi r'^2\, dr' + \int_{R_g}^{r} \rho(r')4\pi r'^2\, dr' = Ar$ $m'(r) = m_R + \int_{R_g}^{r} \rho(r')4\pi r'^2\, dr' = Ar$ $\int_R^r \rho(r')4\pi r'^2\, dr' = Ar - M_0$ $\rho(r)4\pi r^2 = A$, or $\rho(r) = \dfrac{A}{4\pi r^2}$.	0.2 pts
Now to find the constant *A*. $\int_R^r \dfrac{A}{4\pi r'^2} 4\pi r'^2\, dr' = A(r - R_g) = Ar - m_R$ Thus $AR_g = m_g$ and $A = \dfrac{v_0^2}{G}$ We can also find *A* from the following $G\dfrac{m'(r)m_s}{r^2} = G\dfrac{Arm_s}{r^2} = \dfrac{m_s v_0^2}{r}$, thus $A = \dfrac{v_0^2}{G}$. Thus the dark matter mass density (which is also the total mass density since $n \approx 0$ for $r \geq R_g$. $\rho(r) = \dfrac{v_0^2}{4\pi G r^2}$ for $r \geq R_g$	0.3 pts
Total	1.5 pts

C. Interstellar Gas and Dark Matter

Question C.1

Answer	Marks
Consider a very small volume of a disk with area A and thickness Δr, see Fig.1  Figure 1. Hydrostatic equilibrium In hydrostatic equilibrium we have $$(P(r)-P(r+\Delta r))A-\rho g(r)A\Delta r=0$$	0.3 pts
$$\frac{\Delta P}{\Delta r}=-\rho\frac{Gm'(r)}{r^2}$$ $$\frac{dP}{dr}=-\rho\frac{Gm'(r)}{r^2}=-n(r)m_p\frac{Gm'(r)}{r^2}.$$	0.2 pts
Total	0.5 pts

Question C.2

Answer	Marks
Using the ideal gas law $P = n\,kT$ where $n = N/V$ where n is the number density, we have $$\frac{dP}{dr} = kT\frac{dn(r)}{dr} + kn(r)\frac{dT}{dr} = -n(r)m_p\frac{Gm'(r)}{r^2}$$ Thus we have $$m'(r) = -\frac{kT}{Gm_p}\left(\frac{r^2}{n(r)}\frac{dn(r)}{dr} + \frac{r^2}{T(r)}\frac{dT(r)}{dr}\right).$$	0.5 pts
Total	0.5 pts

Question C.3

Answer	Marks
If we have isothermal distribution, we have $dT/dr = 0$ and $$m'(r) = -\frac{kT_0}{Gm_p}\left(\frac{r^2}{n(r)}\frac{dn(r)}{dr}\right)$$	0.2 pts
From information about interstellar gas number density, we have $$\frac{1}{n(r)}\frac{dn(r)}{dr} = -\frac{3r+\beta}{r(r+\beta)}$$ Thus we have $$m'(r) = \frac{kT_0 r}{Gm_p}\frac{3r+\beta}{(r+\beta)}$$	0.2 pts

Mass density of the interstellar gas is $\rho_g(r) = \frac{\alpha m_p}{r(\beta + r)^2}$ Thus $m'(r) = \int_0^r \left(\rho_g(r') + \rho_{dm}(r')\right) 4\pi r'^2\, dr' = \frac{kT_0 r}{Gm_p}\frac{3r+\beta}{(r+\beta)}$ $m'(r) = \int_0^r \left(\frac{\alpha m_p}{r'(\beta + r')^2} + \rho_{dm}(r')\right) 4\pi r'^2\, dr' = \frac{kT_0 r}{Gm_p}\frac{3r+\beta}{(r+\beta)}$	0.3 pts
$\left(\frac{\alpha m_p}{r(\beta + r)^2} + \rho_{dm}(r)\right) 4\pi r^2 = \frac{kT_0}{Gm_p}\frac{3r^2 + 6r\beta + \beta^2}{(r+\beta)^2}$ $\rho_{dm}(r) = \frac{kT_0}{4\pi G m_p}\frac{3r^2 + 6r\beta + \beta^2}{(r+\beta)^2 r^2} - \frac{\alpha m_p}{r(\beta + r)^2}$	0.3 pts
Total	1.0 pts

Earthquake, Volcano and Tsunami

Indonesia is the supermarket of natural hazards. Almost all of the natural hazards have occurred in Indonesia, such as volcano eruptions, earthquakes and tsunami.

A. Merapi Volcano Eruption

(**Fig. 1**: Pyroclastic cloud during Merapi eruption, Courtesy of Volcanological Office of Yogyakarta, BPPTKG)

Merapi volcano in Yogyakarta is one of the most active volcano in Java. Pyroclastic flows are well-known eruption characteristics of the volcano. The pyroclastic flow is a hot mixture of gas and rock which travels away from a volcano. In October 26th, 2010, Merapi showed his explosive character by producing an ash plume that reached 12 km altitude (Fig. 1) and pyroclastic currents displacing more than 20,000 people around the volcano.

Let us look at the causes of the largest eruption of Merapi in 2010. It is known by geophysicists that the influence of the external water into the magma plays an important role to the explosive behavior of volcanic eruptions (hydro magmatic eruptions). Let's assume that we dealt with a volcano as a system that consists of mixture of magmatic particles and water. The volcano vents structures and atmosphere are being boundary of the system. The explosive eruption is considered to be happening in two stages, (1) an instantaneous magma-water interaction, and (2) a system expansion. In the first stage, a mass of magma (m_m) at an absolute temperature (T_m) mixes with a mass of external water (m_w) at an absolute temperature (T_w). The thermal equilibrium is reached almost instantaneously. This interaction can be perceived as a nearly-constant volume process. Latent heat of evaporation of water and latent heat of melting of magma can be neglected.

A.1	Find the equilibrium temperature at the first stage in terms of the masses and heat capacity per unit mass of water c_{Vw} and magma c_{Vm}.	0.5 pt.
A.2	Determine its equilibrium pressure at the first stage by assuming that the mixture can be modeled as ideal gas. Assume that the volume per unit mole of the mixture is v_e.	0.3 pt.

The system expansion (the second stage) can occur through several possibilities, one of which is thermal detonation. Although such process is quite complicated, we can empirically measure the relative velocity of the erupted mixture. The velocity of gas during the eruption depend on the pressure p, the total mass m and the volume V of the mixture in the conduit of a volcano.

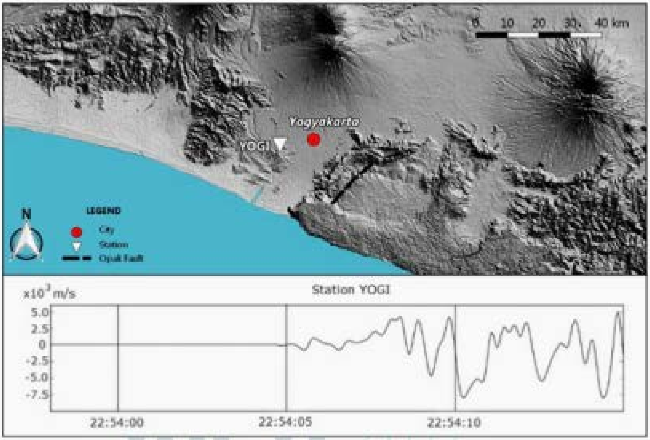
A.3	Express the velocity of gas during the eruption in terms of p, m, and V up to a proportional constant κ.	0.5 pt.

The observed pressure is around the order of 100 MPa. This makes the eruption (relative) velocity can be as large as ballistic velocity.

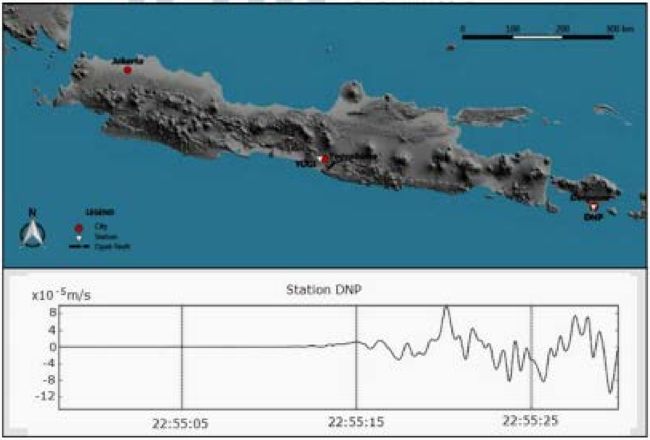
B. The Yogyakarta Earthquake

The 2006 Yogyakarta earthquake of magnitude M_w = 6.4, which destroyed many buildings in the Bantul and Yogyakarta area, occured at 05:54:00.00 local time or 22:54:00.00 UTC. The earthquake was caused by a sudden displacement of the Opak fault segment (see Fig. 2). The hypocenter was located 15 km below the surface.

The seismic wave that propagates on the earth crust can be recorded using seismometer. The diagram from seismometer is called seismogram (Fig. 2 and 3, Lower graph). The seismograms represent the vertical ground velocity as a function of time recorded by the seismic station at Gamping Station Yogyakarta (YOGI) (Fig. 2) and Denpasar, Bali (DNP) (Fig. 3). In general, seismic wave consists of three wave types: the longitudinal or primary (P-wave), the transversal or secondary (S-wave), and the surface wave. The P-wave and S-wave travel in the subsurface while the surface wave travels along the Earth surface. Seismic waves traveling through subsurfaces to the stations can be divided into those that propagate in a straight line, those that are reflected by a layer's boundary, and those that are refracted into the next layer. The longitudinal wave or the primary wave has the highest velocity, while the surface wave has the lowest velocity, around 60% of the P-wave.



(**Fig. 2**: The maps location of YOGI)



(**Fig. 3**: The maps location of DNP (Denpasar))

The distance between the epicenter (the projection of hypocenter on the Earth surface) and the YOGI and DNP stations respectively are 22.5 km and 500 km. The depth of the Earth crust layer in Java, Indonesia, is 30 km. Beneath the Earth's crust is the Earth's mantle layer. Just like other wave phenomena, seismic wave also satisfies the Snell's law. The seismic wave can also be reflected by the mantle layer. In this problem we assume that the earth curvature is neglected.

B.1	Fig. 2 shows the seismogram at the YOGI station. Use the data to find the velocity of the P-wave in the Earth crust.	0.5 pt.
B.2	Find the travel time of the direct P wave and reflected wave due to the Yogyakarta earthquake that arrived at the DNP station in Denpasar.	0.6 pt.

By assuming that the Earth is composed of only two layers: the crust and the mantle, the primary wave propagates in the crust and in the mantle with different constant velocities. The velocity in the mantle is faster than in the crust. Note that P-wave refracted into the mantle at the right angle (90^o) is being partly refracted back into the crust along its entire path of propagation along the crust-mantle boundary.

B.3	Find the velocity of the P-wave in the mantle.	1.2 pt.

For a more realistic Earth structure, the crust can be divided into a number of thin layers so that the velocity of the seismic wave is a function of the depth z according to $v(z) = v_0 + az$ where a is a constant and the hypocenter is approximated on the surface. In this model, the wave ray is curving.

B.4	Let us define the ray parameter $p = \sin\theta(z)/v(z)$, where $\theta(z)$ is the angle between the ray and the normal. Suppose a seismic wave arrives to the station with ray parameter p; express the distance to the hypocenter in terms of p, v_0, and a. Assume that the hypocenter is very close to the ground surface.	1.4 pt.
B.5	Find the travel time T from hypocenter to any station, in form of integral over z.	1.0 pt.

The earth consists of a stack of homogeneous layers with the velocity of each layer is v_i and the thickness of δz_i.

B.6	From the result of the previous problem, approximate the travel time (T) from the hypocenter to DNP station by assuming that the crust consists of only three discretized layers, (i = 1, 2, 3), characterized by v_1 = 6.65 km/sec, v_2 = 6.97 km/sec, v_3 = 6.99 km/sec, p = 0.143 sec/km, δz_1 = 6.0 km, δz_2 = 9.0 km, δz_3 = 15 km.	1.0 pt.

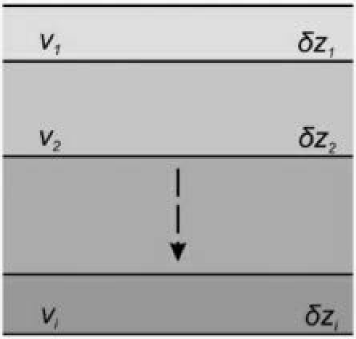


Fig. 4: A simplified model of earth's layers.

C. Java Tsunami

The 2006 Pangandaran earthquake and tsunami occurred on July 17 at 15:19:27 local time off the coast of west and central Java. During the earthquake where the epicenter fault is on the ocean floor, the fault may be displaced producing a remarkably large water wave called tsunami. In other words, a tsunami is a shallow-water wave which is initiated by a tiny amplitude, but with an extremely large wavelength. Consider a sudden fault causing a lifting of some ocean floor as shown in Fig. 5. Assume that the energy of the earthquake is transformed to the potential energy of this raised ocean water. For simple model we approximate that the raised water has a geometry of a box with its area of $\lambda L/2$ (where $L \gg \lambda$) and height of h.

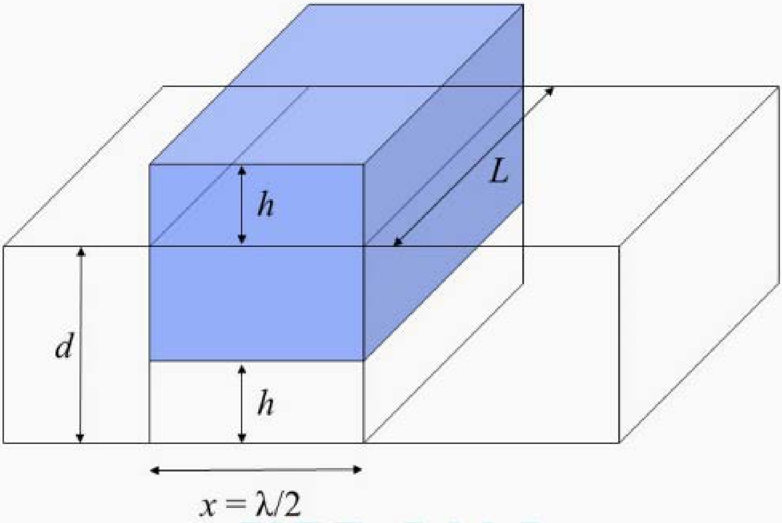


Fig. 5: Illustration for the tsunami wave d is the depth of the ocean.

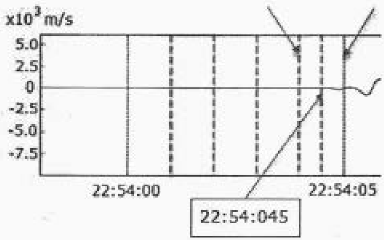
C.1	Find the potential energy stored in the raised ocean water due to the earthquake with respect to ocean surface. Assume that the density of sea water is ρ.	0.5 pt.
C.2	Find the speed of tsunami wave up to dimensionless factor.	1.2 pt.
C.3	Using energy argument, determine the amplitude of the tsunami wave as a function of the depth, assuming that the depth varies slowly and also knowing that at a depth of d_0 the amplitude is A_0.	1.3 pt.

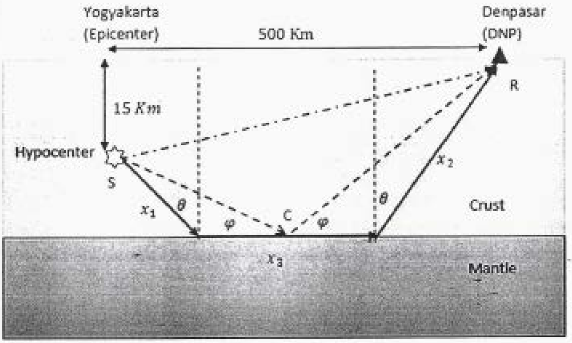
Earthquake, Volcano and Tsunami

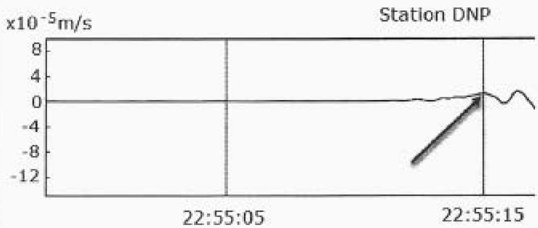
A. Merapi Volcano Eruption

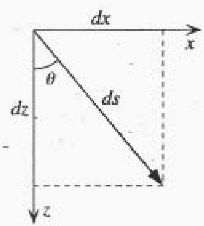
Question	Answer	Marks
A.1	Using Black's Principle the equilibrium temperature can be obtained $$m_w c_{vw}(T_e - T_w) + m_m c_{vm}(T_e - T_m) = 0$$ Thus, $$T_e = \frac{m_w c_{vw} T_w + m_m c_{vm} T_m}{m_w c_{vw} + m_m c_{vm}}$$	0.5 pts
A.2	For ideal gas, $p_e v_e = RT_e$, thus $$p_e = \frac{R}{v_e} \frac{m_w c_{vw} T_w + m_m c_{vm} T_m}{m_w c_{vw} + m_m c_{vm}}$$	0.3 pts
A.3	The relative velocity u_{rel} can be expressed as $$u_{rel} = \kappa\, p^\alpha V^\beta m^\gamma$$ where κ is a dimensionless constant. Using dimensional analysis, one can obtain that $$LT^{-1} = M^{\alpha+\gamma} L^{-\alpha+3\beta} T^{-2\alpha}$$ $$\alpha + \gamma = 0$$ $$-\alpha + 3\beta = 1$$ $$-2\alpha = -1$$ Therefore $$u_{rel} = \kappa\, p^{1/2} V^{1/2} m^{-1/2}$$	0.5 pts
	Total score	1.3 pts

B. The Yogyakarta Earthquake

Question	Answer	Marks	
B.1	From the given seismogram, fig. 2  One can see that the P-wave arrived at 22:54:045 or (4.5 – 5.5) seconds after the earthquake occurred at the hypocenter.	0.3 pts	0.5 pts
	Since the horizontal distance from the epicenter to the seismic station in Gamping is 22.5 km, and the depth of the hypocenter is 15 km, the distance from the hypocenter to the station is $\sqrt{22.5^2+15^2}\ \text{km}=27.04\ \text{km}$	0.1 pts	
	Therefore, the P-wave velocity is $v_P=\dfrac{27.04\ \text{Km}}{4.7\ \text{s}}=5.75\ \text{Km/s}$	0.1 pts	

Question	Answer	Marks	
B.2	Direct wave: $$t_{\text{direct}} = \frac{SR}{v_1} = \frac{\sqrt{500^2+15^2}}{v_1} = \frac{502.021}{5.753}\ \text{s} = 86.9\ \text{s}$$	0.2 pts	0.6 pts
	As in the case of an optical wave, the Snell's law is also applicable to the seismic wave.  Illustration for the traveling seismic Wave Reflected wave: $$t_{\text{reflected}} = \frac{SC}{v_1} + \frac{CR}{v_1}$$ $$SC\cos\varphi + CR\cos\varphi = 500 \Rightarrow \cot\varphi = \frac{500}{45}$$ $$t_{\text{reflected}} = \frac{45}{v_1 \sin\varphi} = 87.3\ \text{s}$$	0.4 pts	

Question	Answer	Marks	
B.3	Velocity of P-wave on the mantle. The fastest wave crossing the mantle is that propagating along the upperpart of the mantle. From the figure on refracted wave, we obtain that $\frac{\sin\theta}{v_1}=\frac{1}{v_2};\quad \sin\theta=\frac{v_1}{v_2};\quad \cos\theta=\sqrt{1-\left(\frac{v_1}{v_2}\right)^2}$ $\cos\theta=\frac{15}{x_1};\quad x_1=\frac{15}{\cos\theta}\text{ km};\quad x_2=\frac{30}{\cos\theta}\text{ km}$ $x_3=500-(x_1+x_2)\sin\theta=500-45\tan\theta$	0.4 pts	1.2 pts
	The total travel time: $t=\frac{x_1+x_2}{v_1}+\frac{x_3}{v_2}=\frac{45}{v_1\cos\theta}+\frac{500}{v_2}-\frac{45\tan\theta}{v_2}$ $t\cos\theta=45u_1+500u_2\cos\theta-45u_2\sin\theta$ where $u_1=1/v_1$ and $u_2=1/v_2$. Arranging the equation, we get $(500^2+45^2)u_2^2-2t\ 500u_2+t^2-45^2\ u_1=0$ whose solution is $v_2=\frac{500tv_1^2+45v_1\sqrt{(45^2+500^2)-t^2v_1^2}}{t^2v_1^2-45^2}$	0.5 pts	
	 From the seismogram, we know that the fastest wave arrived at Denpasar station at 22:55:15, which is $t=75$ s from the origin time of the earthquake in Yogyakarta. Thus $v_2=7.1$ km/s	0.3 pts	

Question	Answer	Marks	
B.4	By using Snell's law and defining $p = \sin\theta/v$ and $u = 1/v$, we obtain $p \equiv u(0)\sin\theta_0 = u(z)\sin\theta; \qquad \sin\theta = \dfrac{p}{u(z)}$	0.2 pts	1.4 pts
	where $u(z) = 1/v(z)$ and θ_0 is the initial angle of the seismic wave direction. $\dfrac{dx}{ds} = \sin\theta = \dfrac{p}{u(z)}; \qquad \dfrac{dz}{ds} = \cos\theta = \sqrt{1-\left(\dfrac{p}{u(z)}\right)^2}$ $\dfrac{dx}{dz} = \dfrac{dx}{ds}\dfrac{ds}{dz} = \dfrac{p}{u}\dfrac{u}{\left(u^2-p^2\right)^{1/2}} = p/\left(u^2-p^2\right)^{1/2}$ $x = \int_{z_1}^{z_2} \dfrac{p}{\left(u^2-p^2\right)^{1/2}}\, dz$	0.5 pts	
	 Illustration for the direction of wave The distance *X* is equal to twice the distance from epicenter to the turning point. The turning point is the point when $\theta = 90°$. Thus $p = u(z_t) = \dfrac{1}{v_0 + az_t}; \qquad z_t = \dfrac{1-pv_0}{ap}$ $X = 2\int_0^{z_t} \dfrac{p(v_0+az)}{(1-p^2(v_0+az)^2)^{1/2}}dz = \dfrac{2}{ap}\left(\sqrt{1-p^2(v_0+az)^2} - \sqrt{1-p^2v_0^2}\right)$	0.7 pts	

Question	Answer	Marks	
B.5	For the travel time, $dt = \dfrac{ds}{v(z)}$; $\dfrac{dt}{ds} = u(z)$. Thus $\dfrac{dt}{dz} = \dfrac{dt}{ds}\dfrac{ds}{dz} = \dfrac{u^2}{(u^2 - p^2)^{1/2}}$ and therefore $T = 2\int_0^{z_t} \dfrac{u^2}{(u^2 - p^2)^{1/2}} dz = 2\int_0^{z_t} \dfrac{1}{(v_0 + az)} \dfrac{1}{(1 - p^2(v_0 + az)^2)^{1/2}} dz$	1.0 pts	1.0 pts
B.6	The total travel time from the source to the Denpasar can be calculated using previous relation $T(p) = 2\int_0^{z_t} \dfrac{u^2(z)}{\left(u^2(z) - p^2\right)^{1/2}} dz$ Which is valid for a continuous $u(z)$. For a simplified stacked of homogeneous layers (Figure F), the integral equation became a summation $T(p) = 2\sum_i^N \dfrac{u_i^2 \Delta z_i}{\left(u_i^2 - p^2\right)^{1/2}}$	0.6 pts	1.0 pts
	$T(p) = 2\dfrac{u_1^2 \Delta z_1}{(u_1^2 - p^2)^{\frac{1}{2}}} + 2\dfrac{u_2^2 \Delta z_2}{(u_2^2 - p^2)^{\frac{1}{2}}} + 2\dfrac{u_3^2 \Delta z_3}{(u_3^2 - p^2)^{\frac{1}{2}}}$ $= \dfrac{2 \times (0.1504)^2 \times 6}{(0.1504^2 - 0.143^2)^{\frac{1}{2}}} + \dfrac{2 \times (0.1435)^2 \times 9}{(0.1435^2 - 0.143^2)^{\frac{1}{2}}} + \dfrac{2 \times (0.1431)^2 \times 15}{(0.1431^2 - 0.143^2)^{\frac{1}{2}}}$ $= 151.64$ second Note that the actual travel time from the epicenter to Denpasar is 75 seconds. By varying the parameters of velocity and depth up to suitable value of observed travel time, physicist can know Earth structure.	0.4 pts	
	Total score		5.7 pts

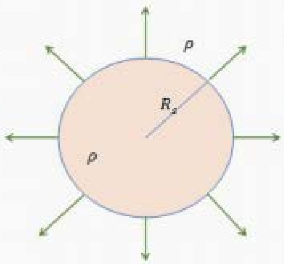
C. Java Tsunami

<table>
<tr><th>Question</th><th>Answer</th><th colspan="2">Marks</th></tr>
<tr><td>C.1</td><td>The center of mass of the raised ocean water with respect to the ocean surface is $h/2$. Thus
$$E_P = \frac{h^2\rho\lambda L g}{4}$$
where ρ is the ocean water density.</td><td>0.5 pts</td><td>0.5 pts</td></tr>
<tr><td rowspan="2">C.2</td><td>Considering a shallow ocean wave in Fig. 5, the whole water (from the surface until the ocean floor) can be considered to be moving due to the wave motion. The potential energy is equal to the kinetic energy.
$$\frac{1}{4}\rho\lambda h^2 L g = \frac{1}{4}\rho d L\lambda U^2$$
Where $x = \lambda/2$ and U is the horizontal speed of the water component.
The water component that was in the upper part $hL\frac{\lambda}{2}$ should be equal to the one that moves horizontally for a half of period of time $^{\tau}/_{2}$, i.e. $hL\,\lambda/2 = dLU\,\tau/2$.
Thus we have
$$U = \frac{h\lambda}{\tau d}$$</td><td>0.7 pts</td><td rowspan="2">1.2 pts</td></tr>
<tr><td>Accordingly,
$\tau = \frac{\lambda}{\sqrt{gd}}$
Thus
$$v = \frac{\lambda}{\tau} = \sqrt{gd}$$</td><td>0.5 pts</td></tr>
<tr><td>C.3</td><td>Using the argument that the wave energy density is proportional to its amplitude $E = kA^2$ with A is amplitude and k is a proportional constant
Because the energy flux is conserve, then
$Eva = E_0 v_0 a$ for an area a where the wave flow though.
Then,
$kA^2\sqrt{gd} = kA_0^2\sqrt{gd_0}$
$$A = A_0\left(\frac{d_0}{d}\right)^{\frac{1}{4}}$$
(Therefore the tsunami wave will increase its amplitude and become narrower as it approaches the beach).</td><td>1.3 pts</td><td>1.3 pts</td></tr>
<tr><td colspan="3" align="right">Total score</td><td>3.0</td></tr>
</table>

Cosmic Inflation

Due to the relative movement of galaxies observed from the earth, the wavelength of visible spectrum of a particular galaxy differs from its original wavelength, which is known as the electromagnetic Doppler effect. One expects, for a collection of galaxies, to a random distributions of wavelength shifts: some positive (red shift) and some negative (blue shift). However, observations show that all, expect for a nearby group of galaxies, are red shifted. This must be true even if the observation take place on different point in the universe. As a conclusion, our universe must be expanding. On the other hand local irregularity of the universe can be neglected on scales of more than 100 Mpc, in which 1 pc = 3.26 light-years. Averaged over large scales, the clumpy distribution of galaxies becomes more and more isotropic (independent of direction) and homogeneous (independent of position). Therefore we can assume the universe as a matter having a uniform mass density ρ and is expanding.

A. Expansion of Universe



For a simple model of our universe, let us consider an expanding uniform-density sphere embedded in a medium of a much larger sphere with the same density. Let say at some time, the radius of the smaller sphere is R_s. To express the expansion of the sphere, the time dependency of the radius $R(t)$ can be expressed by scale factor $a(t)$, that is $R(t) = a(t)R_s$.

Using Newton's law of gravity to evaluate velocity of a mass element on the sphere boundary according the model of our universe, one can obtain a set of Friedmann equations:

$$\left(\frac{\dot{a}}{a}\right)^2 = A_1 \rho(t) - \frac{kc^2}{R_s^2 a^2(t)} \qquad (1)$$

where k a dimensionless constant, and c is velocity of light.

A.1	Determine the constant A_1 in the equation (1)	1.3 pt.

The discussion so far is non-relativistic. But in fact, it can be extended to a relativistic system by reinterpreting $\rho(t)c^2$ as total energy density (excluding the gravitational potential energy). In this relativistic system derives the 2nd Friedmann equation:

$$\dot{\rho} + A_2 \left(\rho + \left(\frac{p}{c^2}\right)\right)\frac{\dot{a}}{a} = 0 \qquad (2)$$

using the 1st law thermodynamics of an adiabatic system, where p denotes the pressure on the sphere.

A.2	Determine the constants A_2 in the equation (2)	0.9 pt.

To solve Eqs. (1) and (2), one should assume a relation $p = p(\rho)$, such as $p(t)/c^2 = w\rho(t)$, where w is a constant. There is also a factor $H = \dot{a}/a$ being called Hubble parameter. The present values of parameters are usually symbolized by subscript 0 such as t_0, ρ_0, H_0, a_0 and so on. For simplicity, we take $a_0 = 1$.

Universe is believed to start from a big explosion called Big-Bang that produces radiation of relativistic particles. During its expansion, the universe is cooling down and the particles in it become non-relativistic. However, the recent observations clarify that the present universe is dominated by cosmological constant energy density. For the case of photon, as the universe is expanding, the photon's wavelength expands proportionally to the scale factor.

A.3	For each of the following three cases determine the resulting value of w: (i) a universe filled only with radiation (i.e. photon energy), (ii) a universe filled only with non-relativistic matter and (iii) a universe with constant energy density.	1.2 pt.
A.4	In the case of $k = 0$, find $a(t)$ for each case of (i) to (iii) being mentioned in A.3. Use the initial condition $a(t = 0) = 0$ for case (i) and (ii), and use the condition $a_0 = 1$ for case (iii).	1.2 pt.

Constant k in Eq. (1) refers to classification of spatial geometry of the universe. Its value can be $k = +1$ for positive-curvature universe (closed), $k = 0$ for flat universe (infinite), and $k = -1$ for negative-curvature universe (open, infinite). Let define a density ratio $\Omega = \rho/\rho_c$, where $\rho_c c^2 = H^2/A_1$ is critical energy density. Note that A_1 is obtained from problem A.1.

A.5	Express k in Eq.(1) in terms of Ω, H, a, and R_0.	0.1 pt.
A.6	Find a range for Ω that corresponds to each value of $k=+1$, $k=0$ and $k=-1$.	0.3 pt.

B. Motivation To Introduce Inflation Phase and Its General Conditions

The observation of cosmic microwave background radiation (CMB) suggests that our present universe is approximately flat. The problem is that if this is true then the present universe should start from exactly flat early universe, otherwise any deviation from the flatness will eventually grow over time and spoil the present flatness.

B.1	Find $(\Omega(t)-1)$ as a function of time for the universe when it is dominated by radiation or non-relativistic matter (see problem A.3).	0.5 pt.

To solve the problem, at some early time in its history, the universe should undergo a constant energy density domination period which leads to an exponential expansion so called inflation period.

B.2	For this constant energy density domination period, find $(\Omega(t)-1)$ as a function of time. Assume that $(\Omega(t)-1) \ll 1$.	0.3 pt.
B.3	Show that condition for inflation implies several following conditions: negative pressure, accelerated expansion $(\ddot{a}>0)$, and decreasing Hubble radius $(d(aH)^{-1}/dt<0)$.	0.9 pt.
B.4	Show that the condition of decreasing Hubble radius can be expressed in terms of parameter $\epsilon = -\dot{H}/H^2$ as $\epsilon < 1$.	0.2 pt.

Inflation occurs as long as $\epsilon < 1$ and then ends when $\epsilon = 1$. We can define e-folding number N, such that $dN = d\ln a = H dt$ and $N=0$ at the end of inflation.

C. Inflation Generated by Homogenously Distributed Matter

As an example of simple physical system that can generate period of inflation is a universe dominated by homogenously distributed matter. This kind of matter is called inflaton and can be characterized by a function $\phi(t)$.

The dynamical equation of the matter can be expressed as

$$\ddot{\phi} + 3H\dot{\phi} = -V', \tag{3}$$

where $V=V(\phi)$ is a potential function and $V' = \frac{\partial V}{\partial \phi}$. The Hubble parameter satisfies

$$H^2 = \frac{1}{3M_{pl}^2}\left[\frac{1}{2}\dot{\phi}^2 + V\right].\qquad(4)$$

with M_{pl} is a constant called the reduced Planck mass. Inflation phase occurs during domination of potential energy V over kinetic energy $\dot{\phi}^2/2$ for sufficient time such that $\ddot{\phi}$ term in equation (3) can be neglected. This condition is called slow-roll approximation.

The quantities ϵ and $\eta_V = \delta + \epsilon$, where $\delta = -\ddot{\phi}/(H\dot{\phi})$, are called 'slow-roll' parameters.

C.1	Estimate parameter ϵ, parameter η_V, $dN/d\phi$ in terms of potential $V(\phi)$ and its first and second derivative (V' and V'').	1.7 pt.

D. Inflation with A Simple Potential

Predictions of any inflation model should be compared to observational constraints from CMB as follow n_s = 0.968 $\pm$ 0.006 and r < 0.12, where $r = 16\epsilon$ and $n_s = 1 + 2\eta_V - 6\epsilon$ are evaluated at $\phi = \phi_{start}$ for inflation model being generated by a dominant matter. Assume that potential of matter takes a simple form $V(\phi) = \Lambda^4 \left(\frac{\phi}{M_{pl}}\right)^n$ where n is any integer and Λ is a constant.

D.1	Calculate ϕ_{end} at the end of inflation.	0.5 pt.
D.2	Express r and n_s in terms of e-folding number N and integer n. Estimate the value of n that is closest to observational values r and n_s. Take N = 60 in your calculation.	0.9 pt.

Cosmic Inflation

A. Expansion of Universe

Question A.1

Answer	Marks
For any test mass m on the boundary of the sphere, $$m\ddot{R}(t) = -GmM_s/R^2(t) \qquad \text{(A.1.1)}$$ where M_s is mass portion inside the sphere	**0.2**
Multiplying equation (A.1.1) with $\dot{R}$ and integrating it gives $$\int \dot{R}\frac{d\dot{R}}{dt}\,dt = \frac{1}{2}\,\dot{R}^2 = \frac{GM_s}{R} + A$$ where A is a integration constant	**0.6**
Taking $M_s = \frac{4}{3}\pi R^3(t)\rho(t)$, and $\dot{R} = \dot{a}\,R_s$	**0.2**
$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\,\rho(t) + \frac{2A}{R_s^2 a^2(t)}$$	**0.2**
Therefore, we have $A_1 = \frac{8\pi G}{3}$	**0.1**
Total	1.3

Question A.2

Answer	Marks

The 2nd Friedmann equation can be obtained from the 1st law of thermodynamics : $$dE = -pdV + dQ.$$	**0.1**
For adiabatic processes $dE + pdV = 0$ and its time derivative is $\dot{E} + p\,\dot{V} = 0$.	**0.1**
For the sphere $\dot{V} = V\,(3\,\dot{a}/a)$	**0.1**
Its total energy is $E = \rho(t)V(t)\,c^2$	**0.2**
Therefore $\dot{E} = \left(\dot{\rho} + 3\,\frac{\dot{a}}{a}\right)Vc^2$	**0.1**
It yields $$\dot{\rho} + 3\left(\rho + \frac{p}{c^2}\right)\frac{\dot{a}}{a} = 0$$	**0.2**
Therefore, we have $A_2 = 3$.	**0.1**
Total	**0.9**

Question A.3

Answer	Marks
Interpreting $\rho(t)c^2$ as total energy density, and substituting $\frac{p(t)}{c^2} = w\,\rho(t)$ in to the 2nd Friedmann equation yields: $$\dot{\rho} + 3\,\rho(1+w)\frac{\dot{a}}{a} = 0$$	**0.1**
$\rho \propto a^{-3(w+1)}$	**0.2**
(i) In case of radiation, photon as example, the energy is given by $E_r = h\nu = hc/\lambda$ then its energy density $\rho_r = \frac{E_r}{V} \propto a^{-4}$ so that $w_r = \frac{1}{3}$	**0.3**
(ii) In case of nonrelativistic matter, its energy density nearly $\rho_m \simeq \frac{m_0 c^2}{V} \propto a^{-3}$ since dominant energy comes from its rest energy $m_0 c^2$, so that $w_m = 0$	**0.3**
(iii) For a constant energy density, let say $\epsilon_\Lambda =$ constant, $\epsilon_\Lambda \propto a^0$ so that $w_\Lambda = -1$.	**0.3**
Total	**1.2**

Question A.4

Answer	Marks
(i) In case of $k=0$, for radiation we have $\rho_r a^4 =$ constant. So by comparing the parameters values with their present value, $\rho_r(t)a^4(t) = \rho_{r0}a_0^4$, $$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_{r0}\left(\frac{a_0}{a}\right)^4.$$ $$\int a\,da = \frac{1}{2}a^2 + K = \left(\frac{8\pi G}{3}\rho_{r0}a_0^4\right)^{\frac{1}{2}} t.$$	0.2
Because $a(t=0)=0, K=0$, then $$a(t) = (2)^{\frac{1}{2}}\left(\frac{8\pi G}{3}\rho_{r0}a_0^4\right)^{\frac{1}{4}} t^{\frac{1}{2}} = (2H_0)^{\frac{1}{2}} t^{\frac{1}{2}}.$$ where $H_0 = \left(\frac{8\pi G}{3}\rho_{r0}\right)^{\frac{1}{2}}$ after taking $a_0 = 1$.	0.2
(ii) for non-relativistic matter domination, using $\rho_m(t)a^3(t) = \rho_{m0}a_0^3$, and similar way we will get $$a(t) = \left(\frac{3}{2}\right)^{\frac{2}{3}}\left(\frac{8\pi G}{3}\rho_{m0}a_0^4\right)^{\frac{1}{3}} t^{\frac{2}{3}} = \left(\frac{3H_0}{2}\right)^{\frac{2}{3}} t^{\frac{2}{3}}.$$ where $H_0 = \left(\frac{8\pi G}{3}\rho_{m0}\right)^{\frac{1}{2}}$.	0.4
(iii) for constant energy density, $$\ln a = H_0 t + K'$$ Where K' is integration constant and $H_0 = \left(\frac{8\pi G}{3}\rho_\Lambda\right)^{\frac{1}{2}}$. Taking condition $a_0 = 1$, $$\ln\left(\frac{a}{a_0}\right) = H_0(t - t_0)$$ $$a(t) = e^{H_0(t-t_0)}$$	0.4
Total	1.2

Question A.5

Answer	Marks
Condition for critical energy condition: $$\rho_c(t) = \frac{3H^2}{8\pi G}$$ Friedmann equation can be written as $$H^2(t) = H^2(\mathrm{t})\Omega(\mathrm{t}) - \frac{kc^2}{R_0^2 a^2(t)}$$ $$\left(\frac{R_0^2}{c^2}\right) a^2 H^2(\Omega - 1) = k \qquad \text{(A.5.1)}$$	**0.1**
Total	**0.1**

Question A.6

Answer	Marks
Because $\left(\frac{R_0^2}{c^2}\right) a^2 H^2 > 0$, then $k = +1$ corresponds to $\Omega > 1$, $k = -1$ corresponds to $\Omega < 1$ and $k = 0$ corresponds to $\Omega = 1$	0.3
Total	0.3

B. Motivation To Introduce Inflation Phase and Its General Conditions

Question B.1

Answer	Marks
Equation (A.5.1) shows that $(\Omega - 1) = \frac{kc^2}{R_0^2}\frac{1}{\dot{a}^2}$.	**0.1**
In a universe dominated by non-relativistic matter or radiation, scale factor can be written as a function of time as $a = a_0\left(\frac{t}{t_0}\right)^p$ where $p < 1$ ($p = \frac{1}{2}$ for radiation and $p = \frac{2}{3}$ for non-relativistic matter)	**0.2**
$(\Omega - 1) = \tilde{k}\, t^{2(1-p)}$	**0.2**
Total	**0.5**

Question B.2

Answer	Marks
For a period dominated by constant energy provides the solution $a(t) = e^{Ht}$ so that $\dot{a} = He^{Ht}$	**0.1**
$$(\Omega - 1) = \frac{k}{H^2}\, t^{-2Ht}$$	**0.2**
Total	**0.3**

Question B.3

Answer	Marks
Inflation period can be generated by constant energy period, therefore it is a phase where $w = -1$ so that $p = w\rho c^2 = -\rho c^2$ (negative pressure).	**0.2**
Differentiating Friedmann equation leads to $$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2 - \frac{kc^2}{R_0^2}$$ $$2\dot{a}\ddot{a} = \frac{8\pi G}{3}(\dot{\rho}a^2 + 2\rho a\,\dot{a}) = \frac{8\pi G}{3}\left(-3\left(\rho + \frac{p}{c^2}\right)a\dot{a} + 2\rho a\dot{a}\right).$$ $$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right)$$	**0.4**
So that because during inflation $p = -\rho c^2$, it is equivalent with condition $\ddot{a} > 0$ (accelerated expansion)	**0.1**
As a result, $\ddot{a} = d(\dot{a})/dt = d(Ha)/dt > 0$ or $d(Ha)^{-1}/dt < 0$ (shrinking Hubble radius).	**0.2**
Total	**0.9**

Question B.4

Answer	Marks
Inflation condition can be written as $\frac{d(aH)^{-1}}{dt} < 0$, with $H = \dot{a}/a$ as such $$\frac{d(aH)^{-1}}{dt} = -\frac{\dot{a}H + a\dot{H}}{(aH)^2} = -\frac{1}{a}(1-\epsilon) < 0 \Rightarrow \epsilon < 1$$	**0.2**
Total	**0.2**

C. Inflation Generated by Homogenously Distributed Matter

Question C.1

Answer	Marks
Differentiating equations (4) and employing equation 4 we can get $$2H\dot{H} = \frac{1}{3M_{pl}^2}\left[\dot{\phi}\ddot{\phi} + \left(\frac{\partial V}{\partial \phi}\right)\dot{\phi}\right] = \frac{1}{3M_{pl}^2}[-3H\,\dot{\phi}^2]$$ $$\dot{H} = -\frac{1}{2}\frac{\dot{\phi}^2}{M_{pl}^2}$$	**0.3**
Therefore $\epsilon = \frac{1}{2}\frac{\dot{\phi}^2}{M_{pl}^2 H^2}$	**0.1**
The inflation can occur when the potential energy dominates the particle's energy ($\dot{\phi}^2 \ll V$) such that $H^2 \approx V/(3M_{pl}^2)$.	**0.2**
Slow-roll approximation: $3H\dot{\phi} \approx -V'$	**0.1**
Implies $$\epsilon \approx \frac{M_{pl}^2}{2}\left(\frac{V'}{V}\right)^2 \qquad \text{(C.1.1)}$$	**0.3**
we also have $$3\dot{H}\dot{\phi} + 3H\ddot{\phi} = -V''\dot{\phi}$$ $\delta = -\frac{\ddot{\phi}}{H\dot{\phi}} = \frac{V''}{3H^2} - \epsilon$ Therefore $$\eta_V \approx M_{pl}^2\frac{V''}{V} \qquad \text{(C.1.2)}$$	**0.4**
$$\mathrm{d}N = H\,dt = \left(\frac{H}{\dot{\phi}}\right)d\phi \approx -\frac{1}{M_{pl}^2}(V/V')\,d\phi \qquad \text{(C.1.3)}$$ $$\frac{dN}{d\phi} \approx -\frac{1}{M_{pl}^2}(V/V')$$	**0.3**
Total	**1.7**

D. **Inflation with A Simple Potential**

Question D.1

Answer	Marks
Inflation ends at $\epsilon = 1$. Using $V(\phi) = \Lambda^4(\phi/M_{pl})^n$ yields $$\epsilon = \frac{M_{pl}^2}{2}\left[\frac{n}{\phi_{\text{end}}}\right]^2 = 1 \Rightarrow \phi_{end} = \frac{n}{\sqrt{2}}M_{pl}$$	**0.5**
Total	**0.5**

Question D.2

Answer	Marks
From equations (C.1.1), (C.1.2) and (C.1.3) we can obtain $$N = -\left[\frac{\phi}{M_{pl}}\right]^2 \frac{1}{2n} + \beta$$ where β is a integration constant. As $N = 0$ at ϕ_{end} then $\beta = \frac{n}{4}$. $$N = -\left[\frac{\phi}{M_{pl}}\right]^2 \frac{1}{2n} + \frac{n}{4}$$	**0.2**
$$\eta_V = n(n-1)\left[\frac{M_{pl}}{\phi}\right]^2 = \frac{2(n-1)}{n-4N}$$	**0.2**
$$\varepsilon = \frac{n^2}{2}\left[\frac{M_{pl}}{\phi}\right]^2 = \frac{n}{n-4N}$$	**0.2**
so that $$r = 16\varepsilon = \frac{16n}{n-4N}$$	**0.1**

$$n_s = 1 + 2\eta_V - 6\epsilon = 1 - \frac{2(n+2)}{(n-4N)}$$	**0.1**
To obtain the observational constraint $n_s = 0.968$ we need $n = -5.93$ which is inconsistent with the condition $r < 0.12$. There is no a closest integer n that can obtains $r < 0.12$. As example, for $n = -6$ leads a contradiction $0 < (-0.27)$ and for $n = -5$ leads a contradiction $0 < (-0.2)$.	**0.1**
Total	0.9

IPhO 2018
Lisbon, Portugal

Solutions to Theory Problem 1

LIGO-GW150914

(V. Cardoso, C. Herdeiro)

July 15, 2018

v6.0

GW150914 (10 points)

Part A. Newtonian (conservative) orbits (3.0 points)

A.1 Apply Newton's law to mass M_1:

$$M_1 \frac{\mathrm{d}^2 \vec{r}_1}{\mathrm{d}t^2} = G \frac{M_1 M_2}{|\vec{r}_2 - \vec{r}_1|^2} \frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_2 - \vec{r}_1|} \,. \tag{1}$$

Use, from eq. (1) of the question sheet

$$\vec{r}_2 = -\frac{M_1}{M_2} \vec{r}_1 \,, \tag{2}$$

in eq. (1) above, to obtain

$$\frac{\mathrm{d}^2 \vec{r}_1}{\mathrm{d}t^2} = -\frac{G M_2^3}{(M_1 + M_2)^2 r_1^2} \frac{\vec{r}_1}{r_1} \,. \tag{3}$$

A.1	1.0pt
$$n = 3, \qquad \alpha = \frac{G M_2^3}{(M_1 + M_2)^2} \,.$$	

A.2 The total energy of the system is the sum of the two kinetic energies plus the gravitational potential energy. For circular motions, the linear velocity of each of the masses reads

$$|\vec{v}_1| = r_1 \Omega \,, \qquad |\vec{v}_2| = r_2 \Omega \,, \tag{4}$$

Thus, the total energy is

$$E = \frac{1}{2}(M_1 r_1^2 + M_2 r_2^2)\Omega^2 - \frac{G M_1 M_2}{L} \,, \tag{5}$$

Now,

$$(M_1 r_1 - M_2 r_2)^2 = 0 \qquad \Rightarrow \qquad M_1 r_1^2 + M_2 r_2^2 = \mu L^2 \,. \tag{6}$$

Thus,

$$E = \frac{1}{2}\mu L^2 \Omega^2 - G\frac{M\mu}{L} \,. \tag{7}$$

A.2	1.0pt
$$A(\mu, \Omega, L) = \frac{1}{2}\mu L^2 \Omega^2 \,.$$	

A.3 Energy (3) of the question sheet can be interpreted as describing a system of a mass μ in a circular orbit with angular velocity Ω, radius L, around a mass M (at rest). Equating the gravitational acceleration to the centripetal acceleration:

$$G\frac{M}{L^2} = \Omega^2 L \,. \tag{8}$$

This is indeed Kepler's third law (for circular orbits). Then, from (7),

$$E = -\frac{1}{2} G \frac{M\mu}{L} \,. \tag{9}$$

A.3	1.0pt
$$\beta = -\frac{1}{2} \,.$$	

Part B - Introducing relativistic dissipation (7.0 points)

B.1 Some simple trigonometry for the x, y motion of the masses (in an appropriate Cartesian system) yields:

$$(x_1, y_1) = r_1(\cos(\Omega t), \sin(\Omega t)) , \qquad (x_2, y_2) = -r_2(\cos(\Omega t), \sin(\Omega t)) . \tag{10}$$

Then,

$$Q_{ij} = \frac{M_1 r_1^2 + M_2 r_2^2}{2} \begin{pmatrix} \frac{4}{3}\cos^2(\Omega t) - \frac{2}{3}\sin^2(\Omega t) & 2\sin(\Omega t)\cos(\Omega t) & 0 \\ 2\sin(\Omega t)\cos(\Omega t) & \frac{4}{3}\sin^2(\Omega t) - \frac{2}{3}\cos^2(\Omega t) & 0 \\ 0 & 0 & -\frac{2}{3} \end{pmatrix} , \tag{11}$$

or, using some simple trigonometry and (6),

$$Q_{ij} = \frac{\mu L^2}{2} \begin{pmatrix} \frac{1}{3} + \cos 2\Omega t & \sin 2\Omega t & 0 \\ \sin 2\Omega t & \frac{1}{3} - \cos 2\Omega t & 0 \\ 0 & 0 & -\frac{2}{3} \end{pmatrix} . \tag{12}$$

B.1 1.0pt

$$k = 2\Omega , \quad a_1 = a_2 = \frac{1}{3} , a_3 = -\frac{2}{3} , \quad b_1 = 1, b_2 = -1, b_3 = 0 , c_{12} = c_{21} = 1, c_{ij} \overset{\text{otherwise}}{=} 0 .$$

B.2 First take the derivatives:

$$\frac{\mathrm{d}^3 Q_{ij}}{\mathrm{d}t^3} = 4\Omega^3 \mu L^2 \begin{pmatrix} \sin 2\Omega t & -\cos 2\Omega t & 0 \\ -\cos 2\Omega t & -\sin 2\Omega t & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{13}$$

Then perform the sum:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{G}{5c^5}(4\Omega^3 \mu L^2)^2 [2\sin^2(2\Omega t) + 2\cos^2(2\Omega t)] = \frac{32}{5}\frac{G}{c^5}\mu^2 L^4 \Omega^6 . \tag{14}$$

B.2 1.0pt

$$\xi = \frac{32}{5} .$$

B.3 Now we assume a sequency of Keplerian orbits, with decreasing energy, which is being taken from the system by the GWs.

First, from (9), differentiating with respect to time,

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{GM\mu}{2L^2}\frac{\mathrm{d}L}{\mathrm{d}t} , \tag{15}$$

Since this loss of energy is due to GWs, we equate it with (minus) the luminosity of GWs, given by (14)

$$\frac{GM\mu}{2L^2}\frac{\mathrm{d}L}{\mathrm{d}t} = -\frac{32}{5}\frac{G}{c^5}\mu^2 L^4 \Omega^6 . \tag{16}$$

We can eliminate the L and $\mathrm{d}L/\mathrm{d}t$ dependence in this equation in terms of Ω and $\mathrm{d}\Omega/\mathrm{d}t$, by using Kepler's third law (8), which relates:

$$L^3 = G\frac{M}{\Omega^2} , \qquad \frac{\mathrm{d}L}{\mathrm{d}t} = -\frac{2}{3}\frac{L}{\Omega}\frac{\mathrm{d}\Omega}{\mathrm{d}t} . \tag{17}$$

Substituting in (16), we obtain:

$$\left(\frac{\mathrm{d}\Omega}{\mathrm{d}t}\right)^3 = \left(\frac{96}{5}\right)^3 \frac{\Omega^{11}}{c^{15}} G^5 \mu^3 M^2 \equiv \left(\frac{96}{5}\right)^3 \frac{\Omega^{11}}{c^{15}} (GM_\mathrm{c})^5 \ . \tag{18}$$

B.3 1.0pt

$$M_\mathrm{c} = (\mu^3 M^2)^{1/5} \ .$$

B.4 Angular and cycle frequencies are related as $\Omega = 2\pi f$. From the information provided above: *GWs have a frequency which is twice as large as the orbital frequency*, we have

$$\frac{\Omega}{2\pi} = \frac{f_\mathrm{GW}}{2} \ . \tag{19}$$

Formula (10) of the question sheet has the form

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = \chi \Omega^{11/3} \ , \qquad \chi \equiv \frac{96}{5}\frac{(GM_\mathrm{c})^{5/3}}{c^5} \ . \tag{20}$$

Thus, from (11) of the question sheet

$$\Omega(t)^{-8/3} = \frac{8}{3}\chi(t_0 - t) \ , \tag{21}$$

or, using (20) and the definition of χ gives

$$f_\mathrm{GW}^{-8/3}(t) = \frac{(8\pi)^{8/3}}{5}\left(\frac{GM_\mathrm{c}}{c^3}\right)^{5/3}(t_0 - t) \ . \tag{22}$$

B.4 2.0pt

$$p = 1 \ .$$

B.5 From the figure, we consider the two Δt's as half periods. Thus, the (cycle) GW frequency is $f_\mathrm{GW} = 1/(2\Delta t)$. Then, the four given points allow us to compute the frequency at the mean time of the two intervals as

	t_AB	t_CD
t (s)	0.0045	0.037
f_GW (Hz)	$(2 \times 0.009)^{-1}$	$(2 \times 0.006)^{-1}$

Now, using (22) we have two pairs of (f_GW,t) values for two unknowns (t_0,M_c). Expressing (22) for both t_AB and t_CD and dividing the two equations we obtain:

$$t_0 = \frac{At_\mathrm{CD} - t_\mathrm{AB}}{A - 1} \ , \qquad A \equiv \left(\frac{f_\mathrm{GW}(t_\mathrm{AB})}{f_\mathrm{GW}(t_\mathrm{CD})}\right)^{-8/3} \ . \tag{23}$$

Replacing by the numerical values, $A \simeq 2.95$ and $t_0 \simeq 0.054$ s. Now we can use (22) for either of the two values t_AB or t_CD and determine M_c. One obtains for the chirp mass

$$M_\mathrm{c} \simeq 6 \times 10^{31}\ \mathrm{kg} \simeq 30 \times M_\odot \ . \tag{24}$$

Thus, the total mass M is

$$M = 4^{3/5} M_\mathrm{c} \simeq 69 \times M_\odot \ . \tag{25}$$

This result is actually remarkably close to the best estimates using the full theory of General Relativity! [Even though the actual objects do not have precisely equal masses and the theory we have just used is not valid very close to the collision.]

B.5	1.0pt
$M_{\text{c}} \simeq 30 \times M_{\odot}\ , \qquad M \simeq 69 \times M_{\odot}\ .$	

B.6 From (8), Kepler's law states that $L = (GM/\Omega^2)^{1/3}$. The second pair of points highlighted in the plot correspond to the cycle prior to merger. Thus, we use (19) to obtain the orbital angular velocity at $t_{\overline{\text{CD}}}$:

$$\Omega_{t_{\overline{\text{CD}}}} \sim 2.6 \times 10^2 \text{ rad/s}\ . \tag{26}$$

Then, using the total mass (25) we find

$$L \sim 5 \times 10^2 \text{ km}\ . \tag{27}$$

Thus, these objects have a maximum radius of $R_{\text{max}} \sim 250$ km. Hence they have over 30 times more mass and,

$$\frac{R_{\odot}}{R_{\text{max}}} \sim 3 \times 10^3 \tag{28}$$

they are 3000 times smaller than the Sun and!

Their linear velocity is

$$v_{\text{col}} = \frac{L}{2}\Omega \simeq 7 \times 10^4 \text{ km/s}\ . \tag{29}$$

They are moving at over 20% of the velocity of light!

B.6	1.0pt
$L_{\text{collision}} \sim 5 \times 10^2 \text{ km}\ , \qquad \frac{R_{\odot}}{R_{\text{max}}} \sim 3 \times 10^3\ , \qquad \frac{v_{\text{col}}}{c} \sim 0.2\ .$	

IPhO 2018
Lisbon, Portugal

Solutions to Theory Problem 2

Where is the neutrino?

(Miguel C N Fiolhais and António Onofre)

July 24, 2018

v1.2

Where is the neutrino? (10 points)

Part A. ATLAS Detector physics (4.0 points)

A.1

The magnetic force is the centripetal force:

$$m\frac{v^2}{r} = evB \Rightarrow r = \frac{mv}{eB}\,.$$

First express the velocity in terms of the kinetic energy,

$$K = \frac{1}{2}mv^2 \Rightarrow v = \sqrt{\frac{2K}{m}}\,,$$

and then insert it in the expression above for the radius to get

A.1	0.5pt
$$r = \frac{\sqrt{2Km}}{eB}\,.$$	

A.2

The radius of the circular motion of a charged particle in the presence of a uniform magnetic field is given by,

$$r = \frac{mv}{eB}\,.$$

This formula is valid in the relativistic scenario if the mass correction, $m \to \gamma m$ is included:

$$r = \frac{\gamma m v}{eB} = \frac{p}{eB} \Rightarrow p = reB\,.$$

Note that the radius of the circular motion is half the radius of the inner part of the detector. One obtains [1 MeV/$c = 5.34 \times 10^{-22}$ m kg s^{-1}]

A.2	0.5pt
$$p = 330 \text{ MeV}/c\,.$$	

A.3

The acceleration for the particle is $a = \frac{evB}{\gamma m} \sim \frac{ecB}{\gamma m}$, in the ultrarelativistic limit. Then,

$$P = \frac{e^4 c^2 \gamma^4 B^2}{6\pi\epsilon_0 c^3 \gamma^2 m^2} = \frac{e^4 \gamma^2 c^4 B^2}{6\pi\epsilon_0 c^5 m^2}.$$

Since $E = \gamma m c^2$ we can obtain $\gamma^2 c^4 = \frac{E^2}{m^2}$ and, finally,

$$P = \frac{e^4}{6\pi\epsilon_0 m^4 c^5} E^2 B^2.$$

Therefore,

A.3 1.0pt

$$\xi = \frac{1}{6\pi}\,, \quad n = 5 \text{ and } \quad k = 4.$$

A.4

The power emitted by the particle is given by,

$$P = -\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{e^4}{6\pi\epsilon_0 m^4 c^5} E^2 B^2\,.$$

The energy of the particle as a function of time can be calculated from

$$\int_{E_0}^{E(t)} \frac{1}{E^2}\mathrm{d}E = -\int_0^t \frac{e^4}{6\pi\epsilon_0 m^4 c^5} B^2 \mathrm{d}t\,,$$

where $E(0) = E_0$. This leads to,

$$\frac{1}{E(t)} - \frac{1}{E_0} = \frac{e^4 B^2}{6\pi\epsilon_0 m^4 c^5}\, t \quad \Rightarrow \quad E(t) = \frac{E_0}{1 + \alpha E_0 t}\,,$$

with

A.4 1.0pt

$$\alpha = \frac{e^4 B^2}{6\pi\epsilon_0 m^4 c^5}\,.$$

A.5

If the initial energy of the electron is 100 GeV, the radius of curvature is extremely large ($r = \frac{E}{eBc} \approx 167$ m). Therefore, in approximation, one can consider the electron is moving in the inner part of the ATLAS detector along a straight line. The time of flight of the electron is $t = R/c$, where $R = 1.1$ m is the radius of the inner part of the detector. The total energy lost due to synchrotron radiation is,

$$\Delta E = E(R/c) - E_0 = \frac{E_0}{1 + \alpha E_0 \frac{R}{c}} - E_0 \approx -\alpha E_0^2 \frac{R}{c}$$

and

A.5 0.5pt

$$\Delta E = -56 \text{ MeV}\,.$$

A.6

In the ultrarelativistic limit, $v \approx c$ and $E \approx pc$. The cyclotron frequency is,

$$\omega(t) = \frac{c}{r(t)} = \frac{ecB}{p(t)} = \frac{ec^2B}{E(t)}$$

A.6 0.5pt

$$\omega(t) = \frac{ec^2B}{E_0}\left(1 + \frac{e^4B^2}{6\pi\epsilon_0 m^4c^5}E_0 t\right) .$$

Part B. Finding the neutrino (6.0 points)

B.1

Since the W^+ boson decays into an anti-muon and a neutrino, one can use principles of conservation of energy and linear momentum to calculate the unknown $p_z^{(\nu)}$ of the neutrino. Moreover, the anti-muon and the neutrino can be considered massless, which implies that the magnitude of their momenta (times c) and their energies are the same. Therefore, the conservation of linear momentum can be expressed as

$$\vec{p}^{\,(W)} = \vec{p}^{\,(\mu)} + \vec{p}^{\,(\nu)},$$

and the conservation of energy as,

$$E^{(W)} = cp^{(\mu)} + cp^{(\nu)}.$$

In addition, one can also relate the energy and the momentum of the W^+ boson through its mass,

$$m_W^2 = (E^{(W)})^2/c^4 - (p^{(W)})^2/c^2$$

which leads to a quadratic equation on $p_z^{(\nu)}$,

$$\begin{aligned} m_W^2 &= \left[(p^{(\mu)} + p^{(\nu)})^2 - (\vec{p}^{\,(\mu)} + \vec{p}^{\,(\nu)})^2\right]/c^2 \\ &= \left(2p^{(\mu)}p^{(\nu)} - 2\vec{p}^{\,(\mu)} \cdot \vec{p}^{\,(\nu)}\right)/c^2 \end{aligned}$$

B.1 1.5pt

$$m_W^2 = \frac{1}{c^2}\left(2p^{(\mu)}\sqrt{(p_{\mathrm{T}}^{(\nu)})^2 + (p_z^{(\nu)})^2} - 2\vec{p}_{\mathrm{T}}^{(\mu)} \cdot \vec{p}_{\mathrm{T}}^{(\nu)} - 2p_z^{(\mu)}p_z^{(\nu)}\right).$$

B.2

The numerical substitution directly in the answer of B.1, using

$$p^{(\mu)} = 37.2\,\text{GeV}/c \qquad m_W^2c^2 = 6464.2\,(\text{GeV}/c)^2 \qquad p_{\mathrm{T}}^{(\nu)\,2} = 10\,864.9\,(\text{GeV}/c)^2$$

$$\vec{p}_{\mathrm{T}}^{(\mu)} \cdot \vec{p}_{\mathrm{T}}^{(\nu)} = 2439.3(\text{GeV}/c)^2 \qquad p_z^{(\mu)} = -12.4\,\text{GeV}/c,$$

leads to

$$6464.2 = 74.4\sqrt{10\,864.9 + p_z^{(\nu)\,2}} - 4878.6 + 24.8p^{(\nu)}.$$

This is a quadratic equation, equivalent to

$$0.88889\,p_z^{(\nu)\,2} + 101.64\,p_z^{(\nu)} - 12378 = 0$$

whose solutions are:

B.2 1.5pt

$$p_z^{(\nu)} = 74.0 \text{ GeV}/c \qquad \text{or} \qquad p_z^{(\nu)} = -188.3 \text{ GeV}/c.$$

The general solution of the equation above in B.1 leads to

$$\begin{aligned} p_z^{(\nu)} &= \frac{2\vec{p}_{\mathrm{T}}^{(\mu)} \cdot \vec{p}_{\mathrm{T}}^{(\nu)} p_z^{(\mu)} + m_W^2 c^2 p_z^{(\mu)}}{2(p_{\mathrm{T}}^{(\mu)})^2} \\ &\pm \frac{p^{(\mu)} \sqrt{-4(p_{\mathrm{T}}^{(\mu)})^2 (p_{\mathrm{T}}^{(\nu)})^2 + 4(\vec{p}_{\mathrm{T}}^{(\mu)} \cdot \vec{p}_{\mathrm{T}}^{(\nu)})^2 + 4\vec{p}_{\mathrm{T}}^{(\mu)} \cdot \vec{p}_{\mathrm{T}}^{(\nu)} m_W^2 c^2 + m_W^4 c^4}}{2(p_{\mathrm{T}}^{(\mu)})^2} \end{aligned}$$

Numerical substitution leads to the above mentioned values for $p_z^{(\nu)}$.

B.3

The final state particles of the top quark decay are the anti-muon, the neutrino and jet 1. Since the neutrino is now fully reconstructed the energy and linear momentum of the top quark can be calculated as,

$$\begin{aligned} E^{(\mathrm{t})} &= cp^{(\mu)} + cp^{(\nu)} + cp^{(j_1)} \\ \vec{p}^{\,(\mathrm{t})} &= \vec{p}^{\,(\mu)} + \vec{p}^{\,(\nu)} + \vec{p}^{\,(j_1)} . \end{aligned}$$

The top quark mass is,

$$\begin{aligned} m_{\mathrm{t}} &= \sqrt{(E^{(\mathrm{t})})^2/c^4 - (\vec{p}^{\,(\mathrm{t})})^2/c^2} \\ &= c^{-1}\sqrt{\left(p^{(\mu)} + p^{(\nu)} + p^{(j_1)}\right)^2 - \left(\vec{p}^{\,(\mu)} + \vec{p}^{\,(\nu)} + \vec{p}^{\,(j_1)}\right)^2} . \end{aligned}$$

The substitution of values leads to two possible masses:

B.3 1.0pt

$$m_{\mathrm{t}} = 169.3 \text{ GeV}/c^2 \qquad \text{or} \qquad m_{\mathrm{t}} = 311.2 \text{ GeV}/c^2$$

B.4

According to the frequency distribution for signal (dashed line), the probability of the $m_{\mathrm{t}} = 169.3$ GeV/c^2 solution is roughly 0.1 while the probability of the $m_{\mathrm{t}} = 311.2$ GeV/c^2 solution is below 0.01. Therefore,

B.4 The most likely candidate is the $m_{\mathrm{t}} = 169.3$ GeV/c^2 solution. 1.0pt

B.5

The top quark energy for the most likely candidate is $E^{(\mathrm{t})} = cp^{(\mu)} + cp^{(\nu)} + cp^{(j_1)} = 272.6$ GeV .

$$d = vt = v\gamma t_0 = \frac{p^{(\mathrm{t})}}{m_{\mathrm{t}}} t_0 = ct_0 \sqrt{\frac{{E^{(\mathrm{t})}}^2}{m_{\mathrm{t}}^2 c^4} - 1}\,.$$

B.5 1.0pt

$d = 2 \times 10^{-16}$ m .

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Solutions to Theory Problem 3

Physics of Live Systems

(Rui Travasso, Lucília Brito)

July 24, 2018

v1.0

Physics of Live Systems (10 points)

Part A. The physics of blood flow (4.5 points)

A.1

Since the vessel network is symmetrical, the flow in a vessel of level $i+1$ is half the flow in a vessel of level i.

In this way, we can sum the pressure differences in all levels:

$$\Delta P = \sum_{i=0}^{N-1} Q_i R_i = Q_0 \sum_{i=0}^{N-1} \frac{R_i}{2^i}.$$

Introducing the radii dependences yields

$$\Delta P = Q_0 \sum_{i=0}^{N-1} \frac{8\ell_i \eta}{2^i \pi r_i^4} = Q_0 \frac{8\ell_0 \eta}{\pi r_0^4} \sum_{i=0}^{N-1} \frac{2^{4i/3}}{2^i 2^{i/3}} = Q_0 N \frac{8\ell_0 \eta}{\pi r_0^4} .$$

Therefore

$$Q_0 = \Delta P \frac{\pi r_0^4}{8N\ell_0 \eta} .$$

Hence, the flow rate for a vessel network in level i is

A.1 $$Q_i = \Delta P \frac{\pi r_0^4}{2^{i+3} N \ell_0 \eta} .$$	1.3pt

A.2

Replace values in the formula and change units appropriately

$$\begin{aligned} Q_0 &= \frac{\Delta P \pi r_0^4}{8N\ell_0 \eta} = \\ &= \frac{(55-30) \times 1.013 \times 10^5 \times 3.1415 \times (6.0 \times 10^{-5})^4}{760 \times 48 \times 2.0 \times 10^{-3} \times 3.5 \times 10^{-3}} = 4.0 \times 10^{-10}\ \text{m}^3/\text{s} \end{aligned}$$

to obtain the final value in the requested unites:

A.2 $$Q_0 \simeq 1.5\ \text{m}\ell/\text{h} .$$	0.5pt

A.3

The current is given by

$$I = \frac{P_{\text{in}}\mathrm{e}^{i\omega t}}{R + i\omega L + \frac{1}{i\omega C}} \,.$$

The pressure difference in the capacitor is

$$P_{\text{out}}\mathrm{e}^{i(\omega t+\phi)} = \frac{P_{\text{in}}\mathrm{e}^{i\omega t}}{R + i\omega L + \frac{1}{i\omega C}} \, \frac{1}{i\omega C} = \frac{P_{\text{in}}\mathrm{e}^{i\omega t}}{i\omega CR - \omega^2 LC + 1} \,.$$

The amplitude is

$$P_{\text{out}} = \frac{P_{\text{in}}}{\sqrt{(1-\omega^2 LC)^2 + \omega^2 C^2 R^2}} \,.$$

To be smaller than P_{in}, for $\omega \to 0$:

$$(1-\omega^2 LC)^2 + \omega^2 C^2 R^2 > 1 \iff -2CL + C^2R^2 > 0 \,.$$

Replacing the expressions for L, C, and R we get: $\frac{64\eta^2\ell^2}{3Ehr^3\rho} > 1$.

A.3 2.0pt

$$P_{\text{out}} = \frac{P_{\text{in}}}{\sqrt{(1-\omega^2 LC)^2 + \omega^2 C^2 R^2}} \,.$$

Condition:

$$\frac{64\eta^2\ell^2}{3Ehr^3\rho} > 1 \,.$$

Alternative way to obtain P_{out}:

The amplitude of the current in the equivalent circuit is $I_0 = \frac{P_{\text{in}}}{Z}$, where

$$Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

is the modulus of the impedance. Hence, the voltage amplitude in the capacitor is

$$P_{\text{out}} = \frac{1}{\omega C} \times I_0 = \frac{P_{\text{in}}}{\sqrt{\omega^2 C^2 R^2 + (\omega^2 LC - 1)^2}}.$$

A.4

The previous condition can also be expressed as

$$h < \frac{64\eta^2\ell^2}{3Er^3\rho} \,.$$

For the network referred to in **A.2**

$$h < \frac{64\eta^2\ell_0^2 \times 2^i}{3 \times 2^{2i/3} E r_0^3 \rho} = \frac{64 \times (3.5\times10^{-3})^2 \times (2.0\times10^{-3})^2}{3 \times 0.06 \times 10^6 \times (6.0\times10^{-5})^3 \times 1.05\times10^3} \times 2^{i/3} = 7.7\times10^{-5} \times 2^{i/3} \,.$$

For $i = 0$, in the worse case scenario,

$$h_{\text{max}} = 7.7 \times 10^{-5} \times 2^0 = 7.7 \times 10^{-5} \text{ m}$$

This value is certainly observed in these vessels since their radius range from 18 μm to 60 μm. A wall width smaller than 80 μm is certainly reasonable.

A.4	Maximum $h = 8 \times 10^{-5}$ m	0.7pt

Part B. Tumor growth (5.5 points)

B.1

The expressions for the masses of tumour and normal tissue are written as:

$$\begin{cases} M_{\text{T}} = V_{\text{T}}\rho_{\text{T}} = V_{\text{T}}\rho_0(1 + \frac{p}{K_{\text{T}}}) \\ \\ M_{\text{N}} = V\rho_0 = (V - V_{\text{T}})\rho_0(1 + \frac{p}{K_{\text{N}}}) \end{cases}$$

The pressure, p, can be expressed as

$$p = \frac{M_{\text{T}}\,K_{\text{T}}}{V_{\text{T}}\,\rho_0} - K_{\text{T}}$$

and, then, used in the equation for M_{N}:

$$M_{\text{N}} = (V - V_{\text{T}})\,\frac{M_{\text{N}}}{V}\left[\left(1 - \frac{K_{\text{T}}}{K_{\text{N}}}\right) + \frac{M_{\text{T}}\,V K_{\text{T}}}{V_{\text{T}}\,M_{\text{N}}\,K_{\text{N}}}\right]$$

Simplifying and rearranging the terms, the equation for v becomes

$$(1 - \kappa)\;v^2 - (1 + \mu)\;v + \mu = 0\,,$$

for which the solution is (the other solution of the quadratic equation is not physically relevant since does not lead to $v = 0$ for $\mu = 0$)

B.1	$$v = \frac{1 + \mu - \sqrt{(1+\mu)^2 - 4\mu\,(1-\kappa)}}{2(1-\kappa)}\,.$$	1.0pt

B.2

For $r < R_{\text{T}}$, the conservation of energy implies that

$$4\pi r^2(-k)\frac{\mathrm{d}T}{\mathrm{d}r} = \mathcal{P}\frac{4}{3}\pi r^3\,.$$

Therefore, the temperature difference to 37 °C = 310.15 K, $\Delta T(r)$, is given by

$$\Delta T(r) = -\frac{\mathcal{P} r^2}{6k} + C\,,$$

where C is a constant.

For $r > R_\mathsf{T}$, the conservation of energy implies that

$$4\pi r^2(-k)\frac{\mathrm{d}T}{\mathrm{d}r} = \mathcal{P}\frac{4}{3}\pi R_\mathsf{T}^3\,.$$

Therefore, the temperature difference to 37 °C is

$$\Delta T(r) = \frac{\mathcal{P} R_\mathsf{T}^3}{3kr}\,.$$

In this case there is no constant, since very far away the increase in temperature is zero.

Matching the two solutions at $r = R_\mathsf{T}$ gives

$$C = \frac{\mathcal{P} R_\mathsf{T}^2}{2k}\,.$$

Therefore the temperature at the centre of the tumour, in SI units, is

B.2	Temperature: $310.15 + \frac{\mathcal{P} R_\mathsf{T}^2}{2k}$.	1.7pt

B.3

The increase in temperature at the tumour surface (the lower temperature in the tumour) is

$$\Delta T(R_\mathsf{T}) = \frac{\mathcal{P} R_\mathsf{T}^2}{3k}\,.$$

This increase should be equal to 6.0 K. Therefore,

$$\mathcal{P} = \frac{3\Delta T k}{R_\mathsf{T}^2} = \frac{3\times 6\times 0.6}{0.05^2} = 4.3 \text{ kW/m}^3.$$

B.3	$\mathcal{P}_\mathsf{min} = 4.3$ kW/m^3.	0.5pt

B.4

We can relate δr with the pressure in the tumour, using the relation given in the text up to leading order in $p - P_\mathsf{cap}$: $\delta r = \frac{p-P_\mathsf{cap}}{2(p_\mathsf{c}-P_\mathsf{cap})}\,\delta r_\mathsf{c}$. Therefore, if $p - P_\mathsf{cap}$ is very small, also it is δr.

The pressure can be related with the volume. We know that

$$\frac{M_\mathsf{N}}{V_\mathsf{N}} = \frac{\rho_0 V}{V - V_\mathsf{T}} = \frac{\rho_0}{1-v} = \rho_0\left(1 + \frac{p}{K_\mathsf{N}}\right)\,.$$

And so $p = \frac{K_\mathsf{N} v}{1-v}$.

When the thinner vessels are narrower, the flow rate in the main vessel is altered:

$$\Delta P = (Q_0 + \delta Q_0) \sum_{i=0}^{N-1} \frac{8\ell_i \eta}{2^i \pi r_i^4} = (Q_0 + \delta Q_0) \frac{8\ell_0 \eta}{\pi r_0^4} \left(\sum_{i=0}^{N-2} \frac{2^{4i/3}}{2^i 2^{i/3}} + \frac{2^{4(N-1)/3}}{2^{N-1} 2^{(N-1)/3} \left(1 - \frac{\delta r}{r_0 / 2^{(N-1)/3}}\right)^4} \right)$$

$$\implies \Delta P \simeq (Q_0 + \delta Q_0) \frac{\Delta P}{N Q_0} \left(N - 1 + 1 + \frac{4\,\delta r}{r_{N-1}} \right)$$

Noting that $\frac{\delta Q_{N-1}}{Q_{N-1}} = \frac{\delta Q_0}{Q_0}$, we obtain

$$1 + \frac{\delta Q_{N-1}}{Q_{N-1}} = \frac{1}{1 + \frac{4\,\delta r}{N r_{N-1}}} \simeq 1 - \frac{4\,\delta r}{N r_{N-1}} \,.$$

And so:

$$\frac{\delta Q_{N-1}}{Q_{N-1}} \simeq -\frac{4}{N} \frac{\delta r}{r_{N-1}} \,.$$

Putting all together

B.4 2.3pt

$$\frac{\delta Q_{N-1}}{Q_{N-1}} \simeq -\frac{2}{N} \frac{K_\mathsf{N} v - (1-v) P_\mathsf{cap}}{(1-v)(p_\mathsf{c} - P_\mathsf{cap})} \frac{\delta r_\mathsf{c}}{r_{N-1}} \,.$$

Zero-length springs and slinky coils

A zero effective length spring (ZLS) is a spring for which the force is proportional to the spring's length, $F = kL$ for $L > L_0$ where L_0 is the minimal length of the spring as well as its unstretched length. Figure 1 shows the relation between the force F and the spring length L for a ZLS, where the slope of the line is the spring constant k.

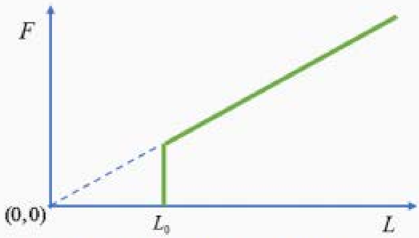


Figure 1: the relation between the force F and the spring length L

A ZLS is useful in seismography and allows very accurate measurement of changes in the gravitational acceleration g. Here, we shall consider a homogenous ZLS, whose weight Mg exceeds kL_0. We define a corresponding dimensionless ratio, $\alpha = kL_0/Mg < 1$ to characterize the relative softness of the spring. The toy known as "slinky" may be (but not necessarily) such a ZLS.

Part A: Statics (3.0 points)

A.1	Consider a segment of length $\Delta\ell$ of the unstretched ZLS spring which is then stretched by a force F, under weightless conditions. What is the length Δy of this segment as a function of F, $\Delta\ell$ and the parameters of the spring?	0.5pt

A.2	For a segment of length $\Delta\ell$, calculate the work ΔW required to stretch it from its original length $\Delta\ell$ to a length Δy.	0.5pt

Throughout this question, we will denote a point on the spring by its distance $0 \leq \ell \leq L_0$ from the bottom of the spring when it is unstretched. In particular, for every point on the spring, ℓ remains unchanged as the spring stretches.

A.3	Suppose that we hang the spring by its top end, so that it stretches under its own weight. What is the total length H of the suspended spring in equilibrium? Express your answers in terms of L_0 and α.	2.0pt

Part B: Dynamics (5.5 points)

Experiments show that when the spring is hung at rest and then released, it gradually contracts from the top, while the lower part remains stationary (see Figure 2). As time advances, the contracting part moves as a solid chunk and accumulates additional turns of the spring, while the stationary part becomes shorter. Every point on the spring begins to move only when the moving part reaches it. The bottom end of the spring starts moving only when the spring is fully collapsed and reaches its unstretched length L_0. After that, the contracted spring continues falling straight downwards, without tumbling, as a rigid body under the influence of gravity.

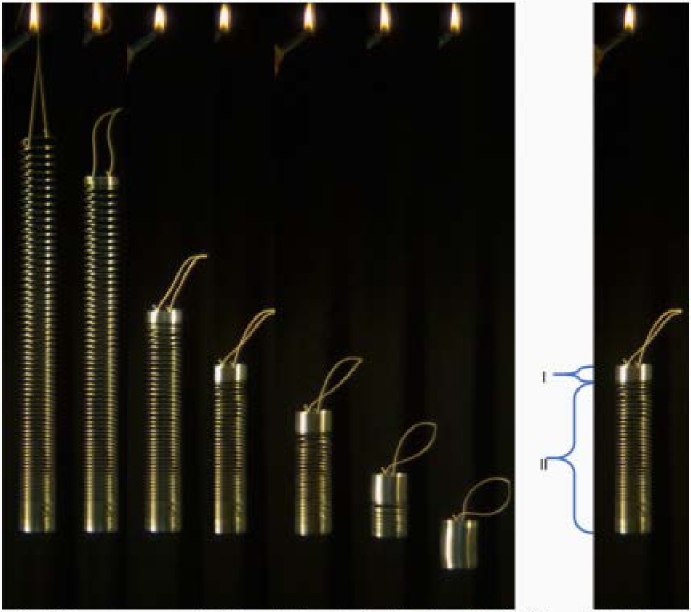


Figure 2: Left: a sequence of pictures taken during the free fall of slinky. Right: the moving part I and the stationary part II during the free fall of the spring.

In the remaining parts of the question, you are asked to base your solution on this described model. You may neglect air resistance, but you are not allowed to neglect L_0.

B.1	Calculate the time t_c it takes from the moment the spring is released, until it fully collapses back to its minimal length L_0. Express your answer in terms of L_0, g and α. Compute the numerical value of t_c for a spring with $k = 1.02\ \mathrm{N/m}$, $L_0 = 0.055\,\mathrm{m}$ and $M = 0.201\ \mathrm{kg}$, while taking g to be $9.80\ \mathrm{m/s^2}$.	2.5pt

B.2	In this task ℓ is used to denote the coordinate of the boundary between parts I (in figure 2, the moving part) and II (the stationary part). At a certain moment, while a stationary part still exists its mass is $m(\ell) = \frac{\ell}{L_0}M$, and the moving part moves with uniform instantaneous velocity $v_I(\ell)$. Show that at this moment (while there exists a stationary part) the velocity of the moving part is $v_I(\ell) = \sqrt{A\ell + B}$. Express the constants A and B in terms of L_0, g and α.	2.5pt

B.3	Based on B.2, find the minimum speed $v_{\min}$ of the moving part of the spring in the course of its motion, after its release and before it hits the ground. Express your answer in terms of L_0, α, A and B.	0.5pt

Part C: Energetics (1.5 points)

C.1	Calculate the amount of mechanical energy Q that was lost by generating heat, from the moment the spring is released until just before the spring hits the ground. Express your answer in terms of L_0, M, g and α.	1.5pt

Zero-length springs and slinky coils – Solution

Part A: Statics

A.1 The force F causes the spring to change its length from L_0 to L. Since equal parts of the spring are extended to equal lengths, we get: $\frac{\Delta y}{\Delta l} = \frac{L}{L_0} \to \Delta y = \frac{L}{L_0}\Delta l$.

Since $L = \max\left\{\frac{F}{k}, L_0\right\}$, we get $\Delta y = \max\{\frac{F}{kL_0}\Delta l, \Delta l\}$. From this result we see that any piece of length Δl the spring behaves as a ZLS with spring constant $k^* = k\frac{L_0}{\Delta l}$.

A.2 Let us compute the work of the force. From Task A.1: $dW = F(x)dx = \frac{kL_0}{\Delta l}xdx$.

Hence, $\Delta W = \int_{\Delta l}^{\Delta y}\frac{kL_0}{\Delta l}xdx = \frac{kL_0}{\Delta l}\frac{x^2}{2}\Big|_{\Delta l}^{\Delta y} = \frac{kL_0}{2\Delta l}(\Delta y^2 - \Delta l^2)$.

A.3. At every point along the statically hanging spring the weight of the mass below is balanced by the tension from above. This implies that at the bottom of the spring there is a section of length l_0 whose turns are still touching each other, as their weight is insufficient to exceed the threshold force kL_0 to pull them apart. The length l_0 can be derived from the equation: $\frac{l_0}{L_0}Mg = kL_0$, hence $l_0 = \frac{kL_0^2}{Mg} = \alpha L_0$.

For $l > l_0$, a segment of the unstretched spring between l and $l + dl$ feels a weight of $\frac{l}{L_0}Mg$ from beneath, which causes its length to stretch from dl to $dy = \frac{F}{kL_0}dl = \frac{l}{L_0}Mg\frac{dl}{kL_0} = \frac{Mg}{kL_0^2}ldl = \frac{l}{l_0}dl$.

Integration of the last expression over the stretched region, up to the point L_0, gives its height when the spring is stretched

$$H = l_0 + \int_{l_0}^{L_0}\frac{l}{l_0}dl = l_0 + \frac{l^2}{2l_0}\Big|_{l_0}^{L_0} = l_0 + \frac{1}{2l_0}(L_0^2 - l_0^2) = \frac{L_0^2}{2l_0} + \frac{l_0}{2} = \frac{L_0}{2}\left(\alpha + \frac{1}{\alpha}\right)$$

Part B: Dynamics

B.1. From Task A.3 we have $H(l)=\frac{l^2}{2l_0}+\frac{l_0}{2}$. We now calculate the position of the center of mass of the suspended spring. The contribution of the unstretched section of height l_0 at the bottom, having a mass of $\frac{l_0}{L_0}M=\alpha M$, is $\alpha M\frac{l_0}{2}$. The position of the center of mass is obtained by summing the contributions of its elements:

$$H_{cm}=\frac{1}{M}\left[\frac{l_0}{2}\alpha M+\int_{l_0}^{L_0}H(l)dm\right]=\frac{1}{M}\left[\frac{\alpha L_0}{2}\alpha M+\int_{l_0}^{L_0}\left(\frac{l^2}{2l_0}+\frac{l_0}{2}\right)\frac{M\,dl}{L_0}\right]$$
$$=\frac{\alpha^2L_0}{2}+\frac{1}{L_0}\left[\frac{l^3}{6l_0}+\frac{l_0}{2}l\right]_{l_0}^{L_0}=\frac{\alpha^2L_0}{2}+\frac{1}{L_0}\left[\frac{L_0^3-l_0^3}{6l_0}+\frac{l_0}{2}(L_0-l_0)\right]$$

Where we have used $dm=\frac{dl}{L_0}M$. Substituting $l_0=\alpha L_0$ yields

$$H_{cm}=L_0\left[\frac{1}{6\alpha}-\frac{\alpha^2}{6}+\frac{\alpha}{2}\right]$$

When the spring is contracted to its free length L_0, its center of mass is located at $\frac{L_0}{2}$. From the falling of the center of mass at acceleration g we get:

$$\frac{g}{2}t_c^2=H_{cm}-\frac{L_0}{2}=L_0\left[\frac{1}{6\alpha}-\frac{\alpha^2}{6}+\frac{\alpha}{2}-\frac{1}{2}\right]=\frac{L_0}{6\alpha}(1-\alpha)^3$$

Hence, $t_c=\sqrt{\frac{L_0}{3g\alpha}(1-\alpha)^3}$.

For $k=1.02$ N/m, $L_0=0.055$ m, $M=0.201$ kg, and $g=9.80$ m/s^2, we have $\alpha=0.0285$, and $t_c=0.245$ s.

B.2. The moving top section of the spring is pulled down by its own weight, $m_{top}g=Mg\frac{(L_0-l)}{L_0}$ and also by the tension in the spring below, which is equal to the weight Mgl/L_0 of the stationary section of the spring. Thus, the moving top section experiences a constant force $F=Mg$ throughout its whole fall. Another way to see that, is that a total force of Mg is exerted on the spring, but only the moving part experiences it. Let's calculate the position of the center of mass at equilibrium of the upper part, i.e., all points with $l'>l$ for some $l>l_0$. From part A, the position of a small portion $\Delta l'$ with coordinate l' is: $H(l') = \frac{l'^2}{2l_0} + \frac{l_0}{2}$ and the center of mass of this part is:

$$H_{cm-upper-i} = \frac{L_0}{M(L_0 - l)}\int_l^{L_0} H(l')dm = \frac{L_0}{M(L_0 - l)}\int_l^{L_0} \left(\frac{l'^2}{2l_0} + \frac{l_0}{2}\right)dm$$
$$= \frac{L_0}{M(L_0 - l)}\int_l^{L_0} \left(\frac{l'^2}{2l_0} + \frac{l_0}{2}\right)\frac{Mdl'}{L_0} = \frac{1}{(L_0 - l)}\int_l^{L_0} \left(\frac{l'^2}{2l_0} + \frac{l_0}{2}\right)dl'$$
$$= \frac{1}{(L_0 - l)}\left[\frac{l'^3}{6l_0} + \frac{l_0 l'}{2}\right]_l^{L\_0} = \frac{L_0^2 + L_0 l + l^2}{6l_0} + \frac{l_0}{2}$$

The position of the upper part of CM when it contracts to a length $L_0 - l$ is $H_{cm-upper-f} = \frac{l^2}{2l_0} + \frac{l_0}{2} + \frac{1}{2}(L_0 - l)$. The change in the CM during the contraction process is: $\Delta H_{cm-upper} = H_{cm-upper-i} - H_{cm-upper-f} = \frac{L_0^2 + L_0 l - 2l^2}{6l_0} - \frac{1}{2}(L_0 - l) = \frac{(L_0 - l)(L_0 + 2l)}{6l_0} - \frac{1}{2}(L_0 - l)$.

The acceleration of the CM of the upper part is $a_{CM} = \frac{FL_0}{M(L_0 - l)} = \frac{gL_0}{L_0 - l}$.
From the work energy theorem we get the equation $v_{upper-f}^2 = 2a_{CM}\Delta H_{cm-upper}$, hence

$$v_{upper-f}^2 = 2\frac{gL_0}{L_0 - l}\left[\frac{(L_0 - l)(L_0 + 2l)}{6\alpha L_0} - \frac{1}{2}(L_0 - l)\right] = 2g\left[\frac{L_0 + 2l}{6\alpha} - \frac{1}{2}L_0\right]$$
$$= \frac{2g}{3\alpha}l + \left(\frac{1}{3\alpha} - 1\right)gL_0$$

Therefore, $A = \frac{2g}{3\alpha}$ and $B = \left(\frac{1}{3\alpha} - 1\right)gL_0$.

Note that for $l = L_0$, we have $v_{upper-f}^2 = L_0 g\,\frac{1-\alpha}{\alpha}$ and for $l = l_0 = \alpha L_0$, we get $v_{upper-f}^2 = L_0 g\,\frac{1-\alpha}{3\alpha}$, hence, the moment we release the spring its velocity is finite (not zero, the meaning is that it accumulate this velocity in time that is much shorter than the contracting time t_c) and it decreases to $\frac{1}{\sqrt{3}}$ of the initial value when $l = l_0$.

B.3. Note that even though the center of mass of the spring accelerates downwards constantly, the moving top section actually decelerates, while the position of the center of mass moves down the spring. The speed of the top section $v(l)$, calculated in Task B2, decreases and approaches the value $\sqrt{A\alpha L_0 + B}$ immediately before it attaches to the bottom section of height $l_0 = \alpha L_0$, which was unstretched and at rest. Once the moving top section attaches to the resting bottom section, its momentum is shared between both sections, so the speed further decreases just before the whole spring starts accelerating downwards as a single mass. Thus, the minimum speed is that of the whole spring immediately after its full collapse. From momentum conservation, we have

$$Mv_{min} = m_{top}v(l_0) = M\left(1 - \frac{l_0}{L_0}\right)\sqrt{A\alpha L_0 + B}$$

$$v_{min} = (1-\alpha)\sqrt{A\alpha L_0 + B}$$

Part C: Energetics

C.1. From the moment the spring is released, the acceleration of its center of mass is governed by the external force Mg and therefore the gravitational potential energy of the spring is fully converted into the kinetic energy of the center of mass of the spring, which just before hitting the ground is equal to the kinetic energy of the spring.

All that is left is the elastic energy stored in the spring, which is converted into heat, sound, etc. To calculate it, we consider the elastic energy stored in a segment dh of the stretched spring, which when unstretched lies between l and $l + \mathrm{d}l$, using the result of Task A.2, $\Delta W = \frac{kL_0}{2\Delta l}(\Delta l_2^2 - \Delta l^2)$, by choosing $\Delta l = dl$ and $\Delta l_2 = dy$, and using $dy = \frac{l}{l_0}dl$ (which was obtained in Task A.3), we get:

$dW = \frac{kL_0}{2}\left(\frac{l^2}{l_0^2} - 1\right)dl$. Integrating from l_0 to L_0 we find

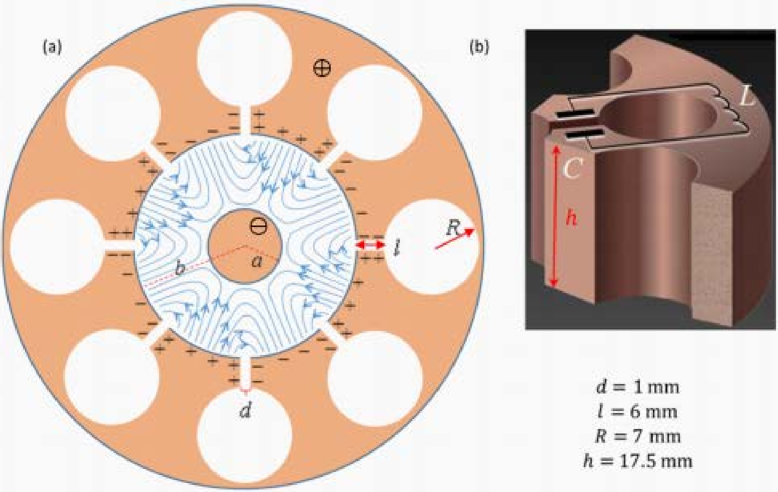
$$W = \int_{l_0}^{L_0} \frac{kL_0}{2}\left(\frac{l^2}{l_0^2} - 1\right)dl = \frac{kL_0}{2}\left[\frac{l^3}{3l_0^2} - l\right]_{l_0}^{L_0} = \frac{kL_0}{2}\left(\frac{L_0^3 - l_0^3}{3l_0^2} - (L_0 - l_0)\right)$$

$$= \frac{kL_0^2}{2}\left(\frac{1-\alpha^3}{3\alpha^2} - (1-\alpha)\right) = \frac{kL_0^2}{6\alpha^2}(1-\alpha)^2(2\alpha+1)$$

$$= MgL_0\frac{(1-\alpha)^2(2\alpha+1)}{6\alpha}$$

The Physics of a Microwave Oven

This question discusses the generation of microwave radiation in a microwave oven, and its use to heat up food. The microwave radiation is generated in a device called "magnetron". Part A concerns the operation of the magnetron, while part B deals with the absorption of microwave radiation in food.



$d = 1\,\text{mm}$
$l = 6\,\text{mm}$
$R = 7\,\text{mm}$
$h = 17.5\,\text{mm}$

Figure 1

Part A: The structure and operation of a magnetron (6.6 points)

A magnetron is a device for the generation of microwave radiation, either in pulses (for radar applications), or continuously (e.g., in a microwave oven). The magnetron has a mode of self-amplifying oscillations. Supplying the magnetron with static (non-alternating) voltage quickly excites this mode. The microwave radiation thus created is transmitted out of the magnetron.

A typical microwave oven magnetron consists of a solid copper cylindrical cathode (with radius a) and a surrounding anode (with radius b). The latter has the shape of a thick cylindrical shell into which cylindrical cavities are drilled. These cavities are known as "resonators". One of the resonators is coupled to an antenna which will transmit the microwave energy out; we will ignore the antenna in the following. All internal spaces are in vacuum. We will consider a typical magnetron with eight resonators, as depicted in Figure 1(a). The three-dimensional structure of a single resonator is shown in Figure 1(b). As indicated there, each of the eight cavities behaves as an inductor-capacitor (LC) resonator, with operating frequency $f = 2.45\,\text{GHz}$.

A static uniform magnetic field is applied along the magnetron's longitudinal axis, pointing out of the page in Figure 1(a). In addition, a constant voltage is applied between the anode (positive potential) and the cathode (negative potential). Electrons emitted from the cathode reach the anode and charge it, such that they excite an oscillation mode in which the sign of the charge is opposite between every two adjacent resonators. The oscillation of the cavities amplify these oscillations.

The process described above creates an alternating electric field with the aforementioned frequency $f = 2.45\ \mathrm{GHz}$ (blue lines in Figure 1(a); the static field is not plotted) in the space between the cathode and the anode, in addition to the static field caused by the applied constant voltage. In the steady state, the typical amplitude of the alternating electric field between the anode and the cathode is approximately $\frac{1}{3}$ of the static electric field there. The electron motion in the space between the cathode and the anode is affected by both the static and the alternating parts of the field. This causes electrons that reach the anode to transfer about 80% of the energy they acquire from the static field into the alternating field. A minority of the ejected electrons returns to the cathode and releases additional electrons, further amplifying the alternating field.

Each resonator can be thought of as a capacitor and an inductor, see Figure 1(b). The capacitance mainly arises from the planar parts of the resonator surface, while the inductance stems from the cylindrical part. Assume that the current in the resonator flows uniformly very close to the surface of its cylindrical cavity, and that the strength of the magnetic field generated by this current is 0.6 times that of an ideal infinite solenoid. The various lengths defining the resonator geometry are given in Figure 1(b). The vacuum permittivity and permeability are $\varepsilon_0 = 8.85 \cdot 10^{-12}\,\frac{\mathrm{F}}{\mathrm{m}}$ and $\mu_0 = 4\pi \cdot 10^{-7}\,\frac{\mathrm{H}}{\mathrm{m}}$, respectively.

A.1	Use the above data to estimate the frequency f_{est} of a single resonator. (Your result may differ from the actual value, $f = 2.45\,\mathrm{GHz}$. Use the **actual** value in the remainder of the question.)	0.4pt

Task A.2 below does not deal with the magnetron itself, but helps to introduce some of the relevant physics. Consider an electron moving in free space under the influence of a uniform electric field directed along the negative y axis, $\overline{E} = -E_0\hat{y}$, and a uniform magnetic field directed along the positive z axis, $\overline{B} = B_0\hat{z}$ (E_0 and B_0 are positive; $\hat{x}, \hat{y}, \hat{z}$ are unit vectors oriented in the conventional manner). Let us denote the electron velocity at time t by $\vec{u}(t)$. The drift velocity $\vec{u}_D$ of the electron is defined as its average velocity. We denote by m and $-e$ the mass and charge of the electron, respectively.

A.2	In each of the following two cases, find $\vec{u}_D$. In addition, draw in the Answer Sheet the electron's trajectory (in the lab frame) during the time interval $0 < t < \frac{4\pi m}{eB_0}$ if: 1. at $t = 0$ the electron velocity is $\vec{u}(0) = (3E_0/B_0)\hat{x}$, 2. at $t = 0$ the electron velocity is $\vec{u}(0) = -(3E_0/B_0)\hat{x}$.	1.5pt

We now resume our discussion of the magnetron. The distance between the cathode and the anode is $15\,\mathrm{mm}$. Assume that, due to the aforementioned energy loss to the alternating fields, the maximal kinetic energy of each electron does not exceed $K_{\max} = 800\,\mathrm{eV}$. The static magnetic field strength is $B_0 = 0.3\,\mathrm{T}$. The electron mass and charge are $m = 9.1 \cdot 10^{-31}\,\mathrm{kg}$ and $-e = -1.6 \cdot 10^{-19}\,\mathrm{C}$, respectively.

A.3	Numerically estimate the maximal radius r of the electron motion trajectory in the reference frame in which this motion is approximately circular, considering this reference frame as approximately inertial.	0.4pt

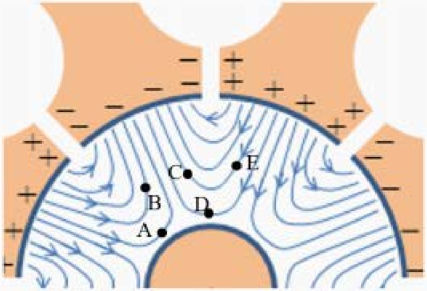


Figure 2

A.4	Figure 2 depicts the alternating electric field lines between the anode and the cathode at a given moment in time (the static field is not plotted). Indicate in the Answer Sheet which of the electrons positioned at A,B,C,D and E will drift towards the anode, which will drift towards the cathode and which will drift at a direction perpendicular to the radius at that moment.	1.2pt

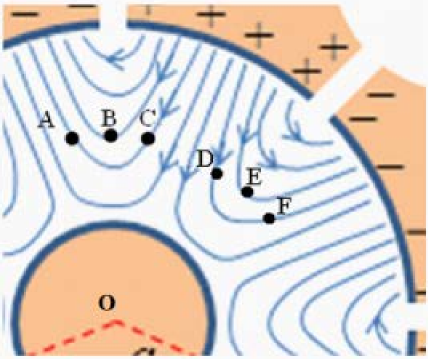


Figure 3

Figure 3 depicts the alternating electric field lines between the anode and the cathode (the static field is not plotted) at a given moment in time. The positions of six electrons at that moment are denoted by A, B, C, D, E and F. All electrons are at the same distance from the cathode.

A.5	Consider the situation shown in Figure 3. For each of the six electron pairs AB, AC, BC, DE, DF, EF, indicate in the Answer Sheet whether their drift will cause the angle between their position vectors (measured from the cathode's center O) to increase or decrease at that moment.	1.2pt

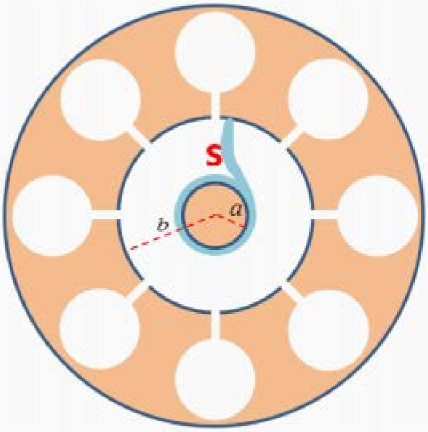


Figure 4

The pattern you have discovered in Task A.5 acts as a focusing mechanism, concentrating the electrons in the space between the cathode and anode into spokes. Figure 4 depicts one such spoke, denoted by S.

A.6	Depict in the Answer Sheet the other spokes at that moment. Indicate by arrows their direction of rotation, and calculate their average angular velocity ω_s.	0.8pt

Make the approximation that the total electric field half-way between the cathode and the anode is equal to its average static value along a radial line from the cathode to the anode, and that the spokes are approximately radial in that region. The cathode and anode radii (a and b, respectively) are defined in Figure 4.

A.7	Find an approximate expression for the static voltage V_0 required for operating the magnetron in the manner described. (The expression you will find gives an approximation for the minimal value required for the magnetron operation; the optimal voltage is somewhat higher.)	1.1pt

Part B: The interaction of microwave radiation with water molecules (3.4 points)

This part deals with the usage of microwave radiation (radiated by the magnetron antenna into the food chamber) for cooking, that is, heating up a lossy dielectric material such as water, either pure or salty (which is our model for, say, soup).

An electric dipole is a configuration of two equal and opposite electric charges q and $-q$ a small distance d apart. The electric dipole vector points from the negative to the positive charge, and its magnitude is $p = qd$.

A time-dependent electric field $\vec{E}(t) = E(t)\hat{x}$ is applied on a single dipole of moment $\vec{p}(t)$ with constant magnitude $p_0 = |\vec{p}(t)|$. The angle between the dipole and the electric field is $\theta(t)$.

B.1	Write expressions for both the magnitude of the torque $\tau(t)$ applied by the electric field on the dipole and the power $H_i(t)$ delivered by the field to the dipole, in terms of p_0, $E(t)$, $\theta(t)$ and their derivatives.	0.5pt

Water molecules are polar, hence can be treated as electric dipoles. Due to the strong hydrogen bonds between water molecules in liquid water, one cannot treat them as independent dipoles. Rather, one should refer to the polarization vector $\vec{P}(t)$, which is the dipole moment density (average dipole moment per unit volume of an ensemble of water molecules). The polarization $\vec{P}(t)$ is parallel to the local applied alternating electric field (of the microwave radiation), $\vec{E}(t)$, and oscillates in time with an amplitude that is proportional to the amplitude of the local alternating electric field, but with a phase lag δ.

The local alternating electric field at a given location inside the water is $\vec{E}(t) = E_0 \sin(\omega t)\hat{x}$, where $\omega = 2\pi f$, giving rise to polarization $\vec{P}(t) = \beta\varepsilon_0 E_0 \sin(\omega t - \delta)\hat{x}$, where the dimensionless constant β is a property of water.

B.2 Find an expression for the time-averaged power $\langle H(t)\rangle$ per unit volume absorbed by the water. (0.5pt)

The time-average for a time dependent periodic variable $f(t)$ over its period T is defined as:

$$\langle f(t)\rangle = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)\,\mathrm{d}t. \tag{1}$$

Let us now consider the propagation of the radiation through the water. The relative dielectric constant of water (at the electromagnetic field frequency) is ε_r, and the corresponding index of refraction of water is $n = \sqrt{\varepsilon_r}$. The momentary energy density of the electric field is given by $\frac{1}{2}\varepsilon_r\varepsilon_0 E^2$. The time-averaged energy density of the electric and magnetic fields are equal.

B.3	Let us denote the time-averaged radiation energy flux density by $I(z)$ (average radiation power flow per unit area). Here z is the depth of penetration into the water, and the radiation propagates in the z direction. Find an expression for the dependence of the flux density $I(z)$ on z. The flux density at the water surface, $I(0)$, may appear in your result.	1.1pt

The phase lag δ is the result of the interaction between the water molecules. It depends on the dimensionless dielectric loss coefficient ε_ℓ and the relative dielectric constant ε_r (both of which depend on the radiation angular frequency ω and the temperature) via the relation $\tan\delta = \varepsilon_\ell/\varepsilon_r$. When δ is small enough, the electric field at penetration depth z into the water is given by:

$$\vec{E}(z,t) = \vec{E}_0 e^{-\frac{1}{2}nk_0 z\tan\delta}\sin(nk_0 z - \omega t) \tag{2}$$

where $k_0 = \omega/c$ and $c = 3.0\cdot 10^8\,\frac{\mathrm{m}}{\mathrm{s}}$ is the speed of light in vacuum.

B.4	Employ the approximation $\tan\delta \approx \sin\delta$ and find an expression for the coefficient β defined in Task B.2 in terms of the other parameters.	0.6pt

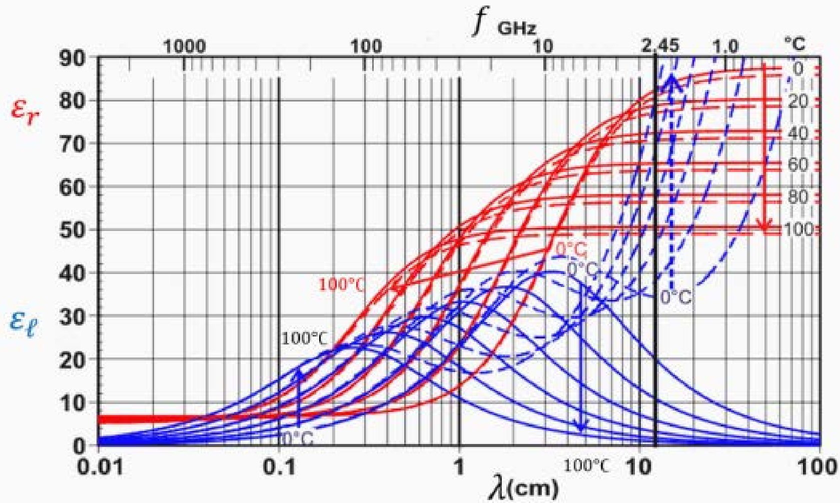


Figure 5. The arrows indicate the variation with temperature across the curves from $0°C$ to $100°C$.

Figure 5 depicts ε_ℓ (blue) and ε_r (red) for both pure water (solid lines) and a dilute solution of salt in water (dashed lines) as functions of wavelength or frequency, at several different temperatures. The angular frequency $\omega = 2\pi \cdot 2.45 \cdot 10^9\,\mathrm{s}^{-1}$ is indicated by a bold vertical line. Below we will consider microwave radiation at this frequency only.

B.5	Use Figure 5 to address the following questions: 1. For water at $20°C$, find the penetration depth $z_{1/2}$ at which the power per unit volume is reduced to half of its value at $z = 0$. 2. Indicate in the Answer Sheet whether the penetration depth of the microwave radiation into water increases, decreases or remains the same with temperature. 3. Indicate in the Answer Sheet whether the penetration depth of the microwave radiation into soup (dilute salt solution) increases, decreases or remains the same with temperature.	0.7pt

The Physics of a Microwave Oven – Solution

Part A: The structure and operation of a magnetron

A.1. The frequency of an LC circuit is $f = \omega/2\pi = 1/(2\pi\sqrt{LC})$. If the total electric current flowing along the boundary of the cavity is I, it generates a magnetic field whose magnitude (by the assumptions of the question) is $0.6\mu_0 I/h$, and a total magnetic flux equal to $\pi R^2 \times 0.6\mu_0 I/h$, hence the inductance of the resonator is $L = 0.6\pi\mu_0 R^2/h$. Approximating the capacitor as a plate capacitor, its capacitance is $C = \varepsilon_0 lh/d$. Putting everything together, we find

$$f_{est} = \frac{1}{2\pi}\frac{1}{\sqrt{LC}} = \frac{1}{2\pi}\sqrt{\frac{h}{0.6\pi R^2\mu_0}\frac{d}{\varepsilon_0 lh}} = \frac{1}{2\pi}\frac{c}{R}\sqrt{\frac{d}{0.6\pi l}} = \frac{1}{2\pi}\frac{3\cdot 10^8}{7\cdot 10^{-3}}\sqrt{\frac{1}{3.6\pi}} = 2.0\cdot 10^9$$

Hz

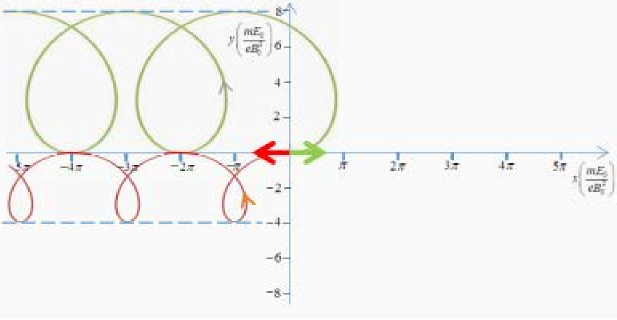
A.2. Denoting the electron velocity by $\vec{u}(t)$, in this case the total force applied on it is

$$\vec{F} = -e(-E_0\hat{y} + \vec{u}(t) \times B_0\hat{z}).$$

Let us write $\vec{u}(t) = \vec{u}_D + \vec{u}'(t)$, with $\vec{u}_D = (-E_0/B_0)\hat{x}$ being the drift velocity of a charged particle in the crossed electric and magnetic fields (the velocity at which the electric and magnetic forces cancel each other exactly). Then $\vec{F} = -e\vec{u}'(t) \times B_0\hat{z}$. Thus, in a frame moving at the drift velocity $\vec{u}_D$, the electron trajectory is a circle with constant-magnitude velocity $u' = |\vec{u}'(t)|$, and radius $r = mu'/eB_0$. In the lab frame this circular motion is superimposed upon the drift at the constant velocity $\vec{u}_D$. Hence:

1. For $\vec{u}(0) = (3E_0/B_0)\hat{x}$ we find $u' = 4E_0/B_0$ and $r = 4mE_0/eB_0^2$.
2. For $\vec{u}(0) = -(3E_0/B_0)\hat{x}$ we find $u' = 2E_0/B_0$ and $r = 2mE_0/eB_0^2$.

This information, together with the independence of the period of the circular motion on u' allows us to plot the electron trajectory in both cases (green and red, for cases 1 and 2, respectively):



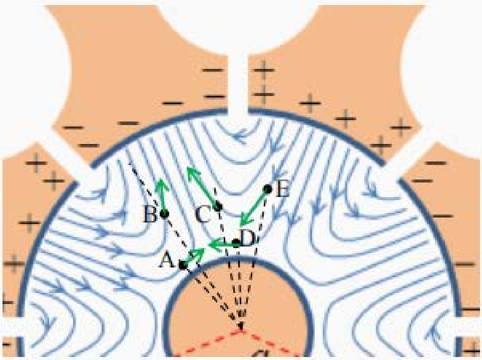
A.3. The velocity of the electron in a frame of reference where the motion is approximately circular is u'. From A.2 we get that $u_D + u' = v_{\max}$ and $u_D - u' = v_{\min}$, hence $u' = (v_{\max} - v_{\min})/2 < v_{\max}$.

The radius of the circular motion of the electron in this frame is $r = mu'/eB_0 < mv_{\max}/eB_0$. The maximal velocity is that corresponding to a kinetic energy, $K_{\max} = mv_{\max}^2/2$, of 800 eV.

Substituting we find $r < \frac{m}{eB}\sqrt{\frac{2eV}{m}} = \frac{1}{B}\sqrt{\frac{2mV}{e}} = \frac{1}{0.3}\sqrt{\frac{2\cdot 9.1\cdot 10^{-31}\cdot 800}{1.6\cdot 10^{-19}}} = 3.18\cdot 10^{-4}\ \mathrm{m} \approx 0.3\ \mathrm{mm}$.

Since this maximal radius is much smaller than the distance between the anode and the cathode, we may ignore the circular component of the electronic motion, and approximate it as pure drift.

A.4. As just explained, we may approximate the electron motion as pure drift. In task A.2 we have found that the direction of the drift velocity $\vec{u}_D$ is in the direction of the vector $\vec{E}\times\vec{B}$. Since we are interested in radial component of the drift velocity, the only contribution is from the azimuthal component of the electric field. The static electric field has no azimuthal component, hence the drift in the radial direction results solely from the azimuthal component of the alternating electric field. What we have to check is if the azimuthal component points clockwise or counterclockwise. From the direction of the field lines it is easy to see (attached figure) that in points A and B the azimuthal component pointing clockwise therefore the electrons there drift towards the cathode, while for points C, D and E the azimuthal component points counterclockwise and the electrons there drift toward the anode.



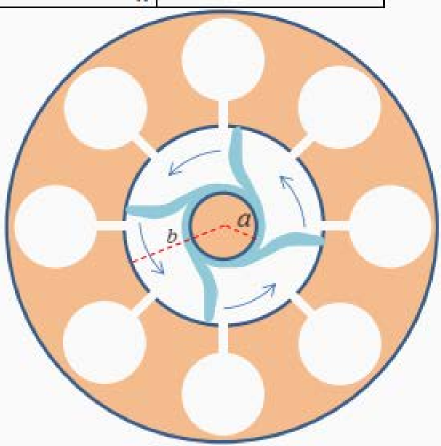
Point	toward the anode	toward the cathode	perpendicular to the radius
A		X	
B		X	

C	X		
D	X		
E	X		

A.5. In this task we need to consider the azimuthal component of the drift velocity, which results from the radial component of the electric field. Since all points are at the same distance from the anode, all electrons experience the same static electric field. Hence only the radial component of the alternating field determines whether the angle between the electrons' position vectors would increase or decrease: If the radial component of the alternating field points inwards (towards the cathode), the azimuthal drift velocity will be positive (counterclockwise) and vice versa. Hence the electrons at A, B and C drift closer to each other in terms of angles, while those at D, E and F drift away from each other.

points	angle decreases	angle increases	indeterminate
AB	X		
BC	X		
CA	X		
DE		X	
EF		X	
DF		X	

A.6. Spokes will be created only in the regions where focusing occurs. By the result of the previous task, there are four spokes, as indicated in the attached Figure.



The electron drift sets the spokes in a counterclockwise rotation. The frequency of the alternating field is $f = 2.45$ GHz. By the time the alternating field flipped its sign (half a period), each spoke moves to the next cavity, corresponding to an angle of $\pi/4$. Therefore, the angular velocity of each spoke is

$\omega = \frac{\pi}{4}/\frac{T}{2} = \frac{\pi}{2}f = 3.85 \cdot 10^9 rad/s$. Each spoke performs a full rotation around the magnetron after four periods of the alternating field.

A.7. The magnitude of the electric field in the region considered, $r = (b+a)/2$, is the magnitude of the static field, that is, $E = V_0/(b-a)$, giving rise to an azimuthal drift velocity of magnitude $u_D = E/B_0 = V_0/[B_0(b-a)]$. Equating u_D/r with the angular velocity found in the previous task we find $V_0 = \pi f B_0 (b^2 - a^2)/4$

Part B: The interaction of microwave radiation with water molecules

B.1. The torque at time t is given by $\tau(t) = -qd\sin[\theta(t)]E(t) = -p_0\sin[\theta(t)]E(t)$, hence the instantaneous power delivered to the dipole by the electric field is

$$H_i(t) = \tau(t)\dot{\theta}(t) = -p_0 E(t)\sin\theta(t)\dot{\theta}(t) = E(t)\frac{d}{dt}\left(p_0\cos\theta(t)\right) = E(t)\frac{dp_x(t)}{dt}$$

B.2. Since the average dipole density (hence the average of each molecular dipole) is parallel to the field, the absorbed power density is (angular brackets, $\langle\cdots\rangle$, denote average over time)

$$\langle H(t)\rangle = \left\langle E_0\sin(\omega_f t)\frac{dP_x}{dt}\right\rangle = \left\langle E_0\sin(\omega_f t)\frac{d}{dt}\left(\beta\varepsilon_0 E_0\sin(\omega_f t-\delta)\right)\right\rangle =$$

$$E_0^2\beta\varepsilon_0\omega_f\left\langle\sin(\omega_f t)\cos(\omega_f t-\delta)\right\rangle = 0.5E_0^2\beta\varepsilon_0\omega_f\left\langle\sin\delta+\sin(2\omega_f t-\delta)\right\rangle = 0.5E_0^2\beta\varepsilon_0\omega_f\sin\delta$$

B.3. The energy density of the electromagnetic field at penetration depth z, which is twice the electric energy density, is $2\times\varepsilon_r\varepsilon_0\langle E^2(z,t)\rangle/2 = \varepsilon_r\varepsilon_0 E_0^2(z)\langle\sin^2(\omega t)\rangle = \varepsilon_r\varepsilon_0 E_0^2(z)/2$. Therefore, the time-averaged flux density at depth z is:

$$I(z) = \frac{1}{2}\varepsilon_r\varepsilon_0 E_0^2(z)\times\frac{c}{n} = \frac{1}{2}\sqrt{\varepsilon_r}\varepsilon_0 c E_0^2(z),$$

where c is the speed of light in vacuum. I decreases with z due to the absorbed power calculated in the previous task we find

$$\frac{dI(z)}{dz} = -\frac{1}{2}\beta\varepsilon_0\omega E_0^2(z)\sin\delta = -\frac{\beta\omega\sin\delta}{c\sqrt{\varepsilon_r}}I(z),$$

hence $I(z) = I(0)\exp[-z\beta\omega\sin\delta/(c\sqrt{\varepsilon_r})]$.

B.4. Similarly to the previous task, the energy flux corresponding to the given field is

$$I(z) = \sqrt{\varepsilon_r}\varepsilon_0 c\langle E^2(z,t)\rangle = \frac{1}{2}\sqrt{\varepsilon_r}\varepsilon_0 c E_0^2 e^{-z\omega\sqrt{\varepsilon_r}\tan\delta/c}.$$

Equating the argument of the exponent in the last expression with the result of the previous task, and using the given approximation $\tan\delta\approx\sin\delta$ leads to $\beta = \varepsilon_r$.

B.5.

1. Using previous results, the radiation power per unit area is reduced to half of its $z = 0$ value at $z_{1/2} = c \ln 2 / (\omega\sqrt{\varepsilon_r} \tan\delta) = c\sqrt{\varepsilon_r} \ln 2 / (\omega\varepsilon_l)$. From the given graph, at the given frequency $\varepsilon_r \approx 78$ and $\varepsilon_l \approx 10$, hence $z_{1/2} \approx 12$ mm.

We have just found that the penetration depth is proportional to $\sqrt{\varepsilon_r}/\varepsilon_l$. From the given graph we thus find that:

2. Heating up pure water (continuous lines) decreases ε_l much more significantly than the corresponding decrease of $\sqrt{\varepsilon_r}$ at the given frequency. Thus, the penetration depth of pure water increases with temperature, allowing deeper penetration of the microwave radiation and heating up the water inner regions.

3. On the contrary, for a soup (dilute salt solution, dashed lines) ε_l at the given frequency increases with temperature while ε_r decreases. Thus, the absorption rate increases with temperature, the penetration depth decreases, and less microwave radiation reaches its inner regions.

material	$z_{1/2}$ increases with temp.	$z_{1/2}$ decreases with temp.	$z_{1/2}$ remains the same
water	X		
soup		X	

Thermoacoustic Engine

A thermoacoustic engine is a device that converts heat into acoustic power, or sound waves - a form of mechanical work. Like many other heat machines, it can be operated in reverse to become a refrigerator, using sound to pump heat from a cold to a hot reservoir. The high operating frequencies reduce heat conduction and eliminate the need for any working chamber confinement. Unlike many other engine types, the thermoacoustic engine has no moving parts except the working fluid itself.

The efficiencies of thermoacoustic machines are typically lower than other engine types, but they have advantages in set up and maintenance costs. This creates opportunities for renewable energy applications, such as solar-thermal power plants and utilization of waste heat. Our analysis will focus on the creation of acoustic energy within the system, ignoring the extraction or conversion for powering external devices.

Part A: Sound wave in a closed tube (3.7 points)

Consider a thermally insulating tube of length L and cross-sectional area S, whose axis lies along the x direction. The two ends of the tube are located at $x = 0$ and $x = L$. The tube is filled with an ideal gas and is sealed on both ends. At equilibrium, the gas has temperature T_0, pressure p_0 and mass density ρ_0. Assume that viscosity can be ignored and that the gas motion is only in the x direction. The gas properties are uniform in the perpendicular y and z directions.

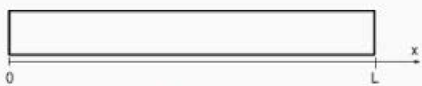


Figure 1

A.1 When a standing sound wave forms, the gas elements oscillate in the x direction with angular frequency ω. The amplitude of the oscillations depends on each element's equilibrium position x along the tube. The longitudinal displacement of each gas element from its equilibrium position x is given by **0.3pt**

$$u(x,t) = a\sin(kx)\cos(\omega t) = u_1(x)\cos(\omega t) \tag{1}$$

(please note the u here describes the displacement of a gas element)

where $a \ll L$ is a positive constant, $k = 2\pi/\lambda$ is the wavenumber and λ is the wavelength. What is the maximum possible wavelength λ_{max} in this system?

We will assume throughout the question an oscillation mode of $\lambda = \lambda_{max}$.

Now, consider a narrow parcel of gas, located at rest between x and $x + \Delta x$ ($\Delta x \ll L$). As a result of the displacement wave of Task A.1, the parcel oscillates along the x axis and undergoes a change in volume and other thermodynamic properties.

Throughout the following tasks assume all these changes to the thermodynamic properties to be small compared to the unperturbed values.

A.2 The parcel volume $V(x,t)$ oscillates around the equilibrium value of $V_0 = S\Delta x$ and has the form **0.5pt**

$$V(x,t) = V_0 + V_1(x)\cos(\omega t). \tag{2}$$

Obtain an expression for $V_1(x)$ in terms of V_0, a, k and x.

A.3 Assume that the total pressure of the gas, as a result of the sound wave, takes the approximate form **0.7pt**

$$p(x,t) = p_0 - p_1(x)\cos(\omega t). \tag{3}$$

Considering the forces acting on the parcel of gas, compute the amplitude $p_1(x)$ of the pressure oscillation to leading order, in terms of the position x, the equilibrium density ρ_0, the displacement amplitude a and the wave parameters k and ω.

At acoustic frequencies, the thermal conductivity of the gas can be neglected. We will treat the expansion and contraction of gas parcels as purely adiabatic, satisfying the relation $pV^{\gamma} = \text{const}$, where γ is the adiabatic constant.

A.4 Use the relation above and the results of the previous tasks to obtain an expression for the speed of sound waves $c = \omega/k$ in the tube, to first order. Express your answer in terms of p_0, ρ_0 and the adiabatic constant γ. **0.3pt**

A.5 The change in the gas temperature due to the adiabatic expansion and contraction, as a result of the sound wave, takes the form: **0.7pt**

$$T(x,t) = T_0 - T_1(x)\cos(\omega t). \tag{4}$$

Compute the amplitude $T_1(x)$ of the temperature oscillations in terms of T_0, γ, a, k and x.

A.6 For the purpose of this task only, we assume a weak thermal interaction between the tube and the gas. As a result, the standing sound wave remains almost unchanged, but the gas can exchange a small amount of heat with the tube. The heating due to viscosity can be neglected. **1.2pt**
For each of the points in Figure 2 (A, C at the edges of the tube, B at the center) state whether the temperature of the tube at that point will increase, decrease or remain the same over a long time.

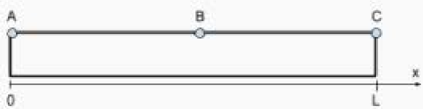


Figure 2

Part B: Sound wave amplification induced by external thermal contact (6.3 points)

A stack of thin well-spaced solid plates is placed inside the tube. The plates of the stack are aligned in parallel to the tube axis, so as not to obstruct the flow of gas along the tube. The center of the stack is positioned at $x_0 = L/4$, and spans a width of $\ell \ll L$ along the tube axis, filling its entire cross section. The right and left edges of the stack are held at temperature difference τ. The left edge of the stack, at $x_H = x_0 - \ell/2$, is held by an external thermal reservoir at temperature $T_H = T_0 + \tau/2$, and at the same time, its right edge, at $x_C = x_0 + \ell/2$, is held at a temperature $T_C = T_0 - \tau/2$.

The plate stack allows a slight longitudinal heat flow to maintain a constant temperature gradient between its edges, such that $T_{\text{plate}}(x) = T_0 - \frac{x - x_0}{\ell}\tau$.

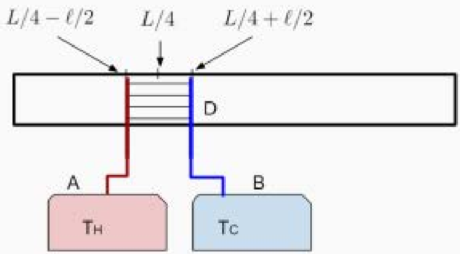


Figure 3. A sketch of the system. (A) and (B) denote the hot and cold heat reservoirs respectively. (D) denotes the stack.

To analyze the effect of the thermal contact between the plate stack and the gas on the sound waves in the tube, make the following assumptions:

- As in the previous part, all changes to the thermodynamic properties are small compared to the unperturbed values.
- The system operates in the fundamental standing-wave mode of the longest possible wavelength. It is only slightly modified by the presence of the plate stack.
- The stack is much shorter than the wavelength $\ell \ll \lambda_{\max}$, and can be positioned far enough from both displacement and pressure nodes, so that the displacement $u(x,t) \approx u(x_0,t)$ and the pressure $p(x,t) \approx p(x_0,t)$ may be considered uniform over the entire length of the stack.
- We may neglect any edge effects, caused by the parcels moving in and out of the stack.
- The temperature difference between the ends of the plate stack, i.e. between the hot and the cold reservoirs, is small compared to the absolute temperature: $\tau \ll T_0$.
- Heat conduction through the stack, through the gas, and along the tube are all negligible. The only significant sources of heat transfer are convection due to the motion of the gas and conduction between the gas and the stack.

B.1	Consider a specific parcel of gas in the region of the stack, originally at $x_0 = L/4$. As the parcel moves within the stack, the local temperature of the nearby part of the stack changes as follows: $$T_{\text{env}}(t) = T_0 - T_{\text{st}}\cos(\omega t). \quad (5)$$ Express T_{st} in terms of a, τ and ℓ.	0.4pt

B.2	Above which critical temperature difference τ_{cr} will the gas be conveying heat from the hot reservoir to the cold one? Express τ_{cr} in terms of T_0, γ, k and ℓ.	1.0pt

B.3	Obtain the general approximate expression for the heat flow $\frac{dQ}{dt}$ into a small parcel of gas as a linear function of its volume and pressure change rates. Express your answer in terms of the rate of volume change $\frac{dV}{dt}$, the rate of pressure change $\frac{dp}{dt}$, the unperturbed equilibrium values of parcel pressure and volume p_0, V_0 and the adiabatic index γ. (You may use the expression for the molar heat capacity at constant volume $c_v = \frac{R}{\gamma - 1}$, where R is the gas constant.)	0.8pt

The limited heat flow rate between the parcel and the stack causes a phase difference between the pressure and volume oscillations of the parcel. We will see how this generates work.

Let the heat flux into the parcel from the stack be proportional to the temperature difference between the parcel and the neighboring element of the stack, given approximately by $\frac{dQ}{dt} = -\beta V_0 (T_{\text{st}} - T_1)\cos(\omega t)$. Here T_1 and T_{st} are the temperature oscillation amplitudes of the gas parcel and the neighbouring stack from Tasks A.5 and B.1, respectively, and $\beta > 0$ is a constant. Assume that at the machine's operating frequencies, the change in gas temperature as a result of this heat flow is insignificant compared to both T_1 and T_{st}.

B.4	In order to calculate work, we will consider a change to the volume of the moving parcel as a result of the thermal contact with the stack. Let us write the pressure and the volume of the parcel under the stack's influence in the form: $$\begin{aligned} p &= p_0 + p_a\sin(\omega t) - p_b\cos(\omega t), \\ V &= V_0 + V_a\sin(\omega t) + V_b\cos(\omega t). \end{aligned} \quad (6)$$ Given p_a and p_b, find the coefficients V_a and V_b. Express your answer in terms of p_a, p_b, p_0, V_0, γ, τ, τ_{cr}, β, ω, a and ℓ.	1.9pt

B.5	Obtain an approximate expression for the acoustic work per unit volume w produced by the gas parcel over one cycle. Integrate over the volume of the stack to obtain the total work W_{tot} generated by the gas over one cycle. Express W_{tot} in terms of γ, τ, τ_{cr}, β, ω, a, k and S.	0.8pt

B.6	Obtain an approximate expression for the heat Q_{tot} transported from the left side of the plane $x = x_0$ to the right, over a cycle. Express your answer in terms of τ, τ_{cr}, β, ω, a, S, ℓ. (Hint: you may use the formula $j = Q\frac{du}{dt}$ for the heat current due convection.)	0.8pt

B.7	Find the efficiency η of the thermoacoustic engine. The efficiency is defined as the ratio of the generated acoustic work to the heat drawn from the hot reservoir. Express your answer in terms of the temperature difference τ between the hot and the cold reservoir, the critical temperature difference τ_{cr} and the Carnot efficiency $\eta_c = 1 - T_C/T_H$.	0.6pt

Thermoacoustic engine – Solution

Part A: Sound wave in a closed tube

A.1. The boundary conditions are: $u(0,t) = u(L,t) = 0$. As a result, $\sin\left(\frac{2\pi}{\lambda}L\right) = 0$, so we get $\lambda_{\max} = 2L$.

A.2. We get

$$V(x,t) = S\cdot\big(\Delta x + u(x+\Delta x,t) - u(x,t)\big) = S\Delta x\cdot(1+u') = V_0 + V_0 u'.$$

Thus,

$$V(x,t) = V_0 + akV_0\cos(kx)\cos(\omega t) \quad\Rightarrow\quad V_1(x) = akV_0\cos(kx).$$

A.3. We use Newton's Second Law $\rho_0\ddot{u} = -p'$ to deduce $p' = -\rho_0\ddot{u} = \rho_0 a\omega^2\sin(kx)\cos(\omega t)$, so that

$$p(x,t) = p_0 - a\frac{\omega^2}{k}\rho_0\cos(kx)\cos(\omega t) \quad\Rightarrow\quad p_1(x) = a\frac{\omega^2}{k}\rho_0\cos(kx).$$

A.4. Using $a \ll L$, we obtain $\frac{p_1(x)}{p_0} = \gamma\frac{V_1(x)}{V_0}$. As a result, $\frac{\rho_0}{p_0}\frac{\omega^2}{k} = \gamma\cdot k$, and $c = \sqrt{\frac{\gamma p_0}{\rho_0}}$.

A.5. The relative change in $T(x,t)$ is the sum of the relative changes in $V(x,t)$ and $p(x,t)$. As a result,

$$T_1(x) = \frac{T_0}{p_0}p_1(x) - \frac{T_0}{V_0}V_1(x) = (\gamma-1)\frac{T_0}{V_0}V_1(x) = ak(\gamma-1)T_0\cos(kx).$$

A.6. The movement of the gas parcels inside the tube conveys heat along its boundary. To determine the direction of the convection, we combining the result of Task A.5 and the expression (1) for $u(x,t)$. We see that when $0 < x < \frac{L}{2}$, the gas is colder when the displacement $u(x,t)$ is positive. Likewise, when $\frac{L}{2} < x < L$, the gas is colder when the displacement $u(x,t)$ is negative. Hence, heat flows into the gas near the point B, cooling it down, and out of the gas near the points A and C, heating them up.

Part B: Sound wave amplification induced by external thermal contact

B.1. We get

$$T_{\text{env}}(t) = T_{\text{plate}}\big(x_0 + u(x_0,t)\big) = T_0 - \frac{\tau}{\ell}\cdot u(x_0,t),$$

so that:

$$T_{st} = \frac{a\tau}{\ell}\sin(kx_0) = \frac{a\tau}{\ell\sqrt{2}}.$$

B.2. The gas will convey heat from the hot reservoir to the cold one if the parcels are colder than the environment when $u(x_0,t) < 0$, and hotter when $u(x_0,t) > 0$. This occurs precisely if

$$T_{st} > T_1.$$

Plugging in the results of Tasks A.5 and B.1, we get

$$\frac{a\tau_{cr}}{\ell}\sin(kx_0) = ak(\gamma-1)T_0\cos(kx_0) \quad \Rightarrow \quad \tau_{cr} = k\ell(\gamma-1)T_0.$$

B.3. Using the first law of thermodynamics, we get

$$\frac{dQ}{dt} = \frac{dE}{dt} + p\frac{dV}{dt}.$$

Plugging in the relation $E = \frac{1}{\gamma-1}pV$, we see that:

$$\frac{dQ}{dt} = \frac{1}{\gamma-1}\frac{d}{dt}(pV) + p\frac{dV}{dt} = \frac{1}{\gamma-1}V\frac{dp}{dt} + \frac{\gamma}{\gamma-1}p\frac{dV}{dt} \approx \frac{1}{\gamma-1}V_0\frac{dp}{dt} + \frac{\gamma}{\gamma-1}p_0\frac{dV}{dt}.$$

B.4. We plug the expression for $\frac{dQ}{dt}$ into the result of Task B.3. This gives:

$$\frac{1}{\gamma-1}V_0\frac{dp}{dt} + \frac{\gamma}{\gamma-1}p_0\frac{dV}{dt} = \beta V_0(T_{st}-T_1)\cdot\cos(\omega t).$$

We now plug in the data given in equation (6), and get (by considering terms with $\cos(\omega t)$ and $\sin(\omega t)$ separately):

$$\frac{1}{\gamma-1}V_0 p_a\omega + \frac{\gamma}{\gamma-1}p_0 V_a\omega = \beta V_0(T_{st}-T_1)$$

$$\frac{1}{\gamma-1}V_0 p_b\omega - \frac{\gamma}{\gamma-1}p_0 V_b\omega = 0$$

and thus, we can already express V_b as

$$V_b = \frac{1}{\gamma}p_b\cdot\frac{V_0}{p_0}.$$

For V_a, we plug in the results of Tasks B.1 and B.2,

$$T_{st} - T_1 = \frac{a}{\ell\sqrt{2}}(\tau - \tau_{cr}),$$

giving:

$$V_a = \left(-\frac{1}{\gamma}p_a - \frac{\gamma-1}{\gamma}\frac{\beta}{\omega}\frac{a}{\ell\sqrt{2}}(\tau - \tau_{cr})\right)\cdot\frac{V_0}{p_0}.$$

B.5. We want to integrate the mechanical work generated, $\int pdV$, and averaging the result over a long time. To do this, we substitute our expressions (6) for the perturbed p and V. Since the average of $\cos(\omega t)\sin(\omega t)$ is 0, and that of $\sin^2(\omega t)$ and $\cos^2(\omega t)$ is $\frac{1}{2}$, we get:

$$\frac{V_0}{S\ell}W_{\text{tot}} = -\pi\cdot(p_a V_b + p_b V_a).$$

Using the result of B.4, we get

$$\frac{V_0}{S\ell}W_{\text{tot}} = \frac{\pi}{\omega}\cdot\frac{\gamma-1}{\gamma}\beta\frac{a}{\ell\sqrt{2}}(\tau-\tau_{cr})\cdot V_0\frac{p_b}{p_0}.$$

To leading order, p_b is the unperturbed wave $p_b \approx p_1(x_0) = a\frac{\omega^2}{k}\rho_0\cos(kx_0) = ak\gamma p_0\frac{1}{\sqrt{2}}$ Simplifying, we get

$$W_{\text{tot}} = \frac{\pi}{\omega}S\cdot\frac{\gamma-1}{\gamma}\beta\frac{a}{\sqrt{2}}(\tau-\tau_{cr})\cdot\frac{p_b}{p_0} = \frac{\pi}{2\omega}(\gamma-1)\beta(\tau-\tau_{cr})ka^2S.$$

B.6. We want to compute the amount of heat convection over one cycle. This means that we need to take the amount of heat moving in or out of the parcel, and weigh it by the position of the parcel at that time. Thus, the total heat conveyed by the parcel, integrated along a cycle, is:

$$Q_{\text{tot}} = \frac{1}{\Delta x}\int\frac{dQ}{dt}u\cdot dt.$$

This expression can be computed to leading order using $\frac{dQ}{dt} = \beta V_0(T_{\text{st}} - T_1)\cdot\cos(\omega t)$ and the unperturbed displacement $u(x_0, t) = \frac{a}{\sqrt{2}}\cos(\omega t)$. This gives

$$Q_{\text{tot}} = \frac{\pi}{\omega}\beta V_0(T_{\text{st}} - T_1)\frac{a}{\sqrt{2}} = \frac{\pi}{\omega}\beta V_0\cdot\frac{a}{\ell\sqrt{2}}(\tau-\tau_{cr})\cdot\frac{a}{\sqrt{2}} = \frac{\pi}{2\omega}\beta(\tau-\tau_{cr})\frac{a^2S}{\ell}.$$

B.7. Dividing the results of Tasks B.5 and B.6, we obtain the expression:

$$\eta = \frac{W_{\text{tot}}}{Q_{\text{tot}}} = (\gamma-1)k\ell = \frac{\tau_{cr}}{T_0} = \frac{\tau_{cr}}{\tau}\cdot\frac{\tau}{T_0} = \frac{\tau_{cr}}{\tau}\cdot\eta_c.$$